On Total Edge Irregularity Strength of Staircase Graphs and Related Graphs

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ABSTRACT. Let $G = (V(G), E(G))$ be a connected simple undirected graph with non empty vertex set $V(G)$ and edge set $E(G)$. For a positive integer k, by an edge irregular total $k-$labeling we mean a function $f : V(G) \cup E(G) \to \{1, 2, ..., k\}$ such that for each two edges ab and cd, it follows that $f(a) + f(ab) + f(b) \neq f(c) + f(cd) + f(d)$, i.e. every two edges have distinct weights. The minimum k for which G has an edge irregular total $k-$labeling is called the total edge irregularity strength of graph G and denoted by $tes(G)$. In this paper, we determine the exact value of total edge irregularity strength for staircase graphs, double staircase graphs and mirror-staircase graphs.

Keywords: Total edge irregularity strength, Staircase graphs, Double staircase graphs, Mirror-staircase graphs.

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1. Introduction

By Wallis [12], a labeling (or valuation) of a graph is defined as a map that relates the graph element to some numbers (usually to the positive or non negative integers). As the graph element, one can take edge set, vertex set or union of vertex set and edge set. In the case that the graph element is the union of vertex set and edge set, then the labeling is called total labeling. From Gallian [4] we observe that there are various kinds of labeling on graphs.

One sort of a graph labeling is proposed by Chartrand, et al. in [3] as the following. Assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, i.e., the weights (label sums) at each vertex are distinct. What is the minimum value of the largest label over all such irregular assignments? The value is then well known as the irregularity strength of the graph. In the case that the labeling is total, and the weighted components of the graph are its edges, then the value is called as the total edge irregularity strength of the graph.

Now, let $G = (V(G), E(G))$ be a connected simple undirected graph with non empty vertex set $V(G)$ and edge set $E(G)$. Let $|E(G)|$ and $|V(G)|$ be the number of the edges and the vertices, respectively. Bača et al. [2] studied the weight $wt_f(xy)$ of an edge xy under a total labeling $f : V(G) \cup E(G) \mapsto \{1, 2, ..., k\}$ of elements of a graph G which is defined by

$$wt_f(xy) = f(x) + f(xy) + f(y).$$

The total labeling f is called an edge irregular total $k-$labeling whenever each two edges ab and cd have distinct weights, that are $f(a) + f(ab) + f(b) \neq f(c) + f(cd) + f(d)$. The minimum k for which G has an edge irregular total $k-$labeling is denoted by $tes(G)$ and is called the total edge irregularity strength of graph G.

The lower bound of the total edge irregularity strength of any graph is given by Bača et al. [2] as follows.

Theorem 1.1. [2] *For any graph G which maximum vertex degree is $\Delta(G)$, it follows that $tes(G) \geq max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}$.*

The following conjecture presented by Ivanco and Jendrol gave the exact value of the total edge of irregularity strength for arbitrary graph.

Conjecture 1.2. [5] *For any graph G which maximum vertex degree is $\Delta(G)$,*

$$tes(G) = max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}.$$

The conjecture has been proved to be true for all trees, complete graphs except K_5, complete bipartite graphs, zig zag graphs, helms, closed helms, fans, tadpoles, armed crowns, webs, flowers, gears, sun flowers, and some other graphs. The *tes* of trees was given by Ivančo and Jendrol ([5]) while the *tes*

for complete graphs and complete bipartite graphs were studied by Jendrol, et al. [6]. The *tes* of zig zag graphs was given by Ahmad in [1]. In [7] and [8], Putra, et al. observed the *tes* of centralized uniform theta graphs and the *tes* of uniform theta graphs, respectively. In [9] and [10], Ratnasari, et al. gave the exact value of the *tes* of some ladder related graphs and arithmetic book graphs, respectively.

Based on [4], aside from total edge irregularity strength, there are many kinds of graph labeling. One among many kinds of labeling is a graceful labeling. We found an interesting work done by Solairaju, et al. [11] about the graceful mirror-staircase graphs. Some graphs that are presented in the paper are staircase graphs, double-staircase graphs, and mirror-staircase graphs. Therefore we are interested in observing the *tes* of the graphs. In this paper, we give the exact value of the total edge irregularity strength of the staircase, double-staircase and mirror-staircase graphs.

2. Results

In this paper, the first graph which the *tes* will be observed is the staircase graph. First, we denote the staircase graph of size $n \geq 1$ by SC_n (see Figure 1). We have

$$V(SC_n) = \{v_{i,j} | i = 0, 1, j = 0, 1, 2, \ldots, n\} \cup \{v_{i,j} | i = 2, \ldots, n, j = i - 1, \ldots, n\}$$

and $E(SC_n)$ which consists of edges as the following.

edges	i	j
$v_{i,j}v_{i+1,j}$	0	$0, 1, 2, \ldots, n$
$v_{i,j}v_{i+1,j}$	$1, \ldots, n-1$	$i, \ldots, n$
$v_{i,j}v_{i,j+1}$	$0, 1$	$0, 1, 2, \ldots, n-1$
$v_{i,j}v_{i,j+1}$	$2, \ldots, n$	$i - 1, \ldots, n - 1$

By a simple calculation we have that $|V(SC_n)| = \frac{1}{2}(n + 1)(n + 2) + n$ and $|E(SC_n)| = n(n + 3)$.
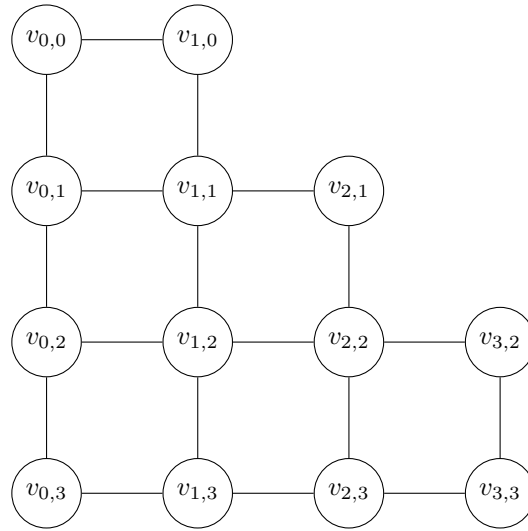
Theorem 2.1. *Let SC_n be the staircase graph of order n. Then*

$$tes(SC_n) = \left\lceil \frac{n(n + 3) + 2}{3} \right\rceil.$$

Proof. It is clear that the maximum degree of the staircase graph is 2 for $n = 1$ and 4 for otherwise. Therefore, by Theorem (1.1), we obtain

$$tes(SC_n) \geq \left\lceil \frac{n(n + 3) + 2}{3} \right\rceil.$$

To complete the proof we show that $tes(SC_n) \leq \left\lceil \frac{n(n+3)+2}{3} \right\rceil$, by showing that there exists a total edge irregularity k-labeling with $k = \left\lceil \frac{n(n+3)+2}{3} \right\rceil$. First, we

FIGURE 1. Staircase Graph SC_3

determine the biggest positive number m such that

$$\frac{m(m+1)}{2} \leq \left\lceil \frac{n(n+3)+2}{3} \right\rceil - 1.$$

(Several n and m are listed on Table 1.)

n	m		n	m		n	m		n	m		n	m
1	1		6	5		11	9		16	13		21	17
2	2		7	6		12	10		17	14		22	18
3	3		8	7		13	11		18	15		23	19
4	3		9	8		14	12		19	16		24	20
5	4		10	8		15	12		20	17		25	21

TABLE 1.

We define a k-labeling

$$f : V(SC_n) \cup E(SC_n) \to \{1, 2, \ldots, \left\lceil \frac{n(n+3)+2}{3} \right\rceil \}$$

to label the graph elements in the following way:

edges and vertices label	i and j
$f(v_{i,0}) = 1$	$i = 0, 1$
$f(v_{i,j}) = \frac{j^2 + j + 2}{2}$	$i = 0, 1, \ldots, j+1$
	$j = 1, 2, \ldots, m$
$f(v_{i,j}v_{i+1,j}) = i + j + 1$	$i = 0, 1, \ldots, j$
	$j = 0, \ldots, m-1$
$f(v_{i,j}v_{i+1,j}) = i + j + 1$	$i = 0, 1, \ldots, m-1$
	$j = m$
$f(v_{i,j}v_{i,j+1}) = i + j + 1$	$i = 0, 1, \ldots, j+1$
	$j = 0, \ldots, m-1$

We stop the process if $m = n$. If $m < n$, then we continue with

edges and vertices label	i and j
$f(v_{i,j}v_{i+1,j}) = i + j + 1$	$i = m$
	$j = m$
$f(v_{i,j}) = \lceil \frac{n(n+3)+2}{3} \rceil$	$i = 0, \ldots, j+1$
	$j = m+1, \ldots, n$
$f(v_{i,j}v_{i,j+1}) = i + \frac{(m+3)(m+2)}{2} - \lceil \frac{n(n+3)+2}{3} \rceil$	$i = 0, \ldots, m+1$
	$j = m$
$f(v_{i,j}v_{i+1,j}) = i + 6 + (m+k-1)(m+k+3) - 2\lceil \frac{n(n+3)+2}{3} \rceil$	$i = 0, \ldots, j$
	$j = m+k$
	$k = 1, \ldots, n-m-1$
$f(v_{i,j}v_{i,j+1}) = i + 4 + (m+k)(m+k+3) - 2\lceil \frac{n(n+3)+2}{3} \rceil$	$i = 0, 1, \ldots, m+k+1$
	$j = m+k$
	$k = 1, \ldots, n-m-1$
$f(v_{i,j}v_{i+1,j}) = i + n^2 + 2n + 3 - 2\lceil \frac{n(n+3)+2}{3} \rceil$	$j = n$
	$i = 0, 1, \ldots, n-1$

We then have the weights of the edges as follows:

weights	i and j
$wt_f(v_{i,0}v_{i+1,0}) = i + 3$	$i = 0$
$wt_f(v_{i,0}v_{i,1}) = i + 4$	$i = 0, 1$
$wt_f(v_{i,j}v_{i+1,j}) = i + j^2 + 2j + 3$	$i = 0, 1, \ldots, j$ $j = 1, 2, \ldots, m$
$wt_f(v_{i,j}v_{i,j+1}) = i + j^2 + 3j + i + 4$	$i = 0, 1, \ldots, j+1$ $j = 1, 2, \ldots, m-1$
$wt_f(v_{i,j}v_{i,j+1}) = i + \frac{j^2+j+2+(m+3)(m+2)}{2}$	$i = 0, \ldots, m+1$ $j = m$
$wt_f(v_{i,j}v_{i+1,j}) = i + 6 + (m+k-1)(m+k+3)$	$i = 0, \ldots, j$ $j = m+k$ $k = 1, \ldots, n-m-1$
$wt_f(v_{i,j}v_{i,j+1}) = i + 4 + (m+k)(m+k+3)$	$i = 0, 1, \ldots, m+k+1$ $j = m+k$ $k = 1, \ldots, n-m-1$
$wt_f(v_{i,j}v_{i+1,j}) = i + n^2 + 2n + 3$	$j = n$ $i = 0, 1, \ldots, n-1$

$\square$

Now we are going to consider the double staircase for the second observation. In this paper, we denote the double staircase (See Figure 2) of order n by DSC_n. We have

$$V(DSC_n) = \{l_{i,j} | i = 1, 2, \ldots, n, j = i - 1, \ldots, n\}$$

$$\cup \{r_{i,j} | i = 1, 2, \ldots, n, j = i - 1, \ldots, n\}$$

and $E(DSC_n)$ which consists of edges as given in the following table.

edges	i	j
$l_{i,j}r_{i,j}$	1	$0, 1, 2, \ldots, n$
$l_{i+1,j}l_{i,j}$	$1, 2, \ldots, n-1$	$1, 2, \ldots, n$
$r_{i,j}r_{i+1,j}$	$1, 2, \ldots, n-1$	$1, 2, \ldots, n$
$l_{i,j}l_{i,j+1}$	$1, 2, \ldots, n$	$i - 1, \ldots, n$
$r_{i,j}r_{i,j+1}$	$1, 2, \ldots, n$	$i - 1, \ldots, n$

By a short calculation we have that $|V(DSC_n)| = n^2 + 3n$ and $|E(DSC_n)| = 2n^2 + 3n - 1$.

Theorem 2.2. *Let DSC_n be the double staircase graph of order n. Then*

$$tes(DSC_n) = \left\lceil \frac{2n^2 + 3n + 1}{3} \right\rceil .$$

Proof. Similar to the staircase graph, we know that the maximum degree of the double staircase graph is 2 for $n = 1$ and 4 for otherwise. Thus, by Theorem
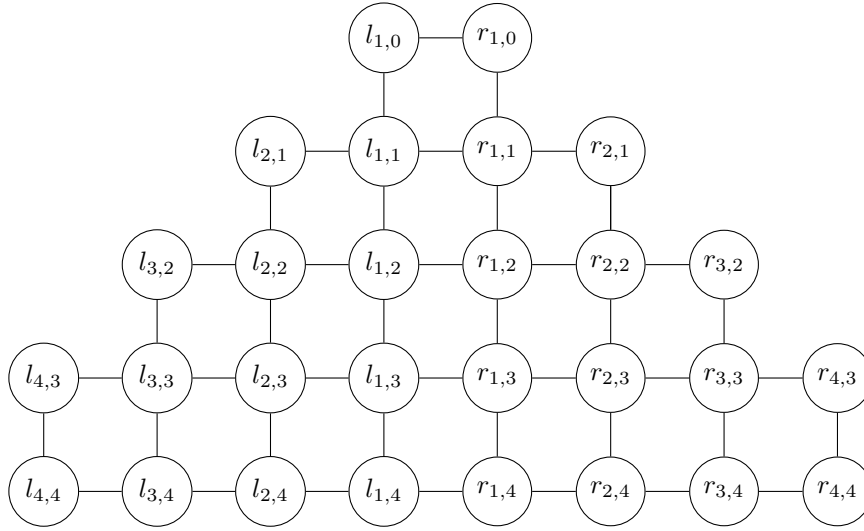
FIGURE 2. Double Staircase Graph DSC_4

(1.1), we have

$$tes(DSC_n) \geq \left\lceil \frac{2n^2 + 3n + 1}{3} \right\rceil.$$

Again, in order to converse the inequality, all we have to do is to show the existence of total edge irregularity k-labeling with $k = \left\lceil \frac{2n^2+3n+1}{3} \right\rceil$.

Let $k = \left\lceil \frac{2n^2+3n+1}{3} \right\rceil$, then we construct an

$$f : V(DSC_n) \cup E(DSC_n) \to \{1, 2, \ldots, k\}$$

as a k-labeling in following manner:

edges and vertices label	i and j
$f(l_{i,j}) = \left\lceil \frac{2j^2+3j+1}{3} \right\rceil$	$i = 1, 2, \ldots, n$
	$j = i - 1, \ldots, n$
$f(r_{i,j}) = \left\lceil \frac{2j^2+3j+1}{3} \right\rceil$	$i = 1, 2, \ldots, n$
	$j = i - 1, \ldots, n$
$f(l_{i,j}r_{i,j}) = 1$	$i = 1, j = 0$
$f(l_{1,j}r_{1,j}) = (2\lceil \frac{j}{3} \rceil - 1)j - \lceil \frac{j}{3} \rceil + 2$	$1 \leq j \leq n, j = 1 \bmod 3$
$f(l_{1,j}r_{1,j}) = 2\lceil \frac{j}{3} \rceil(j - 1) + 2$	$1 \leq j \leq n, j = 2 \bmod 3$
$f(l_{1,j}r_{1,j}) = 2j\lceil \frac{j}{3} \rceil - 3\lceil \frac{j}{3} \rceil + j$	$1 \leq j \leq n, j = 0 \bmod 3$
$f(l_{i+1,i}l_{i,i}) = i^2 - 2i - \lceil \frac{i-4}{3} \rceil(i-1) + 4\lceil \frac{i-1}{3} \rceil + 4$	$1 \leq i \leq n - 1, i = 1 \bmod 3$
$f(l_{i+1,i}l_{i,i}) = i^2 - 4i - \lceil \frac{i-5}{3} \rceil(i-2) + 8\lceil \frac{i-2}{3} \rceil + 10$	$1 \leq i \leq n - 1, i = 2 \bmod 3$
$f(l_{i+1,i}l_{i,i}) = i\left(i - \lceil \frac{i-3}{3} \rceil\right)$	$1 \leq i \leq n - 1, i = 0 \bmod 3$
$f(l_{i+1,j}l_{i,j}) = \lceil \frac{2j^2+3j+1}{3} \rceil - 3\lceil \frac{j-2}{3} \rceil -i - 1$	$1 \leq i \leq n - 1$ $i + 1 \leq j \leq n, j = 2 \bmod 3$
$f(l_{i+1,j}l_{i,j}) = \lceil \frac{2j^2+3j+1}{3} \rceil - 6\lceil \frac{j-1}{3} \rceil +j - i - 1$	$1 \leq i \leq n - 1$ $i + 1 \leq j \leq n, j = 1 \bmod 3$ or $j = 0 \bmod 3$
$f(l_{i,j}l_{i,j+1}) = 2(j+1)\lceil \frac{j-1}{3} \rceil + 2\lceil \frac{j}{3} \rceil -i + 2$	$i = 1, 2, \ldots, n$ $i - 1 \leq j \leq n - 1$ $j = 1 \bmod 3$
$f(l_{i,j}l_{i,j+1}) = 2j\lceil \frac{j}{3} \rceil - i + 2$	$i = 1, 2, \ldots, n$ $i - 1 \leq j \leq n - 1$ $j = 2 \bmod 3$
$f(l_{i,j}l_{i,j+1}) = 2(j+1)\lceil \frac{j}{3} \rceil - i + 2$	$i = 1, 2, \ldots, n$ $i - 1 \leq j \leq n - 1$ $j = 0 \bmod 3$
$f(r_{i,i}r_{i+1,i}) = i^2 - 2i - \lceil \frac{i-4}{3} \rceil(i-1) + 4\lceil \frac{i-1}{3} \rceil + 5$	$1 \leq i \leq n - 1, i = 1 \bmod 3$
$f(r_{i,i}r_{i+1,i}) = i^2 - 4i - \lceil \frac{i-5}{3} \rceil(i-2) + 8\lceil \frac{i-2}{3} \rceil + 11$	$1 \leq i \leq n - 1, i = 2 \bmod 3$
$f(r_{i,i}r_{i+1,i}) = i\left(i - \lceil \frac{i-3}{3} \rceil\right) + 1$	$1 \leq i \leq n - 1, i = 0 \bmod 3$
$f(r_{i,j}r_{i+1,j}) = \lceil \frac{2j^2+3j+1}{3} \rceil - j + i - 1$	$1 \leq i \leq n - 1$ $i + 1 \leq j \leq n, j = 0 \bmod 3$
$f(r_{i,j}r_{i+1,j}) = \lceil \frac{2j^2+3j+1}{3} \rceil - j + i + 1$	$1 \leq i \leq n - 1$ $i + 1 \leq j \leq n, j = 1 \bmod 3$ or $j = 2 \bmod 3$
$f(r_{i,j}r_{i,j+1}) = 2(j+1)\lceil \frac{j-1}{3} \rceil + 2\lceil \frac{j}{3} \rceil + i + 1$	$i = 1, 2, \ldots, n$ $i - 1 \leq j \leq n - 1$ $j = 1 \bmod 3$
$f(r_{i,j}r_{i,j+1}) = 2j\lceil \frac{j}{3} \rceil + i + 1$	$i = 1, 2, \ldots, n$ $i - 1 \leq j \leq n - 1$ $j = 2 \bmod 3$
$f(r_{i,j}r_{i,j+1}) = 2(j+1)\lceil \frac{j}{3} \rceil + i + 1$	$i = 1, 2, \ldots, n$ $i - 1 \leq j \leq n - 1$
$j = 0 \bmod 3$	

Furthermore, to prove that f is an edge irregular total k-labeling, we need to consider the weight of all edges in DSC_n. We obtain the weight of all edges, as follows:

weight	i and j
$wt_f(l_{i,j}r_{i,j}) = 3$	$i = 1, j = 0$
$wt_f(l_{1,j}r_{1,j}) = 2j^2 + 2j + 2$	$1 \le j \le n$
$wt_f(l_{i+1,i}l_{i,i}) = 2i^2 + 3i + 2$	$1 \le i \le n - 1$
$wt_f(l_{i+1,j}l_{i,j}) = 2j^2 + 2j - i + 2$	$1 \le i \le n - 1$
	$i + 1 \le j \le n$
$wt_f(l_{i,j}l_{i,j+1}) = 2j^2 + 4j + 5 - i$	$i = 1, 2, \ldots, n$
	$i - 1 \le j \le n - 1$
$wt_f(r_{i,i}r_{i+1,i}) = 2i^2 + 3i + 3$	$1 \le i \le n - 1$
$wt_f(r_{i,j}r_{i+1,j}) = 2j^2 + 2j + i + 2$	$1 \le i \le n - 1$
	$i + 1 \le j \le n$
$wt_f(r_{i,j}r_{i,j+1}) = 2j^2 + 4j + i + 4$	$i = 1, 2, \ldots, n$
	$i - 1 \le j \le n - 1$

We can list all of the weight of edges by the indices. Then, from the list, we obtain a sequence of integers from 3 up to $2n^2 + 3n + 1$. It means that every different edge have different weight. It also implies that f is an edge irregular total k-labeling with $k = \left\lceil \frac{2n^2 + 3n + 1}{3} \right\rceil$.

Thus, we prove that $tes(DSC_n) = \left\lceil \frac{2n^2 + 3n + 1}{3} \right\rceil$. $\qquad\qquad\square$

For the last observation, we consider the mirror staircase graph of size $n \ge 1$ which we denote by MSC_n (see Figure 3). We have

$$V(MSC_n) = \{v_{i,j} | i = -1, 0, 1, j = 0, 1, 2, \ldots, n\} \cup \{v_{i,j} | i = 2, \ldots, n, j = i - 1, \ldots, n\}$$
$$\cup \{v_{i,j} | i = -2, \ldots, -n, j = -i - 1, \ldots, n\}$$

and $E(MSC_n)$ which consists of edges as listed below

edges	i	j
$v_{i,j}v_{i+1,j}$	$-1, 0$	$0, 1, 2, \ldots, n$
$v_{i,j}v_{i+1,j}$	$1, \ldots, n - 1$	$i, \ldots, n$
$v_{i,j}v_{i+1,j}$	$-2, \ldots, -n$	$-i - 1, \ldots, n$
$v_{i,j}v_{i,j+1}$	$-1, 0, 1$	$0, 1, 2, \ldots, n - 1$
$v_{i,j}v_{i,j+1}$	$2, \ldots, n$	$i - 1, \ldots, n - 1$
$v_{i,j}v_{i,j+1}$	$-2, \ldots, -n$	$-i - 1, \ldots, n - 1.$

By a routine calculation we have that $|V(MSC_n)| = n^2 + 4n + 1$ and $|E(MSC_n)| = n(2n + 5)$.

Theorem 2.3. *Let MSC_n be the mirror staircase graph of order n. Then*

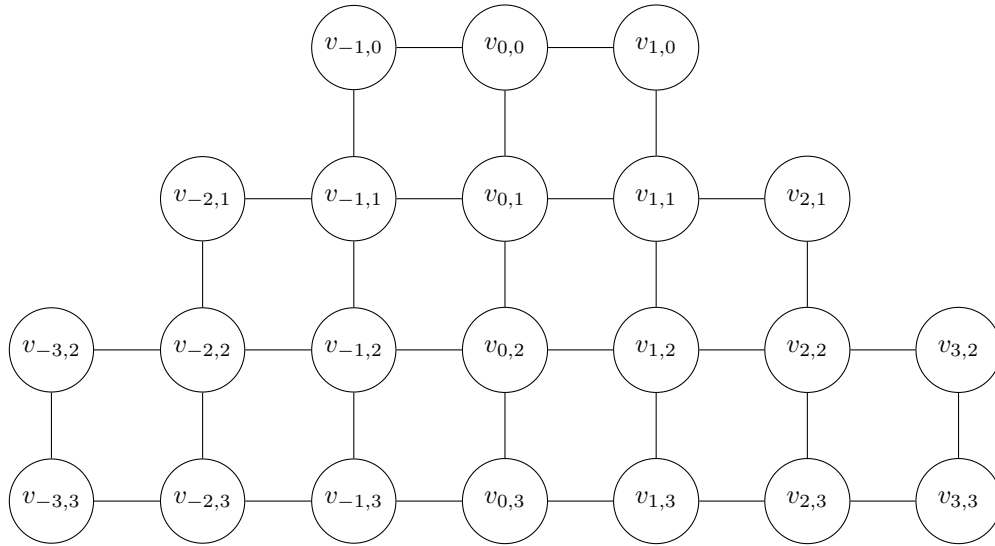$$tes(MSC_n) = \left\lceil \frac{n(2n + 5) + 2}{3} \right\rceil.$$

FIGURE 3. Mirror Staircase Graph MSC_3

Proof. It is clear that the maximum degree of the staircase graph is 3 for $n = 1$ and 4 for otherwise. Therefore, by Theorem (1.1), we obtain

$$tes(MSC_n) \geq \left\lceil \frac{n(2n + 5) + 2}{3} \right\rceil.$$

It remains to show that $tes(MSC_n) \leq \left\lceil \frac{n(2n+5)+2}{3} \right\rceil$ by proving that there exists a total edge irregularity k-labeling with $k = \left\lceil \frac{n(2n+5)+2}{3} \right\rceil$. First, we determine the biggest positive number m such that

$$m^2 + m + 1 \leq \left\lceil \frac{n(2n + 5) + 2}{3} \right\rceil - 1.$$

(Several n and m are listed on Table 2.)

n	m		n	m		n	m		n	m		n	m
1	1		6	5		11	9		16	13		21	17
2	2		7	6		12	10		17	14		22	18
3	2		8	6		13	11		18	15		23	19
4	3		9	7		14	11		19	15		24	20
5	4		10	8		15	12		20	16		25	20

TABLE 2.

We define a k-labeling

$$f : V(MSC_n) \cup E(MSC_n) \to \left\{ 1, 2, \ldots, \left\lceil \frac{n(2n+5)+2}{3} \right\rceil \right\}$$

to label the graph elements in the following way:

edges and vertices label	i and j
$f(v_{i,0}) = 1$	$i = -1, 0, 1$
$f(v_{i,j}) = j^2 + j + 1$	$i = -(j+1), \ldots, j+1$
	$j = 1, 2, \ldots, m$
$f(v_{i,j}) = \lceil \frac{n(2n+5)+2}{3} \rceil$	$i = -(j+1), \ldots, j+1$
	$j = m+1, \ldots, n-1$
$f(v_{i,n}) = \lceil \frac{n(2n+5)+2}{3} \rceil$	$i = -n, \ldots, n$
$f(v_{i,j}v_{i+1,j}) = i + 2j + 2$	$i = -(j+1), \ldots, j$
	$j = 0, \ldots, m$
$f(v_{i,j}v_{i,j+1}) = i + 2j + 2$	$i = -(j+1), \ldots, j+1$
	$j = 0, \ldots, m-1.$

We stop the process if $m = n$. If $m < n$, then we continue with

edges and vertices label	i and j
$f(v_{i,j}v_{i,j+1}) = i + m^2 + 5(m+1) - \lceil \frac{n(2n+5)+2}{3} \rceil$	$i = -(j+1), \ldots, j+1$
	$j = m$
$f(v_{i,j}v_{i+1,j}) = i + 2((m+k+1)^2 - \lceil \frac{n(2n+5)+2}{3} \rceil)$ $+2$	$i = -(j+1), \ldots, j$
	$j = m+k$
	$k = 1, \ldots, n-m-1$
	$k \geq 1$
$f(v_{i,j}v_{i,j+1}) = i + 2((m+k+1)^2 - \lceil \frac{n(2n+5)+2}{3} \rceil)$ $+2(m+k)+4$	$i = -(j+1), \ldots, j+1$
	$j = m+k$
	$k = 1, \ldots, n-m-1$
	$k \geq 1$
$f(v_{i,j}v_{i+1,j}) = i + 2((n+1)^2 - \lceil \frac{n(2n+5)+2}{3} \rceil) + 1$	$i = -n, \ldots, n-1$
	$j = n$

For arbitrary edge xy, the weight of xy is $wt_f(x,y) = f(x) + f(x,y) + f(y)$. Therefore, we obtain the weight of all edges as follows:

weight	i and j
$wt_f(v_{i,j}v_{i+1,j}) = i + 4$	$i = -1, 0$ $j = 0$
$wt_f(v_{i,j}v_{i+1,j}) = i + 2j^2 + 4j + 4$	$i = -(j+1), \ldots, j$ $j = 1, 2, \ldots m$
$wt_f(v_{i,j}v_{i+1,j}) = i + 2(m+k+1)^2 + 2$	$i = -(j+1), \ldots, j$ $j = m + k$ $k = 1, \ldots, n - m - 1$
$wt_f(v_{i,j}v_{i+1,j}) = i + 2(n+1)^2 + 1$	$i = -n, \ldots, n - 1$ $j = n$
$wt_f(v_{i,j}v_{i,j+1}) = i + 6$	$i = -1, 0, 1$ $j = 0$
$wt_f(v_{i,j}v_{i,j+1}) = i + 2j^2 + 6j + 6$	$i = -(j+1), \ldots, j + 1$ $j = 1, 2, \ldots, m$
$wt_f(v_{i,j}v_{i,j+1}) = i + 2(m+k+1)(m+k+2) + 2$	$i = -(j+1), \ldots, j + 1$ $j = m + k$ $k = 1, 2, \ldots, n - m - 1$

We can see that the weight of edge xy, for each $xy \in E(MSC_n)$ is all different. $\qquad\square$

Acknowledgments

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