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Diophantine Equations Related with Linear Binary Recurrences

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Abstract. In this paper we find all solutions of four kinds of the Diophantine equations

$$x^2 \pm V_t xy - y^2 \pm x = 0 \text{ and } x^2 \pm V_t xy - y^2 \pm y = 0,$$

for an odd number t, and,

$$x^2 \pm V_t xy + y^2 - x = 0 \text{ and } x^2 \pm V_t xy + y^2 - y = 0,$$

for an even number t, where V_n is a generalized Lucas number. This paper continues and extends a previous work of Bahramian and Daghigh.

Keywords: Linear recurrences, Generalized Fibonacci and Lucas sequences, Diophantine equations, Continued fractions.

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1. Introduction

Since ancient times mathematicians tried to solve equations over the integers. Some of these equations are called *diophantine equations*. Diophantine equations have been studied by many authors to date. In 1909, A. Thue proved the following important theorem:

> *Let* $f = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ *be an irreducible polynomial of degree* ≥ 3 *with integer coefficients. Consider the corresponding homogeneous polynomial*
>
> $$F(x,y) = a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_1 x y^{n-1} + a_0 y^n.$$
>
> *If* m *is a nonzero integer, then the equation* $F(x,y) = m$ *has either no solution or only a finite number of solutions in integers.*

This result is in contrast to the situation when the degree of F is $n = 2$. In this case, if $F(x,y) = x^2 - Dy^2$, where D is a nonsquare positive integer, then for all nonzero integers m, the general Pell's equation $x^2 - Dy^2 = m$ has either no solution or it has infinitely many integral solutions [2].

Let p be a nonzero integer. The generalized Fibonacci and Lucas numbers are defined by

$$U_{n+1} = pU_n + U_{n-1} \qquad \text{and} \qquad V_{n+1} = pV_n + V_{n-1},$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively. When $p = 1$, $U_n = F_n$ (the n^{th} Fibonacci number) and $V_n = L_n$ (the n^{th} Lucas number). The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad V_n = \alpha^n + \beta^n,$$

where $(\alpha, \beta) := \left(\left(p + \sqrt{p^2+4}\right)/2, \left(p - \sqrt{p^2+4}\right)/2\right)$.

Also we note that $U_{-n} = (-1)^{n+1} U_n$ and $V_{-n} = (-1)^n V_n$ for $n \geq 0$. Kılıç and Stanica [8] showed that for any integers t and n,

$$\begin{aligned} U_{tn} &= V_t U_{t(n-1)} + (-1)^{t+1} U_{t(n-2)}, \\ V_{tn} &= V_t V_{t(n-1)} + (-1)^{t+1} V_{t(n-2)}. \end{aligned} \tag{1.1}$$

It is also known that

$$U_{t(n+1)} U_{t(n-1)} - U_{tn}^2 = (-1)^{t(n+1)+1} U_t^2. \tag{1.2}$$

According to Dickson ([4], p. 405), Lucas proved that if x and y are consecutive Fibonacci numbers, then (x,y) is a lattice point on one of the hyperbolas $y^2 - xy - x^2 = \pm 1$, and Wasteels proved the converse in 1902. Interest in conics whose equations are satisfied by pairs of successive terms of linear recursive sequences has been rekindled.

In [9], Kimberling defined a *Fibonacci hyperbola* and solved some of them. For example, he considered the following type hyperbolas

$$p_n(x,y) = x^2 + (-1)^{n+1} L_n xy + (-1)^n y^2 + F_n^2 = 0, \text{ for } n = 1,2,3,\ldots.$$

In [11], McDaniel proved that, if $P > 0$ and x, y are positive integers, then the pair (x,y) is a solution of $y^2 - Pxy - x^2 = \pm 1$ if and only if there exists a positive integer n such that $x = U_n$ and $y = U_{n+1}$.

In [12], Melham generalized McDaniel's results and obtained new ones. For example, if m is a fixed even integer, then the points with integer coordinates on the conics $y^2 - V_m xy + x^2 = \pm U_m^2$ are precisely the pairs $(x,y) = \pm(U_n, U_{n+m})$.

Marlewski and Zarzycki [10] showed that for $k \in \mathbb{Z}^+$, the equation $x^2 - kxy + y^2 + x = 0$ has an infinite number of positive integer solutions x and y if and only if $k = 3$.

In [7], Kılıç and Ömür considered all given results on special conics mentioned in [11, 12] and then gave more general results.

Bahramian and Daghigh [3] proved that for $k \in \mathbb{Z}$, the equations $x^2 \pm kxy - y^2 \pm x = 0$ have an infinite number of positive integer solutions x and y, and, they gave their solutions in terms of a generalized Fibonacci sequence. Also some authors have studied and solved certain similar equations, for more details see [1, 5, 6, 14].

In this paper, we find all solutions of the following four kinds of Diophantine equations:

(1) For odd t,

$$i)\ x^2 \pm V_t xy - y^2 \pm x = 0 \quad \text{and} \quad ii)\ x^2 \pm V_t xy - y^2 \pm y = 0.$$

(2) For even t,

$$iii)\ x^2 \pm V_t xy + y^2 - x = 0 \quad \text{and} \quad iv)\ x^2 \pm V_t xy + y^2 - y = 0.$$

The case $t = 1$ of $(1.i)$ and, that is $V_1 = p$, was studied by Bahramian and Daghigh in [3]. The Diophantine equations $(1.ii)$–$(2.iv)$ will be examined and solved for the first time in this study according to our best knowledge.

We shall solve these equations by the equation of Thue and continued fraction representation.

2. The Diophantine Equations

In this section, we show that the Diophantine equations given by $(1.i)$–$(2.iv)$ mentioned in the introduction section are solvable in integers for certain integers t. Before proving this we give some preliminary results.

Let D be a positive integer not a perfect square. Suppose that $\sqrt{D}$ is written as an infinite simple continued fraction $\sqrt{D} = [a_0, a_1, a_2, \cdots]$. For each nonnegative n, the rational number $[a_0, a_1, a_2, \cdots, a_n] = h_n/k_n$ is called the

n^{th} convergent to the infinite simple continued fraction $[a_0, a_1, a_2, \cdots]$. Then we get for $n \geq 1$,

$$\begin{aligned} h_n &= a_n h_{n-1} + h_{n-2}, \quad h_{-1} = 1,\ h_0 = a_0, \\ k_n &= a_n k_{n-1} + k_{n-2}, \quad k_{-1} = 0,\ k_0 = 1. \end{aligned} \tag{2.1}$$

If $V_t^2 + 4$ is not a square for any integer t, we have the infinite simple continued fraction representation of $\sqrt{V_t^2+4}$ as

$$\sqrt{V_t^2+4} = \begin{cases} \left[V_t,\ \overline{(V_t-1)/2,\ 1,\ 1,\ (V_t-1)/2,\ 2V_t}\right] & \text{if } V_t \text{ is odd,} \\ \left[V_t,\ \overline{V_t/2,\ 2V_t}\right] & \text{if } V_t \text{ is even.} \end{cases} \tag{2.2}$$

The next two theorems give us the convergents of infinite simple continued fraction representation of $\sqrt{V_t^2+4}$ via terms of the sequence $\{U_{tn}\}$. Similar results are proved in [3].

Theorem 2.1. *Let V_t be a positive odd integer and h_n/k_n be the n^{th} convergent to the infinite simple continued fraction of $\sqrt{V_t^2+4}$. For $n \geq 0$*

a) $h_{10n} = ((-1)^{t+1}U_{6nt} + U_{(6n+2)t})/U_t$,
b) $k_{10n} = U_{(6n+1)t}/U_t$,
c) $h_{10n+4} = 1/2((-1)^{t+1}U_{(6n+2)t} + U_{(6n+4)t})/U_t$,
d) $k_{10n+4} = 1/2(U_{(6n+3)t}/U_t)$,
e) $h_{10n+8} = ((-1)^{t+1}U_{(6n+4)t} + U_{(6n+6)t})/U_t$,
f) $k_{10n+8} = U_{(6n+5)t}/U_t$.

Proof. We give only some sketches for the proof of (a) and (b). Since V_t is a positive odd integer and by (2.2), we have that for $n \geq 1$

$$a_0 = V_t,\ a_{5n-4} = (V_t-1)/2,\ a_{5n-3} = 1,$$
$$a_{5n-2} = 1,\ a_{5n-1} = (V_t-1)/2,\ a_{5n} = 2V_t.$$

The second order recursive sequences $\{h_n\}$ and $\{k_n\}$ given in (2.1) could be written in matrix equality form by using their recursions, for $n \geq 1$

$$\begin{bmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_{n-1} & k_{n-1} \\ h_{n-2} & k_{n-2} \end{bmatrix}.$$

Let $A_n = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$ and $P_n = \begin{bmatrix} h_n & k_n \\ h_{n-1} & k_{n-1} \end{bmatrix}$. Then, for $n \geq 1$, we have $P_n = A_n P_{n-1}$. Here we need h_{10n} and k_{10n}. For this, first we compute P_{5n} and then take $2n$ instead of n. Thus we have the conclusions. □

We have the following result without proof.

Theorem 2.2. *Let V_t be a positive even integer and h_n/k_n be the n^{th} convergent to the infinite simple continued fraction of $\sqrt{V_t^2+4}$. For $n \geq 0$,*

a) $h_{2n} = ((-1)^{t+1}U_{2nt} + U_{(2n+2)t})/U_t$, *b)* $k_{2n} = U_{(2n+1)t}/U_t$.

First we will show that all solutions of the equation

$$x^2 - V_t xy - y^2 + x = 0 \tag{2.3}$$

are

$$\begin{aligned}
&\left(U_{2nt}^2/U_t^2, U_{(2n-1)t}U_{2nt}/U_t^2\right), \\
&\left(U_{2nt}^2/U_t^2, -U_{2nt}U_{(2n+1)t}/U_t^2\right), \\
&\left(-U_{(2n+1)t}^2/U_t^2, U_{(2n+1)t}U_{(2n+2)t}/U_t^2\right), \\
&\left(-U_{(2n+1)t}^2/U_t^2, -U_{2nt}U_{(2n+1)t}/U_t^2\right).
\end{aligned} \tag{2.4}$$

To prove this claim, first we need the following lemma whose proof is straightforward.

Lemma 2.3. *If (x, y) is a solution of the equation (2.3), then $(x, -V_t x - y)$ and $(V_t y - x - 1, y)$ are also solutions of (2.3).*

Clearly $(0, 0)$ is a solution of (2.3), from Lemma 2.3, a sequence of solutions of (2.3) is

$$(0,0), \ (-1,0), \ (-1,V_t), \ \left(V_t^2, V_t\right), \ \left(V_t^2, -V_t\left(V_t^2+1\right)\right), \ldots$$

and these solutions can be rewritten as

$$\begin{aligned}
(0,0) &= \left(\frac{U_0^2}{U_t^2}, -\frac{U_0U_t}{U_t^2}\right), \ (-1,0) = \left(-\frac{U_t^2}{U_t^2}, -\frac{U_0U_t}{U_t^2}\right), \\
(-1,V_t) &= \left(-\frac{U_t^2}{U_t^2}, \frac{U_tU_{2t}}{U_t^2}\right), \ \left(V_t^2, V_t\right) = \left(\frac{U_{2t}^2}{U_t^2}, \frac{U_tU_{2t}}{U_t^2}\right), \ldots .
\end{aligned}$$

Theorem 2.4. *For any integer n and odd t, the pairs in (2.4) satisfy $x^2 - V_t xy - y^2 + x = 0$.*

Proof. (By induction on n.) For $n = 0$, it is seen that $(0,0) = \left(U_0^2/U_t^2, -U_0U_t/U_t^2\right)$ is a solution of (2.3). Suppose that the pair $\left(U_{2tn}^2/U_t^2, -U_{2tn}U_{t(2n+1)}/U_t^2\right)$ satisfies (2.3). By Lemma 2.3 and (1.2), we have that

$$\begin{aligned}
(x,y) &= \left(V_t\left(-U_{2tn}U_{t(2n+1)}/U_t^2\right) - U_{2tn}^2/U_t^2 - 1, -U_{2tn}U_{t(2n+1)}/U_t^2\right) \\
&= \left(-U_{t(2n+1)}^2/U_t^2, -U_{2tn}U_{t(2n+1)}/U_t^2\right)
\end{aligned}$$

is a solution of (2.3). By Lemma 2.3 and since

$$\left(-U_{t(2n+1)}^2/U_t^2, -U_{2tn}U_{t(2n+1)}/U_t^2\right)$$

is a solution of (2.3), we have that

$$\begin{aligned}
(x,y) &= \left(-U_{t(2n+1)}^2/U_t^2, -V_t\left(-U_{t(2n+1)}^2/U_t^2\right) - \left(-U_{2tn}U_{t(2n+1)}/U_t^2\right)\right) \\
&= \left(-U_{t(2n+1)}^2/U_t^2, U_{t(2n+1)}U_{t(2n+2)}/U_t^2\right)
\end{aligned}$$

is also a solution of (2.3). Similarly, if $(-U_{t(2n+1)}^2/U_t^2, U_{t(2n+1)}U_{2t(n+1)}/U_t^2)$ satisfies (2.3), then $(U_{2t(n+1)}^2/U_t^2, U_{t(2n+1)}U_{2t(n+1)}/U_t^2)$ satisfies (2.3). If $(U_{2t(n+1)}^2/U_t^2, U_{t(2n+1)}U_{2t(n+1)}/U_t^2)$ satisfies (2.3) and $(U_{2t(n+1)}^2/U_t^2, U_{t(2n+1)}U_{2t(n+1)}/U_t^2)$ is a solution of (2.3), then $(U_{2t(n+1)}^2/U_t^2, -U_{2t(n+1)}U_{t(2n+3)}/U_t^2)$ satisfies (2.3).

For $n < 0$, the proof could be similarly obtained by using the relations $U_{-n} = (-1)^{n+1}U_n$ and $V_{-n} = (-1)^n V_n$. □

Now we prove that the solutions stated in (2.4) are *all* the solutions of (2.3). First we consider the positive solutions. By the method used in (Theorem 1 of [10], Lemma 3 of [5] and Theorem 3.3 of [3]), proves the following theorem.

Theorem 2.5. *If positive integers x and y satisfy the equation*

$$x^2 - V_t xy - y^2 + x = 0, \tag{2.5}$$

then there exist positive integers c, e such that $x = c^2$, $y = ce$ and $\gcd(c, e) = 1$.

Now we need some properties of the Pell equation $x^2 - Dy^2 = N$, where D is a given square-free positive integer and N is a given integer.

We recall the following three results from [13]**:**

Theorem 2.6. *Let $x_0^2 - Dy_0^2 = N$ be fulfilled for some integers x_0, y_0 and $a_0^2 - Db_0^2 = 1$ for some integers a_0, b_0. If $w = x_0 + y_0\sqrt{D}$, $j = a_0 + b_0\sqrt{D}$, then for any integer n, the pair (x_n, y_n) satisfying the equation $x_n + y_n\sqrt{D} = wj^n$ satisfies the equation $x^2 - Dy^2 = N$.*

Theorem 2.7. *Let N be an integer satisfied $|N| < \sqrt{D}$. Then any positive integer solution (s, t) of $x^2 - Dy^2 = N$ with* $\gcd(s, t) = 1$ *satisfies $s = h_n$, $t = k_n$ for some positive integer n, where $\frac{h_n}{k_n}$ is the n^{th} convergent to the infinite simple continued fraction of $\sqrt{D} = [a_0, a_1, a_2, ...]$.*

Theorem 2.8. *Let $[a_0, a_1, a_2, \cdots]$ be the infinite simple continued fraction of $\sqrt{D}$ and suppose that m_n and q_n are two sequences given by*

$$\begin{aligned} m_{n+1} &= a_n q_n - m_n, \quad m_0 = 0, \\ q_{n+1} &= (D - m_{n+1}^2)/q_n, \quad q_0 = 1. \end{aligned}$$

Then

a) m_n and q_n are integers for any positive integer n,

b) $h_n^2 - Dk_n^2 = (-1)^{n+1}q_{n+1}$ for any integer $n \geq -1$.

Now we are ready to prove the fact that all positive solutions of the equation (2.5) are in the form $(x, y) = \left(U_{2tn}^2/U_t^2, U_{t(2n-1)}U_{2tn}/U_t^2\right)$. Using Theorem 2.5, we note that there exist positive integers c and e such that $x = c^2$, $y = ce$ and $\gcd(c, e) = 1$. Substituting them in equation (2.5), we get

$$c^2 - V_t ce - e^2 + 1 = 0. \tag{2.6}$$

We can consider this equation as a quadratic equation with respect to the indeterminate c. This equation has integer solutions if and only if $\Delta = \left(V_t^2+4\right)e^2 - 4$ is a square. Then there exists an integer f such that

$$f^2 - \left(V_t^2+4\right)e^2 = -4. \tag{2.7}$$

From (2.6), we get

$$c = \left(V_t e \pm f\right)/2. \tag{2.8}$$

Now we solve the equation (2.7). First we assume that V_t is odd. From the equation (2.2), we have

$$\sqrt{V_t^2+4} = \left[V_t,\ \overline{(V_t-1)/2,\ 1,\ 1,\ (V_t-1)/2,\ 2V_t}\right].$$

For $n \geq 1$, define

$$a_0 = V_t,\ a_{5n-4} = (V_t-1)/2,\ a_{5n-3} = 1,$$
$$a_{5n-2} = 1,\ a_{5n-1} = (V_t-1)/2,\ a_{5n} = 2V_t.$$

Then by Theorem 2.8, we get two eventually periodic sequences

$$\{m_n\}_{n=0}^{\infty} = \left\{0, \overline{V_t, V_t-2, 2, V_t-2, V_t}\right\}$$

and

$$\{(-1)^{n+1}q_{n+1}\}_{n=-1}^{\infty} = \left\{1, \overline{-4, V_t, -V_t, 4, -1, 4, -V_t, V_t, -4, 1}\right\}. \tag{2.9}$$

Now we assume that (f,e) is a positive solution of the equation (2.7). From (2.7), we deduce that $\gcd(f,e) = 1$ or 2. For the sequence in (2.9), Theorem 2.8 implies that for all $n \geq 0$

$$\begin{aligned} h_{10n}^2 - (V_t^2+4)k_{10n}^2 &= -4, \\ h_{10n+4}^2 - (V_t^2+4)k_{10n+4}^2 &= -1, \\ h_{10n+8}^2 - (V_t^2+4)k_{10n+8}^2 &= -4. \end{aligned} \tag{2.10}$$

Now from the equation (2.10) we conclude that

$$(2h_{10n+4})^2 - (V_t^2+4)(2k_{10n+4})^2 = -4.$$

Moreover the solutions of the equation (2.7) are

$$(f,e) = (h_{10n}, k_{10n}),\ (2h_{10n+4}, 2k_{10n+4}),\ (h_{10n+8}, k_{10n+8}),\ n \geq 0.$$

From the equation (2.8), the solutions (c,e) are of the forms

$$\begin{aligned} &((V_t k_{10n} + h_{10n})/2,\ k_{10n}), \\ &(V_t k_{10n+4} + h_{10n+4},\ 2k_{10n+4}), \\ &((V_t k_{10n+8} + h_{10n+8})/2,\ k_{10n+8}) \end{aligned} \tag{2.11}$$

for all $n \geq 0$. Now using Theorem 2.1 and rearranging the equation (2.11), we have

$$(c,e) \quad = \quad ((V_t U_{(6n+1)t} + (-1)^{t+1}U_{6nt} + U_{(6n+2)t})/2U_t,\ U_{(6n+1)t}/U_t)$$

$$
\begin{aligned}
&= (U_{(6n+2)t}/U_t,\ U_{(6n+1)t}/U_t),\\
(c,e) &= (V_t(\frac{1}{2}U_{(6n+3)t}) + \frac{1}{2}((-1)^{t+1}U_{(6n+2)t} + U_{(6n+4)t}))/U_t,\ 2(\frac{1}{2}U_{(6n+3)t}/U_t))\\
&= (U_{(6n+4)t}/U_t,\ U_{(6n+3)t}/U_t),\\
(c,e) &= ((V_tU_{(6n+5)t} + (-1)^{t+1}U_{(6n+4)t} + U_{(6n+6)t})/2U_t,\ U_{(6n+5)t}/U_t)\\
&= (U_{(6n+6)t}/U_t,\ U_{(6n+5)t}/U_t),
\end{aligned}
$$

and finally from Theorem 2.5, we get $(x,y) = (c^2, ce)$. For each solutions (c,e), we have the following three pairs

$$
\begin{aligned}
(x,y) &= (U^2_{(6n+2)t}/U^2_t,\ U_{(6n+1)t}U_{(6n+2)t}/U^2_t),\\
&\quad (U^2_{(6n+4)t}/U^2_t,\ U_{(6n+3)t}U_{(6n+4)t}/U^2_t),\\
&\quad (U^2_{(6n+6)t}/U^2_t,\ U_{(6n+5)t}U_{(6n+6)t}/U^2_t)
\end{aligned}
$$

and therefore $(x,y) = \left(U^2_{2tn}/U^2_t,\ U_{t(2n-1)}U_{2tn}/U^2_t\right)$ for all positive integer n.

By the above results, we obtain the following result.

Theorem 2.9. *If V_t is a positive odd integer, then every positive solution of $x^2 - V_txy - y^2 + x = 0$ is given by $(x,y) = \left(U^2_{2nt}/U^2_t,\ U_{(2n-1)t}U_{2nt}/U^2_t\right)$.*

Now we consider the case when V_t is even. In this case from the equation (2.2), we have

$$\sqrt{V_t^2 + 4} = \left[V_t, \overline{V_t/2,\ 2V_t}\right].$$

Let

$$a_0 = V_t,\ a_{2n+1} = V_t/2,\ a_{2n+2} = 2V_t,\ n \geq 0.$$

We get two eventually periodic sequences

$$\{m_n\}_{n=0}^{\infty} = \{0, \overline{V_t}\}$$

and

$$\left\{(-1)^{n+1}q_{n+1}\right\}_{n=-1}^{\infty} = \{\overline{1, -4}\}.$$

From this and Theorem 2.8, we have

$$h^2_{2n} - (V_t^2 + 4)k^2_{2n} = -4,\ n \geq 0.$$

Moreover in this case, all solutions of the equation (2.7) are $(f,e) = (h_{2n}, k_{2n})$, and using the equation (2.8), we get $(c,e) = ((V_tk_{2n} + h_{2n})/2, k_{2n})$. But from Theorem 2.2, we know that $h_{2n} = ((-1)^{t+1}U_{2nt} + U_{(2n+2)t})/U_t$ and $k_{2n} = U_{(2n+1)t}/U_t$. Substituting them into the equation (2.8), we get

$$(c,e) = ((V_tU_{(2n+1)t} + (-1)^{t+1}U_{2nt} + U_{(2n+2)t})/2U_t, U_{(2n+1)t}/U_t).$$

Therefore $(x,y) = (c^2, ce) = \left(U^2_{(2n+2)t}/U^2_t, U_{(2n+1)t}U_{(2n+2)t}/U^2_t\right)$. Thus have the following theorem.

Theorem 2.10. *If V_t is a positive even integer, then every positive solution of $x^2 - V_txy - y^2 + x = 0$ is given by $(x,y) = \left(U^2_{2nt}/U^2_t,\ U_{(2n-1)t}U_{2nt}/U^2_t\right)$.*

Now we find all (not necessarily positive) solutions of the equation $x^2 - V_t xy - y^2 + x = 0$. First assume that $x > 0$ and $y < 0$. By substituting $y \to -y$ in the last equation, that is we consider the equation $x^2 + V_t xy - y^2 + x = 0$ and so we already its all positive solutions

$$(x, y) = \left(U_{2nt}^2/U_t^2,\ U_{2nt}U_{(2n+1)t}/U_t^2\right).$$

Similarly if $x < 0$ and $y > 0$, then by substituting $x \to -x$ and considering the equation $x^2 + V_t xy - y^2 - x = 0$, we have

$$(x, y) = \left(U_{(2n+1)t}^2/U_t^2,\ U_{(2n+1)t}U_{(2n+2)t}/U_t^2\right).$$

Finally if $x < 0$ and $y < 0$, then similarly by substituting $x \to -x$ and $y \to -y$, that is we consider the equation $x^2 - V_t xy - y^2 - x = 0$ and so we have

$$(x, y) = \left(U_{(2n+1)t}^2/U_t^2,\ U_{2nt}U_{(2n+1)t}/U_t^2\right).$$

Using the above discussions we have the result:

Theorem 2.11. *For odd t, all solutions of the equation $x^2 - V_t xy - y^2 + x = 0$ are given by*

$$\left(U_{2nt}^2/U_t^2, U_{(2n-1)t}U_{2nt}/U_t^2\right),$$
$$\left(U_{2nt}^2/U_t^2, -U_{2nt}U_{(2n+1)t}/U_t^2\right),$$
$$\left(-U_{(2n+1)t}^2/U_t^2, U_{(2n+1)t}U_{(2n+2)t}/U_t^2\right),$$
$$\left(-U_{(2n+1)t}^2/U_t^2, -U_{2nt}U_{(2n+1)t}/U_t^2\right).$$

Any solution of the equations $x^2 \pm V_t xy - y^2 \pm x = 0$ corresponds to the solution of the equation $x^2 - V_t xy - y^2 + x = 0$. We summarize our earlier results and related other unexpressed results related with them as follows:

Equation	Solutions
$x^2 - V_t xy - y^2 + x = 0$	$\left(U_{2tn}^2/U_t^2, U_{t(2n-1)}U_{2tn}/U_t^2\right)$ $\left(U_{2tn}^2/U_t^2, -U_{2tn}U_{t(2n+1)}/U_t^2\right)$ $\left(-U_{t(2n+1)}^2/U_t^2, U_{t(2n+1)}U_{t(2n+2)}/U_t^2\right)$ $\left(-U_{t(2n+1)}^2/U_t^2, -U_{2tn}U_{t(2n+1)}/U_t^2\right)$
$x^2 + V_t xy - y^2 + x = 0$	$\left(U_{2tn}^2/U_t^2, U_{2tn}U_{t(2n+1)}/U_t^2\right)$ $\left(U_{2tn}^2/U_t^2, -U_{t(2n-1)}U_{2tn}/U_t^2\right)$ $\left(-U_{t(2n+1)}^2/U_t^2, U_{2tn}U_{t(2n+1)}/U_t^2\right)$ $\left(-U_{t(2n+1)}^2/U_t^2, -U_{t(2n+1)}U_{t(2n+2)}/U_t^2\right)$
$x^2 - V_t xy - y^2 - x = 0$	$\left(U_{t(2n+1)}^2/U_t^2, U_{2tn}U_{t(2n+1)}/U_t^2\right)$ $\left(U_{t(2n+1)}^2/U_t^2, -U_{t(2n+1)}U_{t(2n+2)}/U_t^2\right)$ $\left(-U_{2tn}^2/U_t^2, U_{2tn}U_{t(2n+1)}/U_t^2\right)$ $\left(-U_{2tn}^2/U_t^2, -U_{t(2n-1)}U_{2tn}/U_t^2\right)$
$x^2 + V_t xy - y^2 - x = 0$	$\left(U_{t(2n+1)}^2/U_t^2, U_{t(2n+1)}U_{t(2n+2)}/U_t^2\right)$ $\left(U_{t(2n+1)}^2/U_t^2, -U_{2tn}U_{t(2n+1)}/U_t^2\right)$ $\left(-U_{2tn}^2/U_t^2, U_{t(2n-1)}U_{2tn}/U_t^2\right)$ $\left(-U_{2tn}^2/U_t^2, -U_{2tn}U_{t(2n+1)}/U_t^2\right)$

TABLE 1. The solutions of the equations$x^2 \pm V_t xy - y^2 \pm x = 0$.

Now we will prove that all solutions of the equation

$$x^2 - V_t xy - y^2 + y = 0 \tag{2.12}$$

are of the forms

$$\begin{aligned}
&\left(U_{t(2n+1)}U_{t(2n+2)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right), \\
&\left(-U_{2tn}U_{t(2n+1)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right), \\
&\left(U_{t(2n-1)}U_{2tn}/U_t^2, -U_{2tn}^2/U_t^2\right), \\
&\left(-U_{t(2n+1)}U_{2tn}/U_t^2, -U_{2tn}^2/U_t^2\right).
\end{aligned} \tag{2.13}$$

For later use, we need the following lemma.

Lemma 2.12. *If* (x,y) *is a solution of the equation* $x^2-V_txy-y^2+y=0$, *then the solutions of the same equation are the pairs* (V_ty-x,y) *and* $(x,-V_tx-y+1)$.

Theorem 2.13. *For any integer* n *and odd* t, *the pairs in (2.13) satisfy the equation* $x^2-V_txy-y^2+y=0$.

Proof. The proof is similar to the proof of Theorem 2.4. □

Now we shall give the following theorem whose proof is similar to the proof of Theorem 2.5.

Theorem 2.14. *If positive integers* x *and* y *satisfy the equation* $x^2-V_txy-y^2+y=0$, *then there exist positive integers* c,e *such that* $x=ce$, $y=c^2$ *and* $\gcd(c,e)=1$.

We recall the following auxiliary lemma from [7].

Lemma 2.15. *If* V_t^2+4 *is square-free, then for odd* t, *the integer solutions of* $\left(V_t^2+4\right)x^2+4U_t^2=y^2U_t^2$ *are precisely the pairs* $\left(\pm U_{2tn}, \pm V_{2tn}\right)$.

Theorem 2.16. *For any integer* n *and odd* t, *all solutions of the equation* $x^2-V_txy-y^2+y=0$ *are*

$$\begin{gathered}
\left(U_{t(2n+1)}U_{t(2n+2)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right), \\
\left(-U_{2tn}U_{t(2n+1)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right), \\
\left(U_{t(2n-1)}U_{2tn}/U_t^2, -U_{2tn}^2/U_t^2\right), \\
\left(-U_{t(2n+1)}U_{2tn}/U_t^2, -U_{2tn}^2/U_t^2\right).
\end{gathered}$$

Proof. Using Theorem 2.14, $x = ce$, $y = c^2$ such that $\gcd(c, e) = 1$ satisfy the equation $x^2 - V_t xy - y^2 + y = 0$. Then, we have

$$\begin{aligned} c^2e^2 - V_t c^3 e - c^4 + c^2 &= 0, \\ c^2 + V_t ce - e^2 - 1 &= 0. \end{aligned} \tag{2.14}$$

The last equation has integer solutions if and only if $\Delta = \left(V_t^2 + 4\right) e^2 - 4$ is a square. Then there exists an integer f such that

$$\left(V_t^2 + 4\right) e^2 + 4 = f^2. \tag{2.15}$$

From Lemma 2.15, note that all positive solutions of (2.15) are

$$(e, f) = \left(U_{2tn}/U_t, V_{2tn}\right).$$

From (2.14), we write

$$c = \left(-V_t e \pm f\right)/2$$

and $(c, e) = \left(U_{t(2n-1)}/U_t, U_{2tn}/U_t\right)$. From Theorem 2.14, we get

$$(x, y) = \left(U_{t(2n-1)}U_{2tn}/U_t^2, U_{t(2n-1)}^2/U_t^2\right).$$

Therefore $(x, y) = \left(U_{t(2n+1)}U_{t(2n+2)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right)$ is a solution of the equation $x^2 - V_t xy - y^2 + y = 0$. From Lemma 2.12 and Theorem 2.13, the other claims are obtained. There is no other solution than those shown in Theorem 2.11. □

Any solution of the equations $x^2 \pm V_t xy - y^2 \pm y = 0$ corresponds to the solution of the equation $x^2 - V_t xy - y^2 + y = 0$. We can summarize and state our earlier and unexpressed results as:

Equation	Solutions
$x^2 - V_t xy - y^2 + y = 0$	$\left(U_{t(2n+1)}U_{t(2n+2)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right)$ $\left(-U_{2tn}U_{t(2n+1)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right)$ $\left(U_{t(2n-1)}U_{2tn}/U_t^2, -U_{2tn}^2/U_t^2\right)$ $\left(-U_{t(2n+1)}U_{2tn}/U_t^2, -U_{2tn}^2/U_t^2\right)$
$x^2 + V_t xy - y^2 + y = 0$	$\left(U_{2tn}U_{t(2n+1)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right)$ $\left(-U_{t(2n+1)}U_{t(2n+2)}/U_t^2, U_{t(2n+1)}^2/U_t^2\right)$ $\left(U_{2tn}U_{t(2n+1)}/U_t^2, -U_{2tn}^2/U_t^2\right)$ $\left(-U_{t(2n-1)}U_{2tn}/U_t^2, -U_{2tn}^2/U_t^2\right)$
$x^2 - V_t xy - y^2 - y = 0$	$\left(U_{2tn}U_{t(2n+1)}/U_t^2, U_{2tn}^2/U_t^2\right)$ $\left(-U_{t(2n-1)}U_{2tn}/U_t^2, U_{2tn}^2/U_t^2\right)$ $\left(U_{2tn}U_{t(2n+1)}/U_t^2, -U_{t(2n+1)}^2/U_t^2\right)$ $\left(-U_{t(2n+1)}U_{t(2n+2)}/U_t^2, -U_{t(2n+1)}^2/U_t^2\right)$
$x^2 + V_t xy - y^2 - y = 0$	$\left(U_{2tn}U_{t(2n-1)}/U_t^2, U_{2tn}^2/U_t^2\right)$ $\left(-U_{2tn}U_{t(2n+1)}/U_t^2, U_{2tn}^2/U_t^2\right)$ $\left(U_{t(2n+1)}U_{t(2n+2)}/U_t^2, -U_{t(2n+1)}^2/U_t^2\right)$ $\left(-U_{2tn}U_{t(2n+1)}/U_t^2, -U_{t(2n+1)}^2/U_t^2\right)$

TABLE 2. *The solutions of the equations* $x^2 \pm V_t xy - y^2 \pm y = 0$.

For even t, we will find all the solutions of the equations $x^2 \pm V_t xy + y^2 - x = 0$ and $x^2 \pm V_t xy + y^2 - y = 0$.

Lemma 2.17. *If* (x, y) *is a solution of the equation*

$$x^2 - V_t xy + y^2 - x = 0, \tag{2.16}$$

then the solutions of the same equation are the pairs $(x, V_t x - y)$ *and* $(V_t y - x + 1, y)$.

For example, if $(0,0)$ is a solution of (2.16), then by Lemma 2.17, a sequence of solutions of (2.16) is

$$(0,0)\,,\ \ (1,0)\,,\ \ (1,V_t)\,,\ \ \left(V_t^2,V_t\right),\ \ \left(V_t^2,V_t\left(V_t^2-1\right)\right),\ldots.$$

and these solutions can be rewritten as

$$\begin{aligned}(0,0) &= \left(\frac{U_0^2}{U_t^2},\frac{U_0U_t}{U_t^2}\right),\ (1,0)=\left(\frac{U_t^2}{U_t^2},\frac{U_0U_t}{U_t^2}\right),\ (1,V_t)=\left(\frac{U_t^2}{U_t^2},\frac{U_tU_{2t}}{U_t^2}\right)\\ \left(V_t^2,V_t\right) &= \left(\frac{U_{2t}^2}{U_t^2},\frac{U_tU_{2t}}{U_t^2}\right),\ldots.\end{aligned}$$

Theorem 2.18. *For any integer n and even t, the pairs*

$$\left(U_{t(n+1)}^2/U_t^2,U_{t(n+1)}U_{tn}/U_t^2\right)\ \text{and}\ \left(U_{tn}^2/U_t^2,U_{tn}U_{t(n+1)}/U_t^2\right)$$

satisfy the equation $x^2-V_txy+y^2-x=0$.

Proof. The proof is similar to the proof of Theorem 2.4. □

Theorem 2.19. *If positive integers x and y satisfy the equation $x^2-V_txy+y^2-x=0$, then there exist positive integers c,e such that $x=c^2$, $y=ce$ and* $\gcd(c,e)=1$.

Proof. The proof can be done similar to the proof of Theorem 2.5. □

Again, we recall the another following auxiliary lemma from [7].

Lemma 2.20. *If V_t^2-4 is square-free, then for even t, the integer solutions of $\left(V_t^2-4\right)x^2+4U_t^2=y^2U_t^2$ are precisely the pairs $(\pm U_{tn},\pm V_{tn})$.*

Theorem 2.21. *For any integer n and even t, all solutions of the equation $x^2-V_txy+y^2-x=0$ are $\left(U_{t(n+1)}^2/U_t^2,U_{t(n+1)}U_{tn}/U_t^2\right)$ and $\left(U_{tn}^2/U_t^2,U_{tn}U_{t(n+1)}/U_t^2\right)$.*

Proof. Using Theorem 2.19, $x=c^2$, $y=ce$ such that $\gcd(c,e)=1$ satisfy the equation $x^2-V_txy+y^2-x=0$. Then we have

$$\begin{aligned}c^4-V_tc^3e+c^2e^2-c^2 &= 0,\\ c^2-V_tce+e^2-1 &= 0. \qquad (2.17)\end{aligned}$$

The last equation has integer solutions if and only if $\Delta=\left(V_t^2-4\right)e^2+4$ is a square. Then there exists an integer f such that

$$f^2-\left(V_t^2-4\right)e^2=4. \qquad (2.18)$$

From Lemma 2.20, the positive solutions of (2.18) are $(e,f)=(U_{tn}/U_t,V_{tn})$. From (2.17), we write

$$c=(V_te\pm f)/2,$$

and $(c,e)=\left(U_{t(n+1)}/U_t,U_{tn}/U_t\right)$. From Theorem 2.19, we get

$$(x,y)=\left(U_{t(n+1)}^2/U_t^2,U_{t(n+1)}U_{tn}/U_t^2\right).$$

Therefore $(x, y) = \left(U_{t(n+1)}^2/U_t^2, U_{t(n+1)}U_{tn}/U_t^2\right)$ is a solution of the equation (2.16). Similarly, from Lemma 2.17 and Theorem 2.18, the other claim is obtained. There is no other solution than those shown in Theorem 2.11. □

One can similarly see that equation $x^2 - V_t xy + y^2 - y = 0$ has the solutions $\left(U_{t(n+1)}U_{tn}/U_t^2, U_{t(n+1)}^2/U_t^2\right)$ and $\left(U_{t(n+1)}U_{tn}/U_t^2, U_{tn}^2/U_t^2\right)$.

We can summarize the results as:

Equation	Solutions
$x^2 - V_t xy + y^2 - x = 0$	$\left(U_{t(n+1)}^2/U_t^2, U_{t(n+1)}U_{tn}/U_t^2\right)$ $\left(U_{tn}^2/U_t^2, U_{t(n+1)}U_{tn}/U_t^2\right)$
$x^2 + V_t xy + y^2 - x = 0$	$\left(U_{t(n-1)}^2/U_t^2, -U_{t(n-1)}U_{tn}/U_t^2\right)$ $\left(U_{t(n+1)}^2/U_t^2, -U_{t(n+1)}U_{tn}/U_t^2\right)$
$x^2 - V_t xy + y^2 - y = 0$	$\left(U_{t(n+1)}U_{tn}/U_t^2, U_{t(n+1)}^2/U_t^2\right)$ $\left(U_{t(n+1)}U_{tn}/U_t^2, U_{tn}^2/U_t^2\right)$
$x^2 + V_t xy + y^2 - y = 0$	$\left(-U_{t(n-1)}U_{tn}/U_t^2, U_{t(n-1)}^2/U_t^2\right)$ $\left(-U_{t(n+1)}U_{tn}/U_t^2, U_{t(n+1)}^2/U_t^2\right)$

Table 3. *The solutions of* $x^2 \pm V_t xy + y^2 - x = 0$ *and* $x^2 \pm V_t xy + y^2 - y = 0$.

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