Iranian Journal of Mathematical Sciences and Informatics
Vol. 20, No. 1 (2025), pp 131-149
DOI: 10.61186/ijmsi.20.1.131

Existence Problem for Impulsive Nonlinear Sturm-Liouville Problems on the Whole Line

Bilender P. Allahverdiev[a,b], Hüseyin Tuna[b,c*]

[a]Department of Mathematics, Khazar University, AZ1096 Baku, Azerbaijan
[b]UNCE-Azerbaijan State University of Economics, Baku, Azerbaijan
[c]Department of Mathematics, Burdur Mehmet Akif Ersoy University, 15030 Burdur, Turkey

E-mail: bilenderpasaoglu@gmail.com
E-mail: hustuna@gmail.com

Abstract. The existence problem is studied for the impulsive nonlinear Sturm–Liouville equation on the whole line. Existence and uniqueness results are obtained.

Keywords: Impulsive Sturm-Liouville problem, Singular point, Weyl limit-circle case, Completely continuous operator, Fixed point theorems.

2000 Mathematics subject classification: 34B15, 34B16, 34B37, 34B24, 34B40.

1. Introduction

Nowadays, the study of impulsive differential equations continues to be a very active field of research because they describes processes that are subjected to abrupt changes in their states at certain moments. This processes play an important role in several areas such as chemical technology, biotechnology, theoretical physics and industrial robotics. There is an extensive literature on the impulsive differential equations (see monographs [6, 7, 8, 9, 34, 21], and references therein).

*Corresponding Author

Received 11 May 2020; Accepted 14 February 2022

On the other hand, we can see that most papers studied the existence of solutions of the impulsive nonlinear differential equations in regular case (see [16, 17, 18, 28, 2, 29, 33, 36, 15, 31, 10, 12]), but there are few papers which are concerned with singular case ([22, 21, 26, 27, 30, 24, 13, 25, 32, 1]). Then come to the question "How can we obtain the existence of solutions for singular impulsive nonlinear Sturm–Liouville problems when the Weyl's limit-circle case holds at infinity?". Motivated by this question, we establish the criteria for the existence of solutions of the impulsive nonlinear Sturm–Liouville problems on the whole axis $(-\infty,\infty)$. In the analysis that follows, we will largely follow the development of the theory in [14, 3, 4, 5].

2. The fundamental problem

Let us consider the following nonlinear Sturm–Liouville equation

$$\tau\left(y\right):=-(p(x)y')'+q(x)y=f\left(x,y\right),\ x\in J, \tag{2.1}$$

where $J:=J_1\cup J_2,\ J_1:=(-\infty,c),\ J_2:=(c,\infty),\ c>0$ and $y=y\left(x\right)$ is a desired solution.

Let $L^2(J)$ be a Hilbert space which composes of all real-valued functions y satisfying

$$\int_{-\infty}^{\infty}\left|y\left(x\right)\right|^2dx<\infty$$

in relation to the inner product

$$(y,z):=\int_{-\infty}^{\infty}y\left(x\right)z\left(x\right)dx.$$

Denote by $\mathcal{D}$ the linear set of all functions $y\in L^2(J)$ such that y, py' are locally absolutely continuous functions on J, one-sided limits $y(c\pm),(py')(c\pm)$ exist and are finite and $\tau(y)\in L^2(J)$. The operator L defined by $Ly=\tau(y)$ is called the *maximal operator* on $L^2(J)$.

For two arbitrary functions $y,z\in\mathcal{D}$, we have Green's formula

$$\int_{-\infty}^{\infty}\tau\left(y\right)\overline{z}dx-\int_{-\infty}^{\infty}y\tau\left(z\right)dx=[y,z]_{c-}-[y,z]_{-\infty}+[y,z]_{\infty}-[y,z]_{c+}, \tag{2.2}$$

where $[y,z]_x:=W_x\left(y,z\right)=y(x)(pz')(x)-(py')(x)z(x)\ \ (x\in J)$ and the limits $[y,z]_{\pm\infty}=\lim_{x\to\pm\infty}[y,z]_x$ exist and are finite.

We assume that the following conditions are satisfied.

(A1) The point c is regular (The point c is regular for the differential expression τ if $\frac{1}{p},q\in L^1[c-\epsilon,c+\epsilon]$ for some $\epsilon>0$), p and q are real-valued, Lebesgue measurable functions on J and $\frac{1}{p},q\in L^1_{loc}(J_k),\ k=1,2$. Furthermore, Weyl limit-circle case holds for the differential expression τ, i.e., the functions p and q are such that all solutions of the the equation

$$\tau\left(y\right)=0 \tag{2.3}$$

belong to $L^2\left(J\right)$ (see [35, 11, 23]).

(A2) The function $f(x,y)$ is real-valued and continuous in $(x,\zeta)\in J\times\mathbb{R}$, and

$$|f(x,y)|\leq g(x)+\kappa|\zeta| \tag{2.4}$$

for all (x,ζ) in $J\times\mathbb{R}$, where $g(x)\geq 0$, $g\in L^2(J)$, and κ is a positive constant.

Denote by

$$u=u(x)=\begin{cases} u^{(1)}(x), & x\in J_1\\ u^{(2)}(x), & x\in J_2,\end{cases} \quad v=v(x)=\begin{cases} v^{(1)}(x), & x\in J_1\\ v^{(2)}(x), & x\in J_2\end{cases}$$

the solutions of the equation (2.3) satisfying the initial conditions

$$u(0)=0,\ (pu')(0)=1,\ v(0)=-1,\ (pv')(0)=0, \tag{2.5}$$

and impulsive conditions

$$\begin{aligned} U(c+)&=BU(c-),\ U(x):=\begin{pmatrix} u(x)\\ (pu')(x)\end{pmatrix},\\ V(c+)&=BV(c-),\ V(x):=\begin{pmatrix} v(x)\\ (pv')(x)\end{pmatrix},\end{aligned} \tag{2.6}$$

$$B\in M_2(\mathbb{R}),\ \det B=\sigma>0,$$

where $M_2(\mathbb{R})$ denotes the the 2×2 matrices with entries from $\mathbb{R}:=(-\infty,\infty)$.

Now, we introduce the Hilbert space $H=L^2(J_1)\dot{+}L^2(J_2)$ with the inner product

$$\langle f,g\rangle_H:=\int_{-\infty}^{c} f^{(1)}g^{(1)}dx+\theta\int_c^{\infty} f^{(2)}g^{(2)}dx,\ \theta=\frac{1}{\sigma},$$

where

$$f(x)=\begin{cases} f^{(1)}(x), & x\in J_1\\ f^{(2)}(x), & x\in J_2,\end{cases} \quad g(x)=\begin{cases} g^{(1)}(x), & x\in J_1\\ g^{(2)}(x), & x\in J_2.\end{cases}$$

We set

$$\begin{aligned} W_x^{(i)}:=W_x\left(u^{(i)},v^{(i)}\right)&=u^{(i)}(x)(pv^{(i)\prime})(x)\\ &-(pu^{(i)\prime})(x)v^{(i)}(x)\ \ (x\in J_i,\ i=1,2).\end{aligned}$$

Then the equality $W_x^{(1)}=\theta W_x^{(2)}$ holds. For convenience, we denote $W_x:=W_x^{(1)}=\theta W_x^{(2)}$. Since the Wronskian of any two solutions of equation (2.3) are constant, we have $W_x(u,v)=1$. Then, u and v are linearly independent and they form a fundamental system of solutions of equation (2.3). By the condition (A1), we get $u,\ v\in H$ and moreover, $u,\ v\in\mathcal{D}$. So, the values $[y,u]_{\pm\infty}$ and $[y,v]_{\pm\infty}$ exist and are finite for every $y\in\mathcal{D}$.

From the conditions (2.5)-(2.6) and Green's formula (2.2), we obtain

$$
\begin{aligned}
&[y,u]_{-\infty} = y\,(0) - \int_{-\infty}^{0} u\,(x)\,\tau(y\,(x))dx,\\
&[y,v]_{-\infty} = (py')\,(0) - \int_{-\infty}^{0} v\,(x)\,\tau(y\,(x))dx,\\
&[y,u]_{\infty} = y\,(0) + \int_{0}^{\infty} u\,(x)\,\tau(y\,(x))dx,\\
&[y,v]_{\infty} = (py')\,(0) + \int_{0}^{\infty} v\,(x)\,\tau(y\,(x))dx.
\end{aligned}
\tag{2.7}
$$

Now, we will adjoint to problem (2.1) the following boundary conditions

$$
\begin{aligned}
&[y,u]_{-\infty}\cos\alpha + [y,v]_{-\infty}\sin\alpha = d_1,\ \ \alpha\in\mathbb{R},\\
&[y,u]_{\infty}\cos\beta + [y,v]_{\infty}\sin\beta = d_2,\ \ \beta\in\mathbb{R},
\end{aligned}
\tag{2.8}
$$

and impulsive conditions

$$
Y\,(c+) = BY\,(c-)\,,\ \ Y = \begin{pmatrix} y \\ py' \end{pmatrix},\ \ \det B = \sigma > 0, \tag{2.9}
$$

where d_1, d_2 are arbitrary given real numbers and

(A3) $$W := \cos\alpha\sin\beta - \cos\beta\sin\alpha \ \neq 0.$$

3. Construction of an appropriate Green's function

In this section, we will construct an appropriate Green's function. Hence, we will reduce the boundary-value problem (2.1), (2.8), (2.9) to a fixed point problem.

Let us consider the linear boundary value problem

$$
-(p(x)y')' + q(x)y = h\,(x)\,,\ \ x\in J,\ \ h\in H, \tag{3.1}
$$

$$
\left.
\begin{aligned}
&[y,u]_{-\infty}\cos\alpha + [y,v]_{-\infty}\sin\alpha = 0,\ \ \alpha\in\mathbb{R},\\
&[y,u]_{\infty}\cos\beta + [y,v]_{\infty}\sin\beta = 0,\ \ \beta\in\mathbb{R},\\
&Y\,(c+) = BY\,(c-)\,,\ \ Y(x) := \begin{pmatrix} y\,(x) \\ (py')\,(x) \end{pmatrix},\ \det B = \sigma > 0,
\end{aligned}
\right\}
\tag{3.2}
$$

where y is a desired solution, u and v are solutions of equation (2.3) under the conditions (2.5)-(2.6).

Define

$$\varphi(x) = \cos\alpha u(x) + \sin\alpha v(x), \ \psi(x) = \cos\beta u(x) + \sin\beta v(x), \tag{3.3}$$

where $W_x(\varphi, \psi) = W$. It is clear that these functions are solutions of equation (2.3) and are in H. Besides, we have

$$[\varphi, u]_x = \varphi(0) = -\sin\alpha, \ [\varphi, v]_x = (p\varphi)'(0) = \cos\alpha, \ x \in J_1, \tag{3.4}$$

$$[\psi, u]_x = \psi(0) = -\sin\beta, \ [\psi, v]_x = (p\psi)'(0) = \cos\beta, \ x \in J_1, \tag{3.5}$$

$$\begin{aligned} &[\varphi, u]_{-\infty} = -\sin\alpha, \ [\varphi, v]_{-\infty} = \cos\alpha, \\ &[\psi, u]_{\infty} = -\sigma\sin\beta, \ [\psi, v]_{\infty} = \sigma\cos\beta, \end{aligned} \tag{3.6}$$

$$\Phi(c+) = B\Phi(c-), \ \Phi(x) := \begin{pmatrix} \varphi(x) \\ (p\varphi')(x) \end{pmatrix}, \tag{3.7}$$

$$\Psi(c+) = B\Psi(c-), \ \Psi(x) := \begin{pmatrix} \psi(x) \\ (p\psi')(x) \end{pmatrix}. \tag{3.8}$$

Now,we will define the Green's function of the boundary-value problem (3.1)-(3.2) by

$$G(x,t) = \begin{cases} \frac{\varphi(x)\psi(t)}{W}, & \text{if } -\infty \le x \le t \le \infty, \ x \ne c, \ t \ne c, \\ \frac{\varphi(t)\psi(x)}{W}, & \text{if } -\infty \le t \le x \le \infty, \ x \ne c, \ t \ne c. \end{cases} \tag{3.9}$$

Since $\varphi, \ \psi \in H$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,t)|^2 \, dxdt < \infty, \tag{3.10}$$

that is, $G(x,t)$ is a Hilbert-Schmidt kernel.

Theorem 3.1. *The function*

$$y(x) = \langle G(x,t), h(t) \rangle, \ x \in J, \tag{3.11}$$

is the unique solution of the nonhomogeneous boundary-value problem (3.1)-(3.2).

Proof. By a variation of constants formula, the general solution of equation (3.1) has the form

$$y(x) = \begin{cases} k_1\varphi^{(1)}(x) + k_2\psi^{(1)}(x) + \frac{\psi^{(1)}(x)}{W}\int_{-\infty}^{x}\varphi^{(1)}(t)h(t)\,dt \\ \quad + \frac{\varphi^{(1)}(x)}{W}\int_{x}^{c}\psi^{(1)}(t)h(t)\,dt, \ x \in J_1, \\ k_3\varphi^{(2)}(x) + k_4\psi^{(2)}(x) + \frac{\theta}{W}\psi^{(2)}(x)\int_{c}^{x}\varphi^{(2)}(t)h(t)\,dt \\ \quad + \frac{\theta}{W}\varphi^{(2)}(x)\int_{x}^{\infty}\psi^{(2)}(t)h(t)\,dt, \ x \in J_2, \end{cases} \tag{3.12}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants.

By (3.12), we get

$$(py)'(x) = \begin{cases} k_1\left(p\varphi^{(1)}\right)'(x) + k_2\left(p\psi^{(1)}\right)'(x) + \frac{\left(p\psi^{(1)}\right)'(x)}{W}\int_{-\infty}^{x}\varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{\left(p\varphi^{(1)}\right)'(x)}{W}\int_{x}^{c}\psi^{(1)}(t)\,h(t)\,dt,\ x\in J_1, \\ k_3\left(p\varphi^{(2)}\right)'(x) + k_4\left(p\psi^{(2)}\right)'(x) + \frac{\theta}{W}\left(p\psi^{(2)}\right)'(x)\int_{c}^{x}\varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\theta}{W}\left(p\varphi^{(2)}\right)'(x)\int_{x}^{\infty}\psi^{(2)}(t)\,h(t)\,dt,\ x\in J_2. \end{cases}$$

Further, we have

$$[y,u]_x = y(x)(pu')(x) - (py')(x)u(x)$$

$$= \begin{cases} k_1[\varphi^{(1)},u]_x + k_2\left[\psi^{(1)}(x),u\right]_x \\ \frac{1}{W}\left[\psi^{(1)}(x),u\right]_x\int_{-\infty}^{x}\varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{1}{W}\left[\varphi^{(1)}(x),u\right]_x\int_{x}^{c}\psi^{(1)}(t)\,h(t)\,dt,\ x\in J_1, \\ k_3[\varphi^{(2)},u]_x + k_4[\psi^{(2)},u]_x \\ +\frac{\theta}{W}[\psi^{(2)},u]_x\int_{c}^{x}\varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\theta}{W}[\varphi^{(2)},u]_x\int_{x}^{\infty}\psi^{(2)}(t)\,h(t)\,dt,\ x\in J_2, \end{cases}$$

and

$$[y,v]_x = y(x)(pv')(x) - (py')(x)v(x)$$

$$= \begin{cases} k_1[\varphi^{(1)},v]_x + k_2\left[\psi^{(1)}(x),v\right]_x \\ \frac{1}{W}\left[\psi^{(1)}(x),v\right]_x\int_{-\infty}^{x}\varphi^{(1)}(t)\,h(t)\,dt \\ +\frac{1}{W}\left[\varphi^{(1)}(x),v\right]_x\int_{x}^{c}\psi^{(1)}(t)\,h(t)\,dt,\ x\in J_1, \\ k_3[\varphi^{(2)},v]_x + k_4[\psi^{(2)},v]_x \\ +\frac{\theta}{W}[\psi^{(2)},v]_x\int_{c}^{x}\varphi^{(2)}(t)\,h(t)\,dt \\ +\frac{\theta}{W}[\varphi^{(2)},v]_x\int_{x}^{\infty}\psi^{(2)}(t)\,h(t)\,dt,\ x\in J_2. \end{cases}$$

Hence, we have

$$[y,u]_{-\infty} = k_1[\varphi^{(1)},u]_{-\infty} + k_2\left[\psi^{(1)}(x),u\right]_{-\infty}$$

$$+\frac{1}{W}\left[\varphi^{(1)},u\right]_{-\infty}\int_{-\infty}^{c}\psi^{(1)}(t)\,h(t)\,dt$$

$$= -k_1\sin\alpha - k_2\sin\beta - \frac{1}{W}\sin\alpha\int_{-\infty}^{c}\psi^{(1)}(t)\,h(t)\,dt \tag{3.13}$$

and

$$[y,v]_{-\infty} = k_1[\varphi^{(1)},v]_{-\infty} + k_2\left[\psi^{(1)}(x),v\right]_{-\infty}$$

$$+\frac{1}{W}\left[\varphi^{(1)}(x),v\right]_{-\infty}\int_{-\infty}^{c}\psi^{(1)}(t)\,h(t)\,dt$$

$$= k_1\cos\alpha + k_2\cos\beta + \frac{1}{W}\cos\alpha\int_{-\infty}^{c}\psi^{(1)}(t)\,h(t)\,dt. \tag{3.14}$$

Substituting (3.13)and (3.14) into (3.2), we get

$$k_2(\cos\alpha\sin\beta - \sin\alpha\cos\beta) = 0,\;\; k_2W = 0,$$

i.e., $k_2 = 0$.

Similarly, we get

$$[y,u]_{\infty} = k_3[\varphi^{(2)},u]_{\infty} + k_4[\psi^{(2)},u]_{\infty} + \frac{\theta}{W}[\psi^{(2)},u]_{\infty}\int_{c}^{\infty}\varphi^{(2)}(t)\,h(t)\,dt$$

$$= -k_3\sigma\sin\alpha - k_4\sigma\sin\beta + \frac{\theta}{W}\sigma\sin\beta\int_{-\infty}^{c}\psi^{(1)}(t)\,h(t)\,dt \tag{3.15}$$

and

$$[y,v]_{\infty} = k_3[\varphi^{(2)},v]_{\infty} + k_4[\psi^{(2)},v]_{\infty}$$

$$+\frac{\theta}{W}[\psi^{(2)},v]_{\infty}\int_{c}^{\infty}\varphi^{(2)}(t)\,h(t)\,dt$$

$$= k_3\sigma\cos\alpha + k_2\sigma\cos\beta + \frac{\theta}{W}\sigma\cos\beta\int_{-\infty}^{c}\psi^{(1)}(t)\,h(t)\,dt. \tag{3.16}$$

From the conditions (3.2), we obtain

$$k_3(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = 0.$$

Hence, $k_3 = 0$.

Furthermore, we have

$$
\begin{aligned}
&Y(c+)\\
&= \begin{pmatrix} y(c+) \\ (py')(c+) \end{pmatrix} = \begin{pmatrix} k_4\psi^{(2)}(c+) \\ k_4\left(p\psi^{(2)}\right)'(c+) \end{pmatrix}\\
&+ \begin{pmatrix} \frac{\theta}{W}\varphi^{(2)}(c+)\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \\ \frac{\theta}{W}\left(p\varphi^{(2)}\right)'(c+)\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \end{pmatrix}\\
&= k_4 \begin{pmatrix} \psi^{(2)}(c+) \\ \left(p\psi^{(2)}\right)'(c+) \end{pmatrix}\\
&+ \frac{\theta}{W}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt \begin{pmatrix} \varphi^{(2)}(c+) \\ \left(p\varphi^{(2)}\right)'(c+) \end{pmatrix}\\
&= k_4\Psi(c+) + \left\{\frac{\theta}{W}\int_c^\infty \psi^{(2)}(t)\,h(t)\,dt\right\}\Phi(c+)
\end{aligned}
$$

and

$$
\begin{aligned}
&Y(c-)\\
&= \begin{pmatrix} y(c-) \\ (py')(c-) \end{pmatrix} = \begin{pmatrix} k_1\varphi^{(1)}(c-) \\ k_1\left(p\varphi^{(1)}\right)'(c-) \end{pmatrix}\\
&+ \begin{pmatrix} \frac{\psi^{(1)}(c-)}{W}\int_{-\infty}^c \varphi^{(1)}(t)\,h(t)\,dt \\ \frac{\left(p\psi^{(1)}\right)'(c-)}{W}\int_{-\infty}^c \varphi^{(1)}(t)\,h(t)\,dt \end{pmatrix}\\
&= k_1 \begin{pmatrix} \varphi^{(1)}(c-) \\ \left(p\varphi^{(1)}\right)'(c-) \end{pmatrix} + \frac{1}{W}\int_{-\infty}^c \varphi^{(1)}(t)\,h(t)\,dt \begin{pmatrix} \psi^{(1)}(c-) \\ \left(p\psi^{(1)}\right)'(c-) \end{pmatrix}\\
&= k_1\Phi(c-) + \left\{\frac{1}{W}\int_{-\infty}^c \varphi^{(1)}(t)\,h(t)\,dt\right\}\Psi(c-).
\end{aligned}
$$

By the conditions (3.2), we obtain

$$\begin{aligned}
&k_4\Psi\left(c+\right)+\left\{\frac{\theta}{W}\int_c^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt\right\}\Phi\left(c+\right)\\
&=B\left\{k_1\Phi\left(c-\right)+\left\{\frac{1}{W}\int_{-\infty}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt\right\}\Psi\left(c-\right)\right\}.
\end{aligned}$$

Using the conditions (3.6) and (3.8), we get

$$\begin{aligned}
&\Phi\left(c-\right)\left\{\frac{\theta}{W}\int_c^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt-k_1\right\}\\
&=\Psi\left(c-\right)\left\{\frac{1}{W}\int_{-\infty}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt-k_4\right\}\\
&\begin{pmatrix}\varphi^{(1)}\left(c-\right)\\ (p\varphi^{(1)\prime})\left(c-\right)\end{pmatrix}\left\{\frac{\theta}{W}\int_c^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt-k_1\right\}\\
&=\begin{pmatrix}\psi^{(1)}\left(c-\right)\\ (p\psi^{(1)\prime})\left(c-\right)\end{pmatrix}\left\{\frac{1}{W}\int_{-\infty}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt-k_4\right\}.
\end{aligned}$$

So, we have the following linear equation system

$$\begin{aligned}
&k_4\psi^{(1)}\left(c-\right)-k_1\varphi^{(1)}\left(c-\right)\\
&=\left\{\frac{1}{W}\int_{-\infty}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt\right\}\psi^{(1)}\left(c-\right)\\
&-\left\{\frac{\theta}{W}\int_c^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt\right\}\varphi^{(1)}\left(c-\right),\\
&k_4(p\psi^{(1)\prime})\left(c-\right)-k_1(p\varphi^{(1)\prime})\left(c-\right)\\
&=\left\{\frac{1}{W}\int_{-\infty}^{c}\varphi^{(1)}\left(t\right)h\left(t\right)dt\right\}(p\psi^{(1)\prime})\left(c-\right)\\
&-\left\{\frac{\theta}{W}\int_c^{\infty}\psi^{(2)}\left(t\right)h\left(t\right)dt\right\}(p\varphi^{(1)\prime})\left(c-\right),
\end{aligned}$$

i.e.,

$$\begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{pmatrix} \begin{pmatrix} k_4 \\ -k_1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{pmatrix} \begin{pmatrix} \frac{1}{W}\int_{-\infty}^{c} \varphi^{(1)}(t)\, h(t)\, dt \\ -\frac{\theta}{W}\int_{c}^{\infty} \psi^{(2)}(t)\, h(t)\, dt \end{pmatrix}.$$

Hence, we have the following determinant of this linear equation system

$$\begin{vmatrix} \psi^{(1)}(c-) & \varphi^{(1)}(c-) \\ (p\psi^{(1)\prime})(c-) & (p\varphi^{(1)\prime})(c-) \end{vmatrix} = -W.$$

Since this determinant is different from zero, the solution of this system is unique. If we solve this system, we have the following equalities

$$k_1 = \frac{\theta}{W}\int_{c}^{\infty} \psi^{(2)}(t)\, h(t)\, dt,$$

$$k_4 = \frac{1}{W}\int_{-\infty}^{c} \varphi^{(1)}(t)\, h(t)\, dt.$$

From what has already, we have

$$y(x) = \begin{cases} \varphi^{(1)}(x)\, \frac{\theta}{W}\int_{c}^{\infty} \psi^{(2)}(t)\, h(t)\, dt \\ +\frac{\psi^{(1)}(x)}{W}\int_{-\infty}^{x} \varphi^{(1)}(t)\, h(t)\, dt \\ +\frac{\varphi^{(1)}(x)}{W}\int_{x}^{c} \psi^{(1)}(t)\, h(t)\, dt, \;\; x \in J_1, \\ \psi^{(2)}(x)\, \frac{1}{W}\int_{-\infty}^{c} \varphi^{(1)}(t)\, h(t)\, dt \\ +\frac{\theta}{W}\psi^{(2)}(x)\int_{c}^{x} \varphi^{(2)}(t)\, h(t)\, dt \\ +\frac{\theta}{W}\varphi^{(2)}(x)\int_{x}^{\infty} \psi^{(2)}(t)\, h(t)\, dt, \;\; x \in J_2, \end{cases}$$

i.e., (3.9) and (3.11) hold. □

Thus we have a

Theorem 3.2. *The unique solution of the equation (3.1) under the conditions (2.8)-(2.9) is given by the formula*

$$y(x) = w(x) + \langle G(x,.)\,, h(.)\rangle,$$

where

$$w(x) = \frac{d_1}{W}\varphi(x) - \frac{d_2}{W}\psi(x).$$

Proof. By the conditions (3.4)-(3.8), the function $w(x)$ is a unique solution of the equation (3.1) satisfying the conditions (2.8)-(2.9). By Theorem 3.1 the function $\langle G(x,.), h(.)\rangle$ a unique solution of the equation (3.1) satisfying the conditions (3.2). This finishes the proof. □

From Theorem 3.2, the boundary-value problem (2.1), (2.8), (2.9) in H is equivalent to the nonlinear integral equation

$$y(x) = w(x) + \langle G(x,.), f(., y(.))\rangle,\ x \in J, \tag{3.17}$$

where the functions $w(x)$ and $G(x,t)$ are defined above. Hence, we shall study the equation (3.17).

By (2.4) and (3.10), we can define the operator $T : H \to H$ by the formula

$$(Ty)(x) = w(x) + \langle G(x,.), f(., y(.))\rangle,\ x \in J, \tag{3.18}$$

where $y, w \in H$. Then the equation (3.17) can be written as $y = Ty$.

Now, we search the fixed points of the operator T because it is equivalent to solving the equation (3.17).

4. The fixed points of the operator T

In this section, we investigate the fixed points of the operator T by using the following Banach fixed point theorem

Definition 4.1 ([19])**.** Let A be a mapping of a metric space R into itself. Then x is called a fixed point of A if $Ax = x$. Suppose there exists a number $\alpha < 1$ such that

$$\rho(Ax, Ay) \leq \alpha\rho(x, y)$$

for every pair of points $x, y \in R$. Then A is said to be a contraction mapping.

Theorem 4.2 ([19])**.** *Every contraction mapping A defined on a complete metric space R has a unique fixed point.*

Theorem 4.3. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. Further, let the function $f(x, y)$ satisfy the following Lipschitz condition: there* *exist a constant $K>0$ such that*

$$\int_{-\infty}^{c}\left|f^{(1)}\left(x,y^{(1)}\left(x\right)\right)-f^{(1)}\left(x,z^{(1)}\left(x\right)\right)\right|^{2}dx$$

$$+\theta\int_{c}^{\infty}\left|f^{(2)}\left(x,y^{(2)}\left(x\right)\right)-f^{(2)}\left(x,z^{(2)}\left(x\right)\right)\right|^{2}dx$$

$$\leq K^{2}\left(\int_{-\infty}^{c}\left|y^{(1)}\left(x\right)-z^{(1)}\left(x\right)\right|^{2}dx+\theta\int_{c}^{\infty}\left|y^{(2)}\left(x\right)-z^{(2)}\left(x\right)\right|^{2}dx\right)$$

$$=K^{2}\left\Vert y-z\right\Vert^{2} \tag{4.1}$$

for all $y,z\in H$. If

$$K\left(\int_{-\infty}^{c}\int_{-\infty}^{c}\left|G\left(x,t\right)\right|^{2}dxdt+\theta\int_{c}^{\infty}\int_{c}^{\infty}\left|G\left(x,t\right)\right|^{2}dxdt\right)^{1/2}<1, \tag{4.2}$$

then the boundary-value problem (2.1), (2.8), (2.9) has a unique solution in H.

Proof. It suffices to prove that the operator T is a contraction operator. For $y,z\in H_{1}$, we have

$$\begin{aligned}\left|\left(Ty\right)\left(x\right)-\left(Tz\right)\left(x\right)\right|^{2}&=\left|\left\langle G\left(x,.\right),\left[f\left(.,y\left(.\right)\right)-f\left(.,z\left(.\right)\right)\right]\right\rangle\right|^{2}\\&\leq\left\Vert G\left(x,.\right)\right\Vert^{2}\left\Vert f\left(.,y\left(.\right)\right)-f\left(.,z\left(.\right)\right)\right\Vert^{2}\\&\leq K^{2}\left\Vert G\left(x,.\right)\right\Vert^{2}\left\Vert y-z\right\Vert^{2},\ x\in J.\end{aligned}$$

Thus, we get

$$\left\Vert Ty-Tz\right\Vert\leq\alpha\left\Vert y-z\right\Vert,$$

where

$$\alpha=K\left(\int_{-\infty}^{c}\int_{-\infty}^{c}\left|G\left(x,t\right)\right|^{2}dxdt+\theta\int_{c}^{\infty}\int_{c}^{\infty}\left|G\left(x,t\right)\right|^{2}dxdt\right)^{1/2}<1,$$

i.e., T is a contraction mapping. □

Now, our next claim is that the function $f\left(x,y\right)$ satisfies a Lipschitz condition on a subset of H_{1} but not of the whole space.

Theorem 4.4. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let the function $f\left(x,y\right)$ satisfy the following Lipschitz condition:*

there exist constants $M,\ K > 0$ *such that*

$$\int_{-\infty}^{c} \left| f^{(1)}\left(x, y^{(1)}(x)\right) - f^{(1)}\left(x, z^{(1)}(x)\right)\right|^2 dx$$

$$+ \theta \int_{c}^{\infty} \left| f^{(2)}\left(x, y^{(2)}(x)\right) - f^{(2)}\left(x, z^{(2)}(x)\right)\right|^2 dx$$

$$\leq K^2 \left(\int_{-\infty}^{c} \left| y^{(1)}(x) - z^{(1)}(x) \right|^2 dx + \theta \int_{c}^{\infty} \left| y^{(2)}(x) - z^{(2)}(x) \right|^2 dx \right)$$

$$= K^2 \left\| y - z \right\|^2 \tag{4.3}$$

for all y *and* z *in* $S_M = \{y \in H : \|y\| \leq M\}$, *where* K *may depend on* M. *If*

$$\left\{ \int_{-\infty}^{c} \left| w^{(1)}(x) \right|^2 dx + \theta \int_{c}^{\infty} \left| w^{(2)}(x) \right|^2 dx \right\}^{1/2}$$

$$+ \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| G(x,t) \right|^2 dxdt \right)^{1/2}$$

$$\times \sup_{y \in S_M} \left\{ \begin{array}{c} \int_{-\infty}^{c} \left| f^{(1)}\left(t, y^{(1)}(t)\right) - f^{(1)}\left(t, z^{(1)}(t)\right)\right|^2 dt \\ +\theta \int_{c}^{\infty} \left| f^{(2)}\left(t, y^{(2)}(t)\right) - f^{(2)}\left(t, z^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^{1/2}$$

$$\leq M \tag{4.4}$$

and

$$K \left(\int_{-\infty}^{c} \int_{-\infty}^{c} \left| G(x,t) \right|^2 dxdt + \theta \int_{c}^{\infty} \int_{c}^{\infty} \left| G(x,t) \right|^2 dxdt \right)^{1/2} < 1, \tag{4.5}$$

then the boundary-value problem (2.1), (2.8), (2.9) has a unique solution with

$$\int_{-\infty}^{c} \left| y^{(1)}(x) \right|^2 dx + \theta \int_{c}^{\infty} \left| y^{(2)}(x) \right|^2 dx \leq M^2.$$

Proof. It is clear that S_M is a closed set of H. We first prove that the operator T maps S_M into itself. For $y \in S_M$ we have

$$\|Ty\| = \|w + \langle G(x,.), f(., y(.))\rangle\| \leq \|w\| + \|\langle G(x,.), f(., y(.))\rangle\|$$

$$\leq \|w\| + \left(\int_{-\infty}^{c}\int_{-\infty}^{c} |G(x,t)|^2\, dxdt + \theta \int_{c}^{\infty}\int_{c}^{\infty} |G(x,t)|^2\, dxdt\right)^{1/2}$$

$$\times \sup_{y\in S_M} \left\{ \begin{array}{c} \int_{-\infty}^{c} \left|f^{(1)}\left(t, y^{(1)}(t)\right) - f^{(1)}\left(t, z^{(1)}(t)\right)\right|^2 dt \\ +\theta \int_{c}^{\infty} \left|f^{(2)}\left(t, y^{(2)}(t)\right) - f^{(2)}\left(t, z^{(2)}(t)\right)\right|^2 dt \end{array} \right\}^{1/2}$$

$$\leq M.$$

Consequently, $T : S_M \to S_M$.

We can now proceed analogously to the proof of Theorem 4.3. So, we can get

$$\|Ty - Tz\| \leq \alpha \|y - z\|, \ y, z \in S_M.$$

If we apply the Banach fixed point theorem, then we obtain a unique solution of the boundary-value problem (2.1), (2.8), (2.9) in S_M. ☐

5. An existence theorem without uniqueness

In this section, we get an existence theorem without uniqueness of solution. Therefore, we will use the following Schauder fixed point theorem:

Definition 5.1 ([14])**.** An operator acting in a Banach space is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Theorem 5.2 ([14])**.** *Let* $\mathbf{B}$ *be a Banach space and* $\mathbf{S}$ *a nonemty bounded, convex, and closed subset of* $\mathbf{B}$. *Assume* $A : \mathbf{B} \to \mathbf{B}$ *is a completely continuous operator. If the operator* A *leaves the set* $\mathbf{S}$ *invariant, i.e., if* $A(\mathbf{S}) \subset \mathbf{S}$, *then* A *has at least one fixed point in* $\mathbf{S}$.

A set $S \subset H$ is relatively compact iff S is bounded and for every $\varepsilon > 0$ (i) there exists $\delta > 0$ such that $\|y(x+h) - y(x)\| < \varepsilon$ for all $y \in S$ and all $h \geq 0$ with $h < \delta$, (ii) there exists a number $N > 0$ such that $\int_{-\infty}^{-N} |y(x)|^2 + \theta \int_{N}^{\infty} |y(x)|^2 dx < \varepsilon$ for all $y \in S$ ([14]).

Now, we give a

Theorem 5.3. *The operator* T *defined by (3.18) is completely continuous operator under the conditions (A1), (A2) and (A3).*

Proof. Let $y_0 \in H$. Then, we have

$$
\begin{aligned}
&\left|(Ty)(x)-(Ty_0)(x)\right|^2 \\
&= \left|\langle G(x,.),[f(.,y(.))-f(.,y_0(.))]\rangle\right|^2 \leq \left\|G(x,.)\right\|^2 \\
&\times\left\{\begin{array}{c} \int_{-\infty}^{c}\left|f^{(1)}\left(t,y^{(1)}(t)\right)-f^{(1)}\left(t,y_0^{(1)}(t)\right)\right|^2 dt \\ +\theta\int_{c}^{\infty}\left|f^{(2)}\left(t,y^{(2)}(t)\right)-f^{(2)}\left(t,y_0^{(2)}(t)\right)\right|^2 dt \end{array}\right\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left\|Ty-Ty_0\right\|^2 \\
&\leq K\left\{\begin{array}{c} \int_{-\infty}^{c}\left|f^{(1)}\left(t,y^{(1)}(t)\right)-f^{(1)}\left(t,y_0^{(1)}(t)\right)\right|^2 dt \\ +\theta\int_{c}^{\infty}\left|f^{(2)}\left(t,y^{(2)}(t)\right)-f^{(2)}\left(t,y_0^{(2)}(t)\right)\right|^2 dt \end{array}\right\}, \qquad (5.1)
\end{aligned}
$$

where

$$
K=\int_{-\infty}^{c}\int_{-\infty}^{c}\left|G(x,t)\right|^2 dxdt+\theta\int_{c}^{\infty}\int_{c}^{\infty}\left|G(x,t)\right|^2 dxdt.
$$

We know that an operator F defined by $Fy(x)=f(x,y(x))$ is continuous in H under the condition (A2) (see [20]). Hence, for the given $\epsilon>0$, we can find a $\delta>0$ such that $\|y-y_0\|<\delta$ implies

$$
\left\{\begin{array}{c} \int_{-\infty}^{c}\left|f^{(1)}\left(t,y^{(1)}(t)\right)-f^{(1)}\left(t,y_0^{(1)}(t)\right)\right|^2 dt \\ +\theta\int_{c}^{\infty}\left|f^{(2)}\left(t,y^{(2)}(t)\right)-f^{(2)}\left(t,y_0^{(2)}(t)\right)\right|^2 dt \end{array}\right\}<\frac{\epsilon^2}{K^2}.
$$

From (5.1), we get

$$
\left\|Ty-Ty_0\right\|<\epsilon,
$$

i.e., T is continuous.

Set $Y=\{y\in H:\|y\|\leq C\}$. By (4.4), we have

$$
\|Ty\|\leq\|w\|+\left\{\begin{array}{c} K\int_{-\infty}^{c}\left|f^{(1)}\left(t,y^{(1)}(t)\right)\right|^2 dt \\ +\theta K\int_{c}^{\infty}\left|f^{(2)}\left(t,y^{(2)}(t)\right)\right|^2 dt \end{array}\right\}^{1/2},
$$

for all $y \in Y$. Furthermore, using (2.4), we get

$$\int_{-\infty}^{c} \left| f^{(1)}\left(t, y^{(1)}(t)\right)\right|^2 dt + \theta \int_{c}^{\infty} \left| f^{(2)}\left(t, y^{(2)}(t)\right)\right|^2 dt$$

$$\leq \int_{-\infty}^{c} \left[g^{(1)}(t) + \kappa \left| y^{(1)}(t)\right|\right]^2 dt + \theta \int_{c}^{\infty} \left[g^{(2)}(t) + \kappa \left| y^{(2)}(t)\right|\right]^2 dt$$

$$\leq 2\int_{-\infty}^{c} \left[\left(g^{(1)}\right)^2(t) + \kappa^2 \left| y^{(1)}(t)\right|^2\right] dt + 2\theta \int_{c}^{\infty} \left[\left(g^{(2)}\right)^2(t) + \kappa^2 \left| y^{(2)}(t)\right|^2\right] dt$$

$$= 2\left(\|g\|^2 + \kappa^2 \|y\|^2\right) \leq 2\left(\|g\|^2 + \kappa^2 C^2\right).$$

Thus, for all $y \in Y$, we obtain

$$\|Ty\| \leq \|w\| + \left[2K\left(\|g\|^2 + \kappa^2 C^2\right)\right]^{1/2},$$

i.e., $T(Y)$ is a bounded set in H.

Moreover, for all $y \in Y$, we have

$$\int_{-\infty}^{c} \left| \left(Ty^{(1)}\right)(x+h) - \left(Ty^{(1)}\right)(x)\right|^2 dx$$

$$+\theta \int_{c}^{\infty} \left| \left(Ty^{(2)}\right)(x+h) - \left(Ty^{(2)}\right)(x)\right|^2 dx$$

$$= \|\langle [G(x+h,.) - G(x,.)], f(., y(.))\rangle\|^2 dx$$

$$\leq \left\{ \begin{array}{c} \int_{-\infty}^{c}\int_{-\infty}^{c} |G(x+h,t) - G(x,t)|^2 dxdt \\ +\theta \int_{c}^{\infty}\int_{c}^{\infty} |G(x+h,t) - G(x,t)|^2 dxdt \end{array} \right\}$$
$$\times \left\{ \begin{array}{c} \int_{-\infty}^{c} \left| f^{(1)}\left(t, y^{(1)}(t)\right)\right|^2 dt \\ +\theta \int_{c}^{\infty} \left| f^{(2)}\left(t, y^{(2)}(t)\right)\right|^2 dt \end{array} \right\}$$

$$\leq 2\left(\|g\|^2 + \kappa^2 C^2\right) \left\{ \begin{array}{c} \int_{-\infty}^{c}\int_{-\infty}^{c} |G(x+h,t) - G(x,t)|^2 dxdt \\ +\theta \int_{c}^{\infty}\int_{c}^{\infty} |G(x+h,t) - G(x,t)|^2 dxdt \end{array} \right\}.$$

From (3.10), there exists a $\delta > 0$ such that

$$\int_{-\infty}^{c} \left| Ty^{(1)}(x+h) - Ty^{(1)}(x) \right|^2 dx$$

$$+ \theta \int_{c}^{\infty} \left| Ty^{(2)}(x+h) - Ty^{(2)}(x) \right|^2 dx < \epsilon^2,$$

for given $\epsilon > 0$, all $y \in Y$ and all $h < \delta$.

Further, for all $y \in Y$, we have

$$\int_{-\infty}^{-N} \left| Ty^{(1)}(x) \right|^2 dx + \theta \int_{N}^{\infty} \left| Ty^{(2)}(x) \right|^2 dx$$

$$\leq \int_{-\infty}^{-N} \left| w^{(1)}(x) \right|^2 dx + \theta \int_{N}^{\infty} \left| w^{(2)}(x) \right|^2 dx$$

$$+ 2(\|g\|^2 + \kappa^2 C^2) \left(\int_{-\infty}^{-N} \|G(x,.)\|^2 dx + \theta \int_{N}^{\infty} \|G(x,.)\|^2 dx \right).$$

So, from (3.10), we see that for given $\epsilon > 0$ there exists a positive number N, depending only on ϵ such that

$$\int_{-\infty}^{-N} \left| Ty^{(2)}(x) \right|^2 dx + \theta \int_{N}^{\infty} \left| Ty^{(2)}(x) \right|^2 dx < \epsilon^2,$$

for all $y \in Y$.

Thus $T(Y)$ is a relatively compact in H, i.e., the operator T is completely continuous. □

Theorem 5.4. *Suppose that the conditions (A1), (A2) and (A3) are satisfied. In addition, let there exist constants $M > 0$ such that*

$$\left\{ \int_{-\infty}^{c} \left| w^{(1)}(x) \right|^2 dx + \theta \int_{c}^{\infty} \left| w^{(2)}(x) \right|^2 dx \right\}^{1/2}$$

$$+ \left(\int_{-\infty}^{c} \int_{-\infty}^{c} |G(x,t)|^2 dxdt + \theta \int_{c}^{\infty} \int_{c}^{\infty} |G(x,t)|^2 dxdt \right)^{1/2}$$

$$\times \sup_{y \in S_M} \left\{ \begin{array}{c} \int_{-\infty}^{c} \left| f^{(1)}\left(t, y^{(1)}(t)\right) - f^{(1)}\left(t, z^{(1)}(t)\right) \right|^2 dt \\ +\theta \int_{c}^{\infty} \left| f^{(2)}\left(t, y^{(2)}(t)\right) - f^{(2)}\left(t, z^{(2)}(t)\right) \right|^2 dt \end{array} \right\}^{1/2}$$

$$\leq M, \tag{5.2}$$

where $S_M = \{y \in H : \|y\| \leq M\}$. *Then the boundary-value problem (2.1), (2.8), (2.9) has a unique solution with*

$$\int_{-\infty}^{c} \left| y^{(1)}(x) \right|^2 dx + \theta \int_{c}^{\infty} \left| y^{(2)}(x) \right|^2 dx \leq M^2.$$

Proof. Let us define an operator $T : H \to H$ by (3.18). From theorems 4.4, 5.3 and (5.2), we conclude that T maps the set S_M into itself. It is clear that the set S_M is bounded, convex and closed. Using by Theorem 5.2, the theorem follows. □

Acknowledgments

The authors would like to thank the referee for her/his comments.

References

1. R. P. Agarwal, D. O'Regan, Boundary Value Problems of Nonsingular Type on the Semi-infinite Interval, *Tohoku Math. Journal*, **51**(3), (1999), 391–397.
2. R. P. Agarwal, D. O'Regan, Multiple Nonnegative Solutions for Second Order Impulsive Differential Equations, *Appl. Math. Comput.*, **114**, (2000), 51–59.
3. B. P. Allahverdiev, H. Tuna, Existence of Solutions for Nonlinear Singular q-Sturm–Liouville Problems, *Ufa Math. J.*, **12**(1), (2020), 91–102.
4. B. P. Allahverdiev, H. Tuna, Nonlinear Singular Sturm–Liouville Problems with Impulsive Conditions, *Facta Univ., Ser. Math. Inf.*, **34**(3), (2019), 439–457.
5. B. P. Allahverdiev, H. Tuna, On the Solutions for a Nonlinear Singular q-Sturm–Liouville Problems on the Whole Axis, *Southeast Asian Bull. Math.*, **47**(2), (2023), 151-164.
6. D. D. Bainov, P. S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman: Harlow, 1993.
7. D. D. Bainov, P. S. Simeonov, *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, World Scientific: Singapore, 1995.
8. D. D. Bainov, P. S. Simeonov, *Oscillation Theory of Impulsive Differential Equations*, International Publications: Orlando, 1998.
9. M. Benchohra, J. Henderson, S. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation: New York, 2006.
10. W. Ding, M. Han, Periodic Boundary Value Problem for the Second Order Impulsive Functional Differential Equations, *Appl. Math. Comput.*, **155**(3), (2004), 709–726.
11. N. Dunford, J. T. Schwartz, *Linear Operators*, Part II, Interscience, New York, 1964.
12. M. Feng, D. Xie, Multiple Positive Solutions of Multi-point Boundary Value Problem for Second Order Impulsive Differential Equations, *J. Comput. Appl. Math.*, **223**(1), (2009), 438–448.
13. D. Guo, Second Order Impulsive Integro-differential Equations on Unbounded Domains in Banach Spaces, *Nonlinear Analysis: Theory, Methods & Applications*, **35**(4), (1999), 413–423.
14. G. Sh. Guseinov, I. Yaslan, Boundary Value Problems for Second Order Nonlinear Differential Equations on Infinite Intervals, *J. Math. Anal. Appl.*, **290,** (2004), 620–638.
15. S. G. Hristova, D. D. Bainov, Monotone-iterative Techniques of V. Lakshmikantham for a Boundary Value Problem for Systems of Impulsive Differential-difference Equations, *J. Math. Anal. Appl.*, **197**(1), (1996), 1-13.
16. T. Jankowski, Positive Solutions to Second Order Four-point Boundary Value Problems for Impulsive Differential Equations, *Appl. Math. Comput.*, **202**(2), (2008), 550–561.

17. T. Jankowski, Positive Solutions of Three-point Boundary Value Problems for Second Order Impulsive Differential Equations with Advanced Arguments, *Appl. Math. Comput.*, **197**(1), (2008), 179–189.
18. T. Jankowski, Existence of Solutions for Second Order Impulsive Differential Equations with Deviating Arguments, *Nonlinear Analysis: Theory, Methods & Applications*, **67**(6), (2007), 1764–1774.
19. A. N. Kolmogorov, S. V. Fomin, I*ntroductory Real Analysis*, Translated by R. A. Silverman, Dover Publications, New York, 1970.
20. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Gostekhteoretizdat, Moscow, 1956, English transl. Pergamon Press, New York, 1964.
21. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific: Singapore, 1989.
22. I. K. C. Leung, P. N. Shivakumar, On the Eigenvalue Problem $-y'' + f(x)\, y = \lambda y$ on a Semi Infinite Interval, *Math. Comput. Mod.*, **46**, (2007), 316–330.
23. B. M. Levitan, I. S.Sargsjan, *Sturm–Liouville and Dirac Operators*, Mathematics and its Applications (Soviet Series, translated from the Russian), Kluwer Academic Publishers Group, Dordrecht, 1991.
24. J. Li, J. J. Nieto, Existence of Positive Solutions for Multipoint Boundary Value Problem on the Half-line with Impulses, *Boundary Value Problems*, **2009**, (2009), Article ID 834158, 1–12.
25. J. Li, J. Shen, Existence of Positive Solution for Second-order Impulsive Boundary Value Problems on Infinity Intervals, *Boundary Value Problems*, **2006**, (2006), Article ID 14594, 1-11.
26. H. Lian, W. Ge, Existence of Positive Solutions for Sturm–Liouville Boundary Value Problems on the Half-line, *J. Math. Anal. Appl.*, **321**, (2006), 781–792.
27. H. Lian, H. Pang, W. Ge, Triple Positive Solutions for Boundary Value Problems on Infinite Intervals, *Nonlinear Anal.*, **67**, (2007), 2199–2207.
28. X. Lin, D. Jiang, Multiple Positive Solutions of Dirichlet Boundary Value Problems for Second Order Impulsive Differential Equations, *J. Math. Anal. Appl.*, **321**(2), (2006), 501–514.
29. L. Liu, F. Y. Li, Multiple Positive Solution of Nonlinear Two-point Boundary Value Problems, *J. Math. Anal. Appl.*, **203**, (1996), 610–625.
30. Y. Liu, Boundary Value Problems for Second Order Differential Equations on Unbounded Domains in a Banach Space, *Appl. Math. Comput.*, **135**(2-3), (2003), 569–583.
31. X. Liu, D. Guo, Periodic Boundary Value Problems for a Class of Second-order Impulsive Integro-differential Equations in Banach Spaces, *J. Math. Anal. Appl.*, **216**(1), (1997), 284–302.
32. R. Ma, Existence of Positive Solutions for Second-order Boundary Value Problems on Infinity Intervals, *Appl. Math. Letters*, **16**(1), (2003), 33–39.
33. I. Rachunkova, J. Tomecek, Impulsive BVPs with Nonlinear Boundary Conditions for the Second Order Differential Equations Without Growth Restrictions, *J. Math. Anal. Appl.*, **292**(2), (2004), 525–539.
34. A. M. Samoilenko, N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific: Singapore, 1995.
35. E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, Part I. Second Edition, Clarendon Press, Oxford, 1962.
36. Z. Wei, Periodic Boundary Value Problems for Second Order Impulsive Integrodifferential Equations of Mixed Type in Banach Spaces, *J Math Anal Appl.*, **195**(1), (1995), 214–229.