On Contact and Symplectic Lie Algebroids

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ABSTRACT. In this paper, we will study compatible triples on Lie alge-
broids. Using a suitable decomposition for a Lie algebroid, we construct
an integrable generalized distribution on the base manifold. As a result,
the symplectic form on the Lie algebroid induces a symplectic form on the
leaves of the distribution. Hence, the induced Poisson structure on the
base manifold can be represented by means of the induced Poisson struc-
tures on the integral submanifolds. Moreover, for any compatible triple
with an invariant metric and an admissible almost complex structure, we
show that the bracket annihilates on the kernel of the anchor map.

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Poisson structure.

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1. INTRODUCTION

Lie algebroids have an important place in the context of some different
categories in differential geometry and mathematical physics and represent an
active domain of research([1, 2, 5, 8, 13, 14, 16]). Lie algebroids, are generaliza-
tions of Lie algebras and integrable distributions([10]). In fact a Lie algebroid
is an anchored vector bundle with a Lie bracket on the module of sections and
many geometrical notions which involve the tangent bundle were generalized

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to the context of Lie algebroids.

One of the most important examples of Lie algebroids are cotangent bundles of Poisson manifolds([18]). More over the dual bundle of every Lie algebroids has a canonical Poisson structure([4]). A Poisson structure on a Lie algebroid is usually described as a 2-section. Using this 2-section researchers have studied the induced Poissson bracket on the base manifold([9, 11, 15, 17, 18]).

A compatible triple on a smooth manifold, consists of a Riemannian metric, a symplectic 2-form and an almost complex structure, together with a couple of compatibility conditions. Compatible triples on ordinary manifolds, has been an area of interest for decades and has been studied in a great deal of papers. There exists two main methods concerning the issue of compatibility of Riemannian metric and Lie algebroid structure. Boucetta[2] considers a Riemannian metric as compatible provided that its Levi-Civita connection is compatible with the Lie algebroid structure. Focusing on transitive Lie algebroids, however, Boroojerdian[5] calls a metric, compatible if it preserves all the adjoint connections of the Lie algebroids. Due to the Physical applications of the latter method we will use it in this paper.

On the other hand, Ida[6] discussed the notion of almost complex Lie algebroid and clarified the relations between almost complex structure on the Lie algebroid and the one on the base manifold. We use some of his results to derive the compatibility condition.

In this setting, we will focus on compatible triples on Lie algebroids and represent a specific decomposition of the Lie algebroid using the symplectic form. This decomposition helps us to achieve a better and more accurate image of the Lie algebroid that can be useful in order to investigate the structure on the Lie algebroid.

The current paper contains four sections. In section two we review basic definitions and facts about Lie algebroids and decompositions of transitive Lie algebroids based on [5]. Invariant metrics on transitive Lie algebroids are also mentioned in this section.

Section 3 includes two parts. The first, is dedicated to symplectic Lie algebroids, almost complex structures and compatible triples on Lie algebroids. In the second part we have used a compatible triple to give a new and special decomposition for the Lie algebroid. This decomposition induces an integrable generalized distribution on the base manifold. We investigate the relation between the Poisson structure on the integral sub-manifolds and Poisson structure on the base manifold.We already know that any symplectic structure on a Lie algebroid, induces a Poisson bracket on the base manifold. Thus, in any compatible triple, the cotangent bundle of the base manifold inherits a Lie algebroid structure. Having this in mind, we can use decomposition $\mathcal{A} = E^1 + E^2 + L^1 + L^2$ to show that $T^*M \cong E^2 + L^1$ as two Lie algebroids, provided that $\mathcal{A}$ is transitive.

Finally in section 4 we discuss contact Lie algebroids, mainly based on [7]. We find conditions for the contact form of the Lie algebroid to induce a Poisson structure on the base manifold. In fact, this will happen if the Reeb section is in the kernel of the anchor map.

2. Preliminaries on Lie Algebroids

A Lie algebroid over a manifold M may be regarded as a generalized tangent bundle to M. In fact a Lie algebroid $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ over a smooth manifold M is a vector bundle $\pi : \mathcal{A} \to M$ together with a Lie algebra structure $[\,,\,]_{\mathcal{A}}$ on the space $\Gamma(\mathcal{A})$ of sections and a bundle map $\rho : \mathcal{A} \to TM$ called the anchor map such that

(1) The induced map $\rho : \Gamma(\mathcal{A}) \longrightarrow \mathcal{X}(M)$ is a homomorphism of Lie algebras, that is, $\rho([S_1, S_2]_{\mathcal{A}}) = [\rho(S_1), \rho(S_2)]$ for $S_1, S_2 \in \Gamma(\mathcal{A})$.

(2) For any sections $S_1, S_2 \in \Gamma(\mathcal{A})$ and for every smooth function $f \in C^{\infty}(M)$ the Leibniz identity $[S_1, fS_2]_{\mathcal{A}} = f[S_1, S_2]_{\mathcal{A}} + (\rho(S_1) \cdot f)S_2$ is satisfied.

The basic example of a Lie algebroid over M is the tangent bundle TM with the identity map as the anchor map.

Lie algebroids can also be smaller or larger than TM. Any integrable distribution of TM is a Lie algebroid with the induced bracket and the inclusion as the anchor map. On the other hand, any Lie algebra $\mathfrak{g}$ is a Lie algebroid over a point.

Example 2.1. Let $(M, \{\,,\,\})$ be a Poisson manifold equipped with a bivector π. Consider the Lie bracket $[\,,\,]_{\pi}$ on $\Gamma(T^*M)$ as follows

$$[\alpha, \beta]_{\pi} = \mathcal{L}_{\pi^{\sharp}(\alpha)}\beta - \mathcal{L}_{\pi^{\sharp}(\beta)}\alpha - d(\pi(\alpha, \beta)$$

for $\alpha, \beta \in \Gamma(T^*M)$. Then $(T^*M, \pi^{\sharp}, [\,,\,]_{\pi})$ is a Lie algebroid over $M([9])$.

An important operator associated with a Lie algebroid $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ over a manifold M is the exterior derivative $d^{\mathcal{A}} : \Gamma(\wedge^k \mathcal{A}^*) \longrightarrow \Gamma(\wedge^{k+1} \mathcal{A}^*)$ of $\mathcal{A}$ which is defined as follows

$$d^{\mathcal{A}}(\eta)(S_0, ..., S_k) = \sum_{i=0}^{k} (-1)^i \rho(S_i) \cdot \eta(S_1, ..., \widehat{S_i}, ..., S_k)$$

$$+ \sum_{i<j=1}^{k} (-1)^{i+j} \eta([S_i, S_j]_{\mathcal{A}}, S_1, ..., \widehat{S_i}, ..., \widehat{S_j}, ..., S_k)$$

for $\eta \in \Gamma(\wedge^k \mathcal{A}^*)$ and $S_0, ..., S_k \in \Gamma(\mathcal{A})$. It follows that $(d^{\mathcal{A}})^2 = 0([8])$.
Moreover, if S is a section of $\mathcal{A}$, one may introduce, in a natural way, the Lie derivative with respect to S, as the operator $\mathcal{L} : \Gamma(\wedge^k \mathcal{A}^*) \longrightarrow \Gamma(\wedge^k \mathcal{A}^*)$ given by

$$\mathcal{L}_S = i_S \circ d^{\mathcal{A}} + d^{\mathcal{A}} \circ i_S$$

where i_S is the inner contraction with S. Exactly like ordinary manifolds, the usual property $\mathcal{L}_S \circ d = d \circ \mathcal{L}_S$ holds here(see[4]), as well as the relations

$$i_{[S,T]_{\mathcal{A}}} = [\mathcal{L}_S, i_T]_{\mathcal{A}} \qquad [\mathcal{L}_S, \mathcal{L}_T]_{\mathcal{A}} = \mathcal{L}_{[S,T]_{\mathcal{A}}}.$$

Let $(\mathcal{A}, [,]_{\mathcal{A}}, \rho_{\mathcal{A}})$ and $(\mathcal{B}, [,]_{\mathcal{B}}, \rho_{\mathcal{B}})$ be two Lie algebroids over smooth manifolds M and M', respectively. A vector bundle map $(\Phi, \phi) : \mathcal{A} \to \mathcal{B}$ is called a *morphism of Lie algebroids* if for every $\alpha \in \Gamma(\wedge^k \mathcal{B}^*)$ we have

$$\Phi^*\big(d^{\mathcal{B}}\alpha\big) = d^{\mathcal{A}}\big(\Phi^*(\alpha)\big) \tag{2.1}$$

where $\Phi^* : \wedge^k \mathcal{B}^* \longrightarrow \wedge^k \mathcal{A}^*$ is defined by

$$\Phi^*(\alpha)_p(S_1, ..., S_n) = \alpha_{\phi(p)}\big(\Phi(S_1), ..., \Phi(S_n)\big)$$

for $\alpha \in \Gamma(\wedge^k \mathcal{B}^*)$, $p \in M$ and $S_1, ..., S_n \in \mathcal{A}_p$. It is easy to see that (Φ, ϕ) is a morphism of Lie algebroids if for every $S \in \Gamma\mathcal{A}$, $\rho_{\mathcal{B}}(\Phi(S)) = \phi_*(\rho_{\mathcal{A}}(S))$ and equation (2.1) holds for every 1-form section of $\mathcal{B}$. Moreover, if ϕ is diffeomorphism then Φ maps any section of $\mathcal{A}$ to a section of $\mathcal{B}$. In this case, Φ is a morphism of Lie algebroids if $\rho_{\mathcal{B}} \circ \Phi = \phi_* \circ \rho_{\mathcal{A}}$ and $\Phi([S_1, S_2]_{\mathcal{A}}) = [\Phi(S_1), \Phi(S_2)]_{\mathcal{B}}$ for any $S_1, S_2 \in \Gamma\mathcal{A}([4])$.

A Lie algebroid $(\mathcal{A}, \rho, [,]_{\mathcal{A}})$ over smooth manifold M is called *transitive* (respectively *regular*) if ρ is surjective (respectively constant rank)([4]). An immediate consequence of this definition is that, for any p in M, there is an induced Lie bracket say $([,]_{\mathcal{A}})_p$ on

$$L_p = Ker(\rho_p) \subseteq A_p$$

which makes it into a Lie algebra[10]. For a Lie algebroid $(\mathcal{A}, \rho, [,]_{\mathcal{A}})$ over M, the image of ρ defines a smooth integrable generalized distribution in $M([3])$. The derived foliation is called the *characteristic foliation* of $\mathcal{A}$. Let N be any leaf of the characteristic foliation of $\mathcal{A}$ on M, it is easy to see that the bracket on $\Gamma(\mathcal{A})$ deduce the bracket on the space of section of the restriction $\mathcal{A}_N$ of $\mathcal{A}$ to N. Then $\rho_{|_{\mathcal{A}_N}} : \mathcal{A}_N \to TN$ is a transitive Lie algebroid([2]).

For a transitive Lie algebroid, $L = ker\rho$ is a bundle of Lie algebras [10]. Suppose $(\mathcal{A}, \rho, [,]_{\mathcal{A}})$ is a transitive Lie algebroid, then a vector bundle map $\lambda : TM \to \mathcal{A}$ such that $\rho \circ \lambda = 1_{TM}$ is a splitting of $(\mathcal{A}, \rho, [,]_{\mathcal{A}})$ i,e., we can decompose $\mathcal{A}$ to $E + L$ of vector bundles, where $E = \lambda(TM)$.

Fix a $\lambda : TM \to \mathcal{A}$ splitting of ρ. The map λ defines a linear connection on L, called an adjoint connection as follows.

$$\nabla^\lambda : \mathcal{X}(M) \times \Gamma(L) \to \Gamma(L)$$
$$\nabla^\lambda_X T := [\lambda(X), T]_{\mathcal{A}}$$

The 2-form Ω^λ in $A^2(M, L)$ is defined as follows

$$2\Omega^\lambda(X, Y) = [\lambda(X), \lambda(Y)]_{\mathcal{A}} - \lambda([X, Y])$$

and is called the curvature 2-form. The Lie bracket on $\Gamma(\mathcal{A})$ with respect to the decomposition $\mathcal{A} = E + L$ is written as follows

$$[\lambda(X) + S, \lambda(Y) + T]_{\mathcal{A}} = \lambda([X,Y]) + \nabla_X^\lambda T - \nabla_Y^\lambda S + [S,T]_L + \Omega(X,Y).$$

Conversely if L is a bundle of Lie algebras, and ∇ is a connection on L that preserves the Lie bracket and the curvature of ∇ is in the form $[2\Omega(X,Y), S]_{\mathcal{A}}$ for $S \in \Gamma(L)$ and some $\Omega \in A^2(M,L)$, then we can make $TM + L$ into a transitive Lie algebroid by defining a Lie bracket on $\Gamma(TM + L)$ as follows

$$[\![X + S, Y + T]\!] = [X,Y] + \nabla_X T - \nabla_Y S + [S,T]_L + \Omega(X,Y).$$

So all transitive Lie algebroids have the above structure (for more details, see[5]).

A Riemannian metric g on Lie algebroid $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ is said to be *invariant* if all adjoint connections of $\mathcal{A}$ preserve the restriction g_L of g to L, i.e., for every λ and $X \in \mathcal{X}(M)$, $\nabla_X^\lambda g_L = 0$. Having an invariant Riemannian metric g one can write

$$g([S_1, S_2]_{\mathcal{A}}, S_3) = g(S_1, [S_2, S_3]_{\mathcal{A}})$$

for $S_1, S_2, S_3 \in \Gamma(L)$.

Proposition 2.2 ([5])**.** *If g is an invariant metric on $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ and ∇ is the Levi-Civita connection of $\mathcal{A}$ then*

$$\nabla_{S_1} S_2 = \frac{1}{2}[S_1, S_2]_{\mathcal{A}} \tag{2.2}$$

for every $S_1, S_2 \in \Gamma(L)$.

3. Compatible Triples on Lie Algebroids

This section includes three parts. The first, is dedicated to almost complex Lie algebroids. And the second reviewed symplectic Lie algebroids and compatible triples on Lie algebroids. In the last part, we have used a compatible triple to give a new and special decomposition for the Lie algebroid.

3.1. **Almost complex Lie algebroids.**

An *almost complex structure* $J_{\mathcal{A}}$ on $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ is an endomorphism $J_{\mathcal{A}} : \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$, over the identity, such that $J_{\mathcal{A}}^2 = -\mathrm{id}_{\Gamma(\mathcal{A})}$. A Lie algebroid $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}}, J_{\mathcal{A}})$ endowed with such a structure is called an almost complex Lie algebroid.

Definition 3.1. We call an almost complex structure $J_{\mathcal{A}}$ on $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$, *admissible* (or called admissible with respect to J_M) if there exists an almost complex structure J_M on M such that

$$\rho \circ J_{\mathcal{A}} = J_M \circ \rho.$$

On transitive Lie algebroids, we can derive a better and more useful criterion to investigate the admissibility of an almost complex structure.

Proposition 3.2. *An almost complex structure $J_\mathcal{A}$ on transitive Lie algebroid $\mathcal{A}$ is admissible if and only if $J_\mathcal{A}(L) \subseteq L$.*

Proof. Let $\lambda : TM \longrightarrow \mathcal{A}$ be a splitting of ρ and $E := \lambda(TM)$, i.e., $\mathcal{A} = E + L$. For $X \in TM$ put

$$J_M(X) := \rho\big(J_\mathcal{A}(\lambda(X))\big).$$

We show that J_M is an almost complex structure on M. To prove this we need to show that $J_M^2 = -id_{TM}$. For $X \in \mathcal{X}(M)$

$$J_M^2(X) = \rho\Big(J_\mathcal{A}\big(\lambda(\rho\big(J_\mathcal{A}(\lambda(X))\big))\big)\Big) = \rho\Big(J_\mathcal{A}\big((J_\mathcal{A}(\lambda(X)))^E\big)\Big) \qquad (3.1)$$

where $\big(J_\mathcal{A}(\lambda(X))\big)^E$ is the E-part of $J_\mathcal{A}(\lambda(X))$ with respect to the given decomposition of $\mathcal{A}$. On the other hand, $J_\mathcal{A}(\lambda(X)) = J_\mathcal{A}(\lambda(X))^E + J_\mathcal{A}(\lambda(X))^L$. Thus by applying $J_\mathcal{A}$ we have $J_\mathcal{A}(J_\mathcal{A}(\lambda(X))^E) = -X - J_\mathcal{A}(J_\mathcal{A}(\lambda(X))^L)$. Again by applying ρ we have $\rho(J_\mathcal{A}(J_\mathcal{A}(\lambda(X))^E)) = -\rho(\lambda(X)) = -X$. So by (3.1), $J_M^2 = -Id_{TM}$, i.e., J_M is an almost complex structure on M that clearly satisfies $J_M \circ \rho = \rho \circ J_\mathcal{A}$.
The converse is obvious.   □

Remark 3.3.

(1) We will use the notion of integrability of almost complex structures on Lie algebroids as Popescu developed in [6].
(2) If $J_\mathcal{A}$ is admissible with respect to J_M, and $N_{J_\mathcal{A}}$ and N_{J_M} are Nijenhuis tensors of $(\mathcal{A}, J_\mathcal{A})$ and (M, J_M) respectively, we have

$$\rho(N_{J_\mathcal{A}}(a,b)) = N_{J_M}(\rho(a), \rho(b)) \qquad (3.2)$$

for every a, b in $\Gamma(\mathcal{A})$([6]).

EXAMPLE 3.4. Let (M, J_M, g_M) be an almost Hermitian manifold with the Levi- Civita connection ∇. We denote the induced connection on $L(TM)$ again by ∇. This connection preserves Lie bracket of $L(TM)$. We also denote the curvature of ∇ on $L(TM)$ by R'. So for $T \in L(TM)$, we have $R'(X,Y)T = [R(X,Y), T]_{L(TM)}$. Consequently, we can define an algebroid structure on $\mathcal{A} := TM + L(TM)$ by the following Lie bracket on $\Gamma(TM + L(TM))$.([5])
For every $X, Y \in \mathcal{X}(M)$ and T, S in $\Gamma(L(TM))$

$$[X + T, Y + S]_\mathcal{A} = [X, Y] + \nabla_X S - \nabla_Y T + [T, S]_{L(TM)} + R(X, Y).$$

Now we can define an almost complex structure on $\Gamma(L(TM))$, induced by the almost complex structure J_M on M, as

$$J_{L(TM)} : \quad \Gamma(L(TM)) \longrightarrow \Gamma(L(TM))$$

$$J_{L(TM)}(T)(X) := T(J_M(X)) \qquad (X \in \mathcal{X}(M)).$$

Using these two almost complex structures, we define an almost complex structure on $TM + L(TM))$ as $J(X + T) = J_M(X) + J_{T(M)}(T)$ for $X \in \mathcal{X}(M)$ and

$T \in \Gamma(L(TM))$. Clearly J is admissible with respect to J_M. It is easy to see that $g_{L(TM)}(T, S) := trace(TS^T)$ is a Riemannian metric on $L(TM)$, compatible with $J_{L(TM)}$. Thus $(TM + L(TM), J, g)$ is an almost Hermitian Lie algebroid where

$$g(X + T, Y + S) = g_M(X, Y) + g_{L(TM)}(T, S).$$

Moreover, (M, J_M, g_M) is a Hermitian manifold if and only if $(TM + L(TM), J, g)$ is a Hermitian Lie algebroid. In fact, if (M, J_M, g_M) is Hermitian, then the Nijenhuis tensor of J can be calculated in the following two conditions:

(1) If $T, S \in \Gamma(L(TM))$ and $X \in \mathcal{X}(M)$ then by direct calculus we have

$$N(T, X)(Y) = N(T, S)(X) = 0.$$

(2) If $X, Y \in \mathcal{X}(M)$ then

$$
\begin{aligned}
N(X, Y) &= [J_M(X), J_M(Y)] - [X, Y] - J_M[J_M(X), Y] \\
&\quad - J_M[X, J_M(Y)] + R(J_M(X), J_M(Y)) - R(X, Y) \quad (3.3) \\
&\quad - J_M(R(X, J_M(Y))) - J_M(R(J_M(X), Y)).
\end{aligned}
$$

The first line in the above equation is clearly zero. So we need to show that the second line is zero.

Integrability of J_M leads to the following calculation

$$
\begin{aligned}
J_M(R(X, Y)Z) &= J_M(\nabla_X \nabla_Y Z) - J_M(\nabla_Y \nabla_X Z) - J_M(\nabla_{[X,Y]} Z) \\
&= \nabla_X \nabla_Y J_M(Z) - \nabla_Y \nabla_X J_M(Z) - \nabla_{[X,Y]} J_M(Z) \\
&= R(X, Y) J_M(Z)
\end{aligned}
$$

For $X, Y, Z \in \mathcal{X}(M)$. Using the property of curvature tensor R for $U, V \in \mathcal{X}(M)$ we have

$$
\begin{aligned}
g_M(R(X, J_M(Y))U, V) &= g_M(R(U, V)X, J_M(Y)) \\
&= -g_M(R(U, V)J_M(X), Y) \\
&= -g_M(R(J_M(X), Y)U, V).
\end{aligned}
$$

One can easily show that $R(X, J(Y)) = -R(J(X), Y)$. By (3.3) we see that $N(X, Y) = 0$ for every X, Y in $\mathcal{X}(M)$, i.e., $(TM + L(TM), J, g)$ is Hermitian. Conversely if $(TM + L(TM), J, g)$ is a Hermitian Lie algebroid then by equation (3.3) we have $N_{J_M} = 0$ and so (M, J_M, g_M) is a Hermitian manifold.

EXAMPLE 3.5. Consider the almost contact metric manifold $(M, \xi, \eta, \varphi.g)$ and the Lie algebroid $TM + L(TM)$. Let J act on vector fields as follows

$$
\begin{aligned}
J : \mathcal{X}(M) &\longrightarrow \Gamma(TM + L(TM)) \\
J(X) &= \varphi(X) + \eta(X)\eta \otimes \xi
\end{aligned}
$$

where $\eta \otimes \xi$ is a section of $L(TM)$ such that

$$\eta \otimes \xi : \mathcal{X}(M) \to \mathcal{X}(M)$$
$$X \mapsto \eta(X)\xi.$$

J also acts on sections of $L(TM)$ as

$$J : \Gamma(L(TM)) \longrightarrow \Gamma\big(TM + L(TM)\big)$$
$$J(T) = T \circ \varphi + \eta \otimes \varphi(T(\xi)) - \eta(T(\xi))\xi$$

The map J defines an almost complex structure on $TM + L(TM)$.

3.2. Symplectic Lie algebroids.

A *symplectic Lie algebroid* is a Lie algebroid $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ together with a closed and non-degenerate 2-form ω on $\mathcal{A}([12])$.

EXAMPLE 3.6. Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra, i.e., $\mathfrak{g}$ is a Lie algebra and ω is a non-degenerate 2- form on it and

$$\omega([S_1, S_3]_{\mathfrak{g}}, S_2) - \omega([S_1, S_2]_{\mathfrak{g}}, S_3) - \omega([S_2, S_3]_{\mathfrak{g}}, S_1) = 0 \qquad (S_1, S_2, S_3 \in \mathfrak{g}),$$

then $(M \times \mathfrak{g}, 0, [\,,\,]_{\mathfrak{g}}, \omega)$ is a symplectic Lie algebroid.

For a symplectic Lie algebroid $(\mathcal{A}, \omega)$ and every smooth function f on M there exists a unique section $a_f \in \Gamma(\mathcal{A})$ such that

$$\mathrm{d}^{\mathcal{A}} f(b) = \omega(a_f, b) \qquad (b \in \Gamma(\mathcal{A})).$$

This is called the *Hamiltonian section* of f. Using this, one can define a Poisson structure on M as follows([12]):

$$\{f, g\} := \omega(a_f, a_g).$$

There is no analogue to the Lie's third theorem, i.e., not every Lie algebroid can be integrated globally to a Lie groupoid, although there is a local version (see [3]). We will introduce a specific symplectic Lie algebroids that are globally integrable.

Proposition 3.7. *For a symplectic Lie algebroid* $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}}, \omega)$ *if* $\omega|_L$ *is non-degenerate, then* $\mathcal{A}$ *is integrable.*

Proof. Let $\mathcal{N}$ be a leaf of the characteristic foliation of $\mathcal{A}$ then $(\mathcal{A}_{\mathcal{N}}, \rho_{\mathcal{N}}, [\,,\,]_{\mathcal{N}}, \omega_{\mathcal{N}})$ is a symplectic Lie algebroid over $\mathcal{N}$. Suppose that E is the symplectic complement of $L_{\mathcal{N}}$, i.e., $E = L_{\mathcal{N}}^{\omega}$. Since $\omega|_L$ is nondegenerate, $L \cap L^{\omega} = 0$. So $\mathcal{A}_{\mathcal{N}} = E + L_{\mathcal{N}}$ is a decomposition of $\mathcal{A}_{\mathcal{N}}$. We claim that E is closed under the bracket. Considering Ω as the curvature 2-form with respect to the given

decomposition of $\mathcal{A}_{\mathcal{N}}$, we have

$$
\begin{aligned}
0 \;=\; & \mathrm{d}^{\mathcal{A}_{\mathcal{N}}}\omega_{\mathcal{N}}(a,b,s) = \rho_{\mathcal{N}}(a)\cdot\omega_{\mathcal{N}}(b,s) + \rho_{\mathcal{N}}(b)\cdot\omega_{\mathcal{N}}(a,s) - \rho_{\mathcal{N}}(s)\cdot\omega_{\mathcal{N}}(a,b) \\[2mm]
& +\omega_{\mathcal{N}}([a,s]_{\mathcal{N}},b) - \omega_{\mathcal{N}}([a,b]_{\mathcal{N}},s) - \omega_{\mathcal{N}}([b,s]_{\mathcal{N}},a) \\[2mm]
=\; & \omega_{\mathcal{N}}(\Omega(a,b),s),
\end{aligned}
$$

for all $a,b \in \Gamma E$ and $s \in \Gamma L_{\mathcal{N}}$.

This means that $\Omega = 0$. Thus ΓE is closed under the bracket. Hence by Corollary 5.2, of [3], $\mathcal{A}$ is integrable. $\qquad\square$

EXAMPLE 3.8. Let (L,ω_L) is a symplectic Lie algebra bundle over a symplectic manifold (M,ω_M). Consider the flat connection ∇ on L that preserves ω_L. Then (L,∇) together with the zero curvature 2-form construct a transitive Lie algebroid, $\mathcal{A} := TM + L$. Put

$$
\begin{aligned}
\omega :\ & \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \longrightarrow C^{\infty}(M) \\
& \omega(X + S, Y + T) = \omega_M(X,Y) + \omega_L(S,T)
\end{aligned}
$$

clearly ω is nondegenerate. Moreover, ω is closed, since for every $X,Y \in \mathcal{X}(M)$ and $S,T,T' \in \Gamma(L)$ we have

$$
\mathrm{d}^{(TM+L)}\omega(X,S,T) = d^{(TM+L)}\omega(X,Y,Z) = d^{(TM+L)}\omega(S,T,T') = 0
$$

Thus $(TM + L, \omega)$ is a symplectic Lie algebroid over M.

In order to complete the procedure of extending tangent bundle notions to Lie algebroids, after investigating Riemannian metric, almost complex structure and symplectic form, we will discuss the compatibility condition of these three notions.

Definition 3.9. Over a symplectic Lie algebroid $(\mathcal{A},\rho,[\,,\,]_{\mathcal{A}},\omega)$, the triple (ω,J,g) is called compatible if J is an almost complex structure and g is a Riemannian metric on $\mathcal{A}$ such that

$$
g(J(S),J(T)) = g(S,T) \ \ and \ \omega(S,T) = g(S,J(T)) \qquad (S,T \in \Gamma(\mathcal{A})).
$$

For such a triple, we also have([6])

$$
2g((\nabla_a J)b,c) = g(N(b,c),J(a)), \tag{3.4}
$$

where ∇ is the Levi-Civita connection on $\mathcal{A}$ and N is the Nijenhuis tensor of J.

Moreover $(\mathcal{A},\omega,J,g)$ is called a Kähler Lie algebroid if $N_J = 0$. One can easily see that on a Kähler Lie algebroid, $\nabla J = 0$.

3.3. **Decomposition of Lie algebroids by compatible triple.**

In proposition 3.7 we discussed the symplectic Lie algebroid $(\mathcal{A}, \omega)$ in which $\omega_{|_L}$ was nondegenerated. But, what happens when $\omega_{|_L}$ is degenerated? In such case, one can find a linear subspace of L consists of elements $s \in L$ satisfying $i_s \omega_{|_L} = 0$. This leads us to a useful new decomposition of $\mathcal{A}$.

Now, suppose that (ω, J, g) be a compatible triple on a Lie algebroid $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ over M. For every $p \in M$ put $L_p^1 = L_p \cap L_p^\omega$ and $L_p^2 = (L_p^1)^\perp$ under the g_L restriction of g on L. Note that $\omega|_{L_p^2}$ is nondegenerate. Putting $E_p^1 = J(L_p^1)$, for $J(T_p) \in E_p^1$ and $S_p \in L_p$ we have

$$g(S_p, J(T_p)) = \omega(S_p, T_p) = 0.$$

Thus E_p^1 is perpendicular to L_p. Finally taking $E_p^2 = (L_p + E_p^1)^\omega$ one can see that $E_p^2 \cap (L_p + E_p^1) = 0$, since for $a_p \in E_p^2 \cap (L_p + E_p^1)$ there is $S_p \in L_p$ and $T_p \in L_p^1$ such that $a_p = S_p + J(T_p)$. Clearly

$$0 = \omega(a_p, T_p) = \omega(S_p + J(T_p), T_p) = g(T_p, T_p).$$

Thus $T_p = 0$, and so $a_p = S_p$. Since $a_p \in L_p^\omega$, we have $S_p \in L_p^1$. On the other hand, $S_p \in (E_p^1)^\omega$, hence

$$0 = \omega(J(S_p), S_p) = g(S_p, S_p),$$

thus $a_p = 0$. Therefore, $(E_p^1 + E_p^2) \oplus L_p$ is a decomposition of $\mathcal{A}_p$, i.e., $\mathcal{A}_p = (E_p^1 + E_p^2) \oplus (L_p^1 + L_p^2)$. Note that these sets (including E^1, E^2, L^1, L^2) may not be sub-bundles or distributions, and they may have not constant rank. However, under certain circumstances, some combinations of these sets are sub-bundles For instance, if $\mathcal{A}$ is transitive then $E^2 + L^1$ is a sub Lie algebroid. The restriction of $[\,,\,]_{\mathcal{A}}$ to L_p^1 is zero. In fact ω is closed so, for every $S_p, T_p \in L_p^1$ and $Z_p \in L_p$ we have

$$0 = d^{\mathcal{A}} \omega(S_p, T_p, Z_p) = -\omega([S_p, T_p]_{\mathcal{A}}, Z_p).$$

Therefore, $[S_p, T_p]_{\mathcal{A}} \in L_p^1$. Again for $Z_p \in L_p^1$ we have

$$0 = d^{\mathcal{A}} \omega(S_p, T_p, J(Z_p)) = -\omega([S_p, T_p]_{\mathcal{A}}, J(Z_p)) = g([S_p, T_p]_{\mathcal{A}}, Z_p)$$

thus $[S_p, T_p]_{\mathcal{A}} \in L_p^2$. Therefore, $[S_p, T_p]_{\mathcal{A}} = 0$.

Moreover, $\rho(E^2 + L^1)$ is an integrable generalized distribution on M. If $\mathcal{N}$ is a leaf of $\mathcal{A}$ then $(E^2 + L^1)_{|_\mathcal{N}}$ is a Lie sub algebroid of $\mathcal{A}_{|_\mathcal{N}} \to \mathcal{N}$, because for every $a, b \in \Gamma(E^2 + L^1)$ and $S \in \Gamma(\mathcal{A})$ such that $S_p \in L$, we have

$$0 = d^{\mathcal{A}} \omega(a, b, S)(p) = -\omega([a, b]_{\mathcal{A}}, S)(p),$$

which means that

$$[a, b]_{\mathcal{A}} \in \Gamma(E^2 + L^1). \tag{3.5}$$

The above equation shows that $E^2 + L^1$ is closed under the bracket. Now we put $\Lambda := \rho(E^2 + L^1)$ and call it the *symplectic generalized distribution* of $(\mathcal{A}, \omega)$

on M. For $x \in M$ if $\mathcal{O}_x$ is the integral sub-manifold of Λ at x, we can define $\omega_{\mathcal{O}_x}$ as a nondegenerate 2-form on $\mathcal{O}_x$ by

$$\omega_{\mathcal{O}_x} : \quad \mathcal{X}(\mathcal{O}_x) \times \mathcal{X}(\mathcal{O}_x) \longrightarrow C^\infty(\mathcal{O}_x)$$
$$\omega_{\mathcal{O}_x}(X,Y)(p) := \omega(a,b)(p),$$

where $a, b \in \Gamma(E^2 + L^1)$ satisfy $\rho(a_p) = X_p, \rho(b_p) = Y_p$ for every p in $\mathcal{O}_x$. Clearly $\omega_{\mathcal{O}_x}$ is a 2-form on $\mathcal{O}_x$. Since $\omega_{|_{E^2}}$ is nondegenerate, so is $\omega_{\mathcal{O}_x}$. Moreover $\omega_{\mathcal{O}_x}$ is closed and so is a symplectic form on $\mathcal{O}_x$.

Theorem 3.10. *Let (ω, J, g) be a compatible triple on Lie algebroid $\mathcal{A}$ and Λ be the symplectic generalized distribution of $(\mathcal{A}, \omega)$ on M. Then for every integral sub-manifold $\mathcal{O}$ of Λ we have*

$$\{f, g\}_{|_\mathcal{O}} = \{f_{|_\mathcal{O}}, g_{|_\mathcal{O}}\}_\mathcal{O},$$

where $\{\,,\}$ is the Poisson structure induced by ω and $\{\,,\}_\mathcal{O}$ is the Poisson structure induced by $\omega_\mathcal{O}$ on $\mathcal{O}$.

Proof. For $f, g \in C^\infty(M)$ let a_f and a_g be the Hamiltonian sections of f, g with respect to ω and $X_{f_{|_\mathcal{O}}}, X_{g_{|_\mathcal{O}}}$ be the Hamiltonian vector fields of $f_{|_\mathcal{O}}$ and $g_{|_\mathcal{O}}$ with respect to $\omega_\mathcal{O}$. Then

$$\rho(a_f(x)) = X_{f_{|_\mathcal{O}}}(x), \quad \rho(a_g(x)) = X_{g_{|_\mathcal{O}}}(x) \qquad (x \in \mathcal{O}).$$

Thus for every x in $\mathcal{O}$ we have

$$\{f_{|_\mathcal{O}}, g_{|_\mathcal{O}}\}_\mathcal{O}(x) = \omega_\mathcal{O}\big(X_{f_{|_\mathcal{O}}}(x), X_{g_{|_\mathcal{O}}}(x)\big) = \omega\big(a_f(x), a_g(x)\big) = \{f, g\}_{\omega_{|_\mathcal{O}}}(x),$$

which completes the proof. $\qquad\qquad\square$

Corollary 3.11. *With the above notations, if f is constant on $\mathcal{O}_x$ for some $x \in M$, then*

$$\{f, g\}(y) = 0,$$

for every $g \in C^\infty(M)$ and $y \in \mathcal{O}_x$.

Suppose that (ω, J, g) be a compatible triple on Lie algebroid $(\mathcal{A}, \rho, [\,,]_\mathcal{A})$ such that J preserves L^2. Then for any $a_p \in E_p^2$ and $S_p \in L_p^2$ we have

$$0 = \omega(J(S_p), a_p) = g(S_p, a_p),$$

which means that E_p^2 is perpendicular to L_p^2. Our claim is that E^2 is invariant under J. To prove this, for a in E_p^2 consider the equation

$$J(a) = \big(J(a)\big)^{E_p^2} + \big(J(a)\big)^{L_p^2},$$

where E_p^2 is perpendicular to $L_p^1 + E_p^1$ and J is compatible with g. Thus $J(a)$ has no component through L_p^1 and E_p^1. Therefore,

$$0 = \omega\Big(\big(J(a)\big)^{L_p^2}, a\Big)$$
$$= \| J\big((J(a))^{L_p^2}\big) \| .$$

Hence $\left(J(a)\right)^{L_p^2}$ is zero, for every a in E_p^2.

Next, for an integral sub-manifold $\mathcal{O}$ of Λ, we define $g_{\mathcal{O}}$ and $J_{\mathcal{O}}$ as follows

$$J_{\mathcal{O}}(X) = \rho(J(a)) \qquad , \quad g_{\mathcal{O}}(X,Y) = g(a,b),$$

where $X, Y \in T\mathcal{O}$ and $a, b \in E^2$ such that $\rho(a) = X$ and $\rho(b) = Y$.

Clearly $J_{\mathcal{O}}$ is an almost complex structure and $g_{\mathcal{O}}$ a Riemannian metric on M such that the triple $(\omega, J_{\mathcal{O}}, g_{\mathcal{O}})$ is compatible on $\mathcal{O}$. We have proved the following result:

Theorem 3.12. *Let* (ω, J, g) *be a compatible triple on* $(\mathcal{A}, \rho, [\,,]_{\mathcal{A}})$ *together with decomposition* $(E_p^1 + E_p^2) + L_p$ *for* $\mathcal{A}_p$, *such that* J *preserves* L^2. *Then for every integral sub-manifold* $\mathcal{O}$ *of* $\Lambda = \rho(E^2 + L^1)$, $(\omega_{\mathcal{O}}, J_{\mathcal{O}}, g_{\mathcal{O}})$ *is a compatible triple on* $\mathcal{O}$.

Corollary 3.13. *Let* (ω, J, g) *be a compatible triple on a transitive Lie algebroid* $(\mathcal{A}, \rho, [\,,]_{\mathcal{A}})$ *such that* J *is admissible. Then* $\omega_{|_L}$ *is nondegenerate,* L^1 *and* E^1 *are null, and* $\Lambda = TM$. *Also* (M, ω_M) *is a symplectic manifold and the triple* (ω_M, g_M, J_M) *is compatible. Moreover the Poisson structure induced by* ω *is equal to the Poisson structure induce by* ω_M *on* M.

Our decomposition of Lie algebroid, enables us to somehow regard the cotangent bundle of the base manifold as a Lie sub-algebroid of our Lie algebroid.

Theorem 3.14. *Let* (ω, J, g) *be a compatible triple on a Lie algebroid* $(\mathcal{A}, \rho, [\,,]_{\mathcal{A}})$ *with the decomposition* $\mathcal{A}_p = (E_p^1 + E_p^2) + (L_p^1 + L_p^2)$. *Then for any leaf* $\mathcal{N}$ *of the characteristic foliation of* $\mathcal{A}$, *the Lie algebroid* $\left((E^2 + L^1)_{\mathcal{N}}, \rho_{|_{(E^2 + L^1)_{\mathcal{N}}}}, [\,,]_{|_{(E^2 + L^1)_{\mathcal{N}}}}\right)$ *and* $(T^*\mathcal{N}, \pi^*, [\,,]_{T^*\mathcal{N}})$ *are isomorphic over* $\mathcal{N}$.

Proof. Note that $(\mathcal{A}_{\mathcal{N}}, \rho_{\mathcal{N}}, [\,,]_{\mathcal{N}})$ is a transitive Lie algebroid over $\mathcal{N}$. Also $L_{\mathcal{N}} = Ker(\rho_{\mathcal{N}})$ is a Lie algebra sub-bundle of $(\mathcal{A}_{\mathcal{N}}, \rho_{\mathcal{N}}, [\,,]_{\mathcal{N}})$. Since $E_{\mathcal{N}}^2 + L_{\mathcal{N}}^1 = L_{\mathcal{N}}^{\omega_{\mathcal{N}}} = -J(L_{\mathcal{N}}^\perp)$, where $\omega_{\mathcal{N}}$ is the restriction of ω to $\mathcal{A}_{\mathcal{N}}$, $(E^2 + L^1)_{\mathcal{N}}$ is a vector sub-bundle of $\mathcal{A}_{\mathcal{N}}$. Moreover, by (3.5), $\Gamma(E^2 + L^1)_{\mathcal{N}}$ is closed under the bracket and thus inherits the Lie algebroid properties.

Looking at $\mathcal{A}_{\mathcal{N}}$ as $L_{\mathcal{N}} + L_{\mathcal{N}}^\perp$, one can consider the vector bundle map

$$J^\perp : \mathcal{A}_{\mathcal{N}} \longrightarrow \mathcal{A}_{\mathcal{N}}$$

$$a \mapsto J(a)^{L_{\mathcal{N}}^\perp}.$$

It is easy to see that $J^\perp\left((E^2 + L^1)_{\mathcal{N}}\right) = (E^1 + E^2)_{\mathcal{N}}$. Thus $(E^1 + E^2)_{\mathcal{N}}$ is smooth and so $(E^1 + E^2)_{\mathcal{N}} + L_{\mathcal{N}}$ is a decomposition of $\mathcal{A}_{\mathcal{N}}$. Let $\lambda : T\mathcal{N} \to \mathcal{A}_{\mathcal{N}}$ be the corresponding splitting with respect to the decomposition $\mathcal{A}_{\mathcal{N}} = L_{\mathcal{N}} + L_{\mathcal{N}}^\perp$. We define

$$\psi : (E^2 + L^1)_{\mathcal{N}} \longrightarrow T^*\mathcal{N}$$

$$a \longmapsto -\lambda^*(i_a \omega)$$

Since ω is nondegenerate, ψ is one to one. Thus ψ is an isomorphism of vector bundles.

To complete the proof we need to prove that ψ preserves the bracket and anchor maps. To do this, we take an arbitrary $f \in C^\infty(\mathcal{N})$ and denote its Hamiltonian section on $\mathcal{N}$ by a_f. Then

$$
\begin{aligned}
\big(\rho(a_f) + \pi^*(\mathrm{d}f)\big) \cdot g &= dg(a_f) + \mathrm{d}g(\pi^*(\mathrm{d}f)) \\
&= \omega(a_g, a_f) + \pi(\mathrm{d}f, \mathrm{d}g) \\
&= \{g, f\} - \{g, f\} \\
&= 0,
\end{aligned}
$$

i.e., $\rho(a_f) = -\pi^*(\mathrm{d}f)$. Thus $a_f + \lambda(\pi^*(\mathrm{d}f)) \in L_\mathcal{N}$. Now for $a \in (E^2 + L^1)_\mathcal{N}$ and $f \in C^\infty(\mathcal{N})$, we have

$$
\begin{aligned}
\big(\rho(a) - \pi^*(\psi(a))\big) \cdot f &= \omega(a_f, a) - \pi(\psi(a), \mathrm{d}f) \\
&= \omega(a_f, a) + \psi(a)(\pi^*(\mathrm{d}f)) \\
&= \omega(a_f, a) - \omega(a, \lambda(\pi^*(\mathrm{d}f))) \\
&= \omega(a_f + \lambda(\pi^*(\mathrm{d}f)), a) \\
&= 0.
\end{aligned}
$$

Thus $\rho(a) = \pi^*(\psi(a))$, i.e., ψ preserves the anchor maps.

On the other hand, for $a, b \in \Gamma\big((E^2 + L^1)_\mathcal{N}\big)$ and $X \in \mathcal{X}(\mathcal{N})$ we have

$$
[\psi(a), \psi(b)]_{\pi^*}(X) = \psi\Big([a, b]_{(E^2 + L^1)_\mathcal{N}}\Big)(X),
$$

which means that ψ preserves the bracket and so it is a Lie algebroid isomorphism. $\qquad\square$

Transitive Lie algebroids that admit compatible triple structures, encounter especial limitations provided that the Riemannian metric is invariant. Here, we will mention some cases.

Theorem 3.15. *If (ω, J, g) is a compatible triple on transitive Lie algebroid $\mathcal{A}$ such that g is invariant and J is admissible then*

    (1) *The bracket of every two sections of L is zero.*
    (2) *$(\mathcal{A}, \omega, J, g)$ is Kähler if and only if (M, ω_M, J_M, g_M) is Kähler.*

Proof. (1) ω is closed. Therefore, for $S_1, S_2, S_3 \in \Gamma(L)$ we have

$$
0 = \mathrm{d}^\mathcal{A}\omega(S_1, S_2, S_3) = -\omega([S_1, S_2]_\mathcal{A}, S_3) + \omega([S_1, S_3]_\mathcal{A}, S_2) - \omega([S_2, S_3]_\mathcal{A}, S_1).
$$

Since (ω, J, g) are compatible we can rewrite the above equation as

$$
-g([S_1, S_2]_\mathcal{A}, JS_3) + g([S_1, S_3]_\mathcal{A}, JS_2) - g([S_2, S_3]_\mathcal{A}, JS_1) = 0.
$$

Since g is invariant we have

$$
-g([S_1, S_2]_\mathcal{A}, JS_3) - g([S_1, JS_2]_\mathcal{A}, S_3) - g([JS_1, S_2]_\mathcal{A}, S_3) = 0. \tag{3.6}
$$

This shows that

$$-[S_1, S_2]_{\mathcal{A}} - J([S_1, JS_2]_{\mathcal{A}}) - J([JS_1, S_2]_{\mathcal{A}}) = 0.$$

Using the above equation, one can calculate $N(S_1, S_2)$ as follows

$$N(S_1, S_2) = [JS_1, JS_2]_{\mathcal{A}} - [S_1, S_2]_{\mathcal{A}} - J[JS_1, S_2]_{\mathcal{A}} - J[S_1, JS_2]_{\mathcal{A}} = [JS_1, JS_2]_{\mathcal{A}}. \tag{3.7}$$

Replacing S_1 by JS_1, in (3.6) we have

$$g(J[JS_1, S_2]_{\mathcal{A}}, S_3) - g([JS_1, JS_2]_{\mathcal{A}}, S_3) + g([S_1, S_2]_{\mathcal{A}}, S_3) = 0,$$

which means that

$$-[JS_1, JS_2]_{\mathcal{A}} + [S_1, S_2]_{\mathcal{A}} + J[JS_1, S_2]_{\mathcal{A}} = 0.$$

Therefore,

$$N(S_1, S_2) = -J[S_1, JS_2]_{\mathcal{A}}. \tag{3.8}$$

By applying (3.8) and (2.2) in (3.4) we have

$$
\begin{aligned}
0 &= 2g\big((\nabla_{S_1} J)S_2, S_3\big) - g\big(N(S_2, S_3), JS_1\big) \\
&= g([S_1, JS_2]_{\mathcal{A}}, S_3) - g(J[S_1, S_2]_{\mathcal{A}}, S_3) + g(J[S_2, JS_3]_{\mathcal{A}}, JS_1) \\
&= g([S_1, JS_2]_{\mathcal{A}}, S_3) - g(J[S_1, S_2]_{\mathcal{A}}, S_3) - g(J[S_1, S_2]_{\mathcal{A}}, S_3) \\
&= g([S_1, JS_2]_{\mathcal{A}} - 2J[S_1, S_2]_{\mathcal{A}}, S_3),
\end{aligned}
\tag{3.9}
$$

i.e., for every $S_1, S_2 \in \Gamma(L)$

$$[S_1, JS_2]_{\mathcal{A}} = 2J[S_1, S_2]_{\mathcal{A}}.$$

Applying this to (3.7) and (3.8) we get $N(S_1, S_2) = [JS_1, JS_2]_{\mathcal{A}} = -4[S_1, S_2]_{\mathcal{A}}$. On the other hand, $N(S_1, S_2) = -J[S_1, JS_2]_{\mathcal{A}} = 2[S_1, S_2]_{\mathcal{A}}$. Therefore, $[S_1, S_2]_{\mathcal{A}} = 0$ and $N(S_1, S_2) = 0$.

(2) Now look at $\mathcal{A}$ as $L^{\perp} + L$. By the proof of Theorem 3.14, $L^{\perp}$ is closed under the restriction bracket. Suppose that (M, ω_M, J_M, g_M) is Kähler, therefore, $N_{J_M} = 0$, i.e.,

$$N_{J_M}(X, Y) = 0 \qquad (X, Y \in \mathcal{X}(M)).$$

By (3.2)

$$N_J(a, b) = 0 \qquad (a, b \in \Gamma(L^{\perp})).$$

We have just proved, in the previous part, that $N(S, T) = 0$ for every $S, T \in \Gamma(L)$. To complete the proof, it suffices to show that $N(S, a) = 0$ for every $S \in \Gamma(L)$ and $a \in \Gamma(L^{\perp})$. Using (3.4) we have

$$g(N(S, a), T) = -g((\nabla_{JT} J)(S), a) = 0 \qquad (T \in \Gamma(L)).$$

On the other hand, $g(N(S, a), b) = 0$ for every $b \in \Gamma(L^{\perp})$ and so $N(S, a) = 0$. Thus $(\mathcal{A}, J, \omega, g)$ is Kähler. The converse is trivial. $\qquad \square$

Theorem 3.16. *On a transitive Kähler Lie algebroid $(\mathcal{A}, \omega, J, g)$ if g is invariant then the restriction of the bracket on $\Gamma(L)$ is zero.*

Proof. Fix a point $p \in M$. Using the decomposition mentioned at the beginning of the section, one can write $\mathcal{A}_p = (E_p^1 + E_p^2) + (L_p^1 + L_p^2)$. Since $(\mathcal{A}, \omega, J, g)$ is Kähler, $\nabla \omega = \nabla J = 0$, where ∇ is the Levi-Civita connection of g. Now for $S, T \in L_p^2$ and $Z \in L_p^1$ we have

$$
\begin{aligned}
0 =& (\nabla_S \omega)(T, Z) \\
=& \omega(\nabla_S T, Z) + \omega(T, \nabla_S Z) \\
=& \omega(\frac{1}{2}[S, T]_{\mathcal{A}}, Z) + \omega(T, \frac{1}{2}[S, Z]_{\mathcal{A}}) \\
=& \omega(T, \frac{1}{2}[S, Z]_{\mathcal{A}}).
\end{aligned}
$$

This means that

$$[L_p^1, L_p^2]_{\mathcal{A}} \subseteq L_p^1. \tag{3.10}$$

Using the above equation and the fact that g is invariant, for $S, T \in L_p^2$ and $Z \in L_p^1$ we have

$$g([S, T]_{\mathcal{A}}, Z) = g(S, [T, Z]_{\mathcal{A}}) = 0, \tag{3.11}$$

and so

$$[L_p^2, L_p^2]_{\mathcal{A}} \subseteq L_p^2. \tag{3.12}$$

Moreover for $S, T, Z \in L_p^2$

$$
\begin{aligned}
0 =& \mathrm{d}^{\mathcal{A}} \omega(S, T, Z) \\
=& -\omega([S, T]_{\mathcal{A}}, Z) + \omega([S, Z]_{\mathcal{A}}, T) - \omega([T, Z]_{\mathcal{A}}, S) \\
=& -\omega([S, T]_{\mathcal{A}}, Z) - 2(\nabla_Z \omega)(S, T) \\
=& -\omega([S, T]_{\mathcal{A}}, Z),
\end{aligned}
$$

i.e., $[L_p^2, L_p^2]_{\mathcal{A}} \subseteq L_p^1$. Hence by (3.12) we have $[L_p^2, L_p^2]_{\mathcal{A}} = 0$.
Furthermore, we know that $[L_p^1, L_p^1]_{\mathcal{A}} = 0$. Thus for $S, T \in L_p^1$ and $Z \in L_p^2$ we have

$$0 = g([S, T]_{\mathcal{A}}, Z) = g(S, [T, Z]_{\mathcal{A}}),$$

i.e., $[L_p^1, l_p^2]_{\mathcal{A}} \subseteq L_p^2$. Using (3.10) one can easily see that $[L_p^1, L_p^2]_{\mathcal{A}} = 0$. This proves that the restriction of the bracket on L is zero.

$\square$

4. Contact Lie algebroid

Let $(\mathcal{A}, \rho, [\,,\,]_{\mathcal{A}})$ be a Lie algebroid of rank $2n+1$ over a smooth m-dimensional manifold M. A 1-form $\eta \in \Gamma(\mathcal{A}^*)$ is called *contact* if $\eta \wedge (\mathrm{d}^{\mathcal{A}} \eta)^n \neq 0$. In this case, $(\mathcal{A}, \eta)$ is called a *contact Lie algebroid*([7]). For a contact Lie algebroid $(\mathcal{A}, \eta)$ there exists a unique section $\xi \in \Gamma \mathcal{A}$ called the *Reeb section* such that

$$\eta(\xi) = 1, \qquad i_{\xi} \mathrm{d}^{\mathcal{A}} \eta = 0.$$

A triple (φ, ξ, η) is called an *almost contact structure* on $\mathcal{A}$ if φ is a $(1,1)$ tensor section of $\Gamma(\mathcal{A} \otimes \mathcal{A}^*)$, $\xi \in \Gamma \mathcal{A}$, $\eta \in \Gamma(\mathcal{A}^*)$ and

$$\eta(\xi) = 1, \qquad \varphi^2 = -id_{\mathcal{A}} + \eta \otimes \xi.$$

Moreover $(\mathcal{A}, \varphi, \xi, \eta)$ is called an *almost contact Lie algebroid.*

A Riemannian metric g on $\mathcal{A}$ is said to be compatible with an almost contact structure (φ, ξ, η) if

$$g(\varphi(S), \varphi(T)) = g(S, T) - \eta(S)\eta(T) \qquad (S, T \in \Gamma \mathcal{A}).$$

In this case, $(\mathcal{A}, \varphi, \xi, \eta, g)$ is called an *almost contact Riemannian Lie algebroid.* For an almost contact Riemannian Lie algebroid $(\mathcal{A}, \varphi, \xi, \eta, g)$ if

$$\mathrm{d}^{\mathcal{A}}\eta(S, T) = g(S, \varphi(T))$$

then η is a contact form, ξ is the Reeb section and $(\mathcal{A}, \varphi, \xi, \eta, g)$ is called a *contact Riemannian Lie algebroid*([7]).

Let $(\mathcal{A}, \varphi, \xi, \eta)$ be an almost contact Lie algebroid. Then $D := Ker(\eta)$ is a vector sub-bundle of $\mathcal{A}$ of rank $2n$. If η is contact then $\mathrm{d}\eta_{|_D}$ is nondegenerate. Let $(\mathcal{A}, \eta)$ be a contact Lie algebroid, consider the vector bundle morphism

$$\psi : \mathcal{A} \longrightarrow \mathcal{A}^*$$

$$S \longmapsto i_S \mathrm{d}^{\mathcal{A}}\eta.$$

It is easy to see that

$$\mathcal{A}^* = Im(\psi) \oplus \langle \eta \rangle.$$

Thus for $f \in C^{\infty}(M)$, there exists $S_f \in \Gamma(\mathcal{A})$ and $h \in C^{\infty}(M)$ such that

$$\mathrm{d}^{\mathcal{A}} f = i_{S_f} \mathrm{d}^{\mathcal{A}} \eta + h\eta. \tag{4.1}$$

In fact

$$h = \rho(\xi) \cdot f$$

If $S_f, \bar{S}_f$ satisfy (4.1), there exist $k \in C^{\infty}(M)$ such that

$$S_f - \bar{S}_f = k\xi. \tag{4.2}$$

Putting $a_f := S_f - \eta(S_f)\xi$ one can easily see that a_f satisfies the (4.1). Also a_f is independent of the choise of S_f. In fact, if $S_f, \bar{S}_f$ satisfy (4.1), then by (4.2) we have

$$S_f - \eta(S_f)\xi = \bar{S}_f - \eta(\bar{S}_f)\xi.$$

We call a_f, the *Hamiltonian section* of f.

Now we can define

$$\{ , \} : C^{\infty}(M) \times C^{\infty}(M) \longrightarrow C^{\infty}(M)$$

$$(f, g) \longmapsto \mathrm{d}^{\mathcal{A}}\eta(a_f, a_g).$$

In fact for $f, g \in C^{\infty}(M)$

$$\{f, g\} = \rho(a_g) \cdot f = -\rho(a_f) \cdot g \tag{4.3}$$

Theorem 4.1. *On a contact Lie algebroid* $(\mathcal{A}, \eta)$ *if* $\rho(\xi) = 0$ *then the above bracket is a Poisson structure on* M.

Proof. $\{\,,\}$ is $\mathbb{R}$-bilinear and skew-symmetric. Now if a_f, a_g, a_h are Hamiltonian sections of $f, g, h \in C^\infty(M)$, respectively, then

$$\begin{aligned}
\mathrm{d}^\mathcal{A}(fg) &= f\mathrm{d}^\mathcal{A}g + g\mathrm{d}^\mathcal{A}f \\
&= fi_{a_g}\mathrm{d}^\mathcal{A}\eta + gi_{a_f}\mathrm{d}^\mathcal{A}\eta \\
&= i_{(fa_g + ga_f)}\mathrm{d}^\mathcal{A}\eta.
\end{aligned}$$

Moreover, $\eta(fa_g + ga_f) = 0$. Thus $fa_g + ga_f$ is the Hamiltonian section of fg. Hence

$$\begin{aligned}
\{fg, h\} &= \mathrm{d}^\mathcal{A}\eta(fa_g + ga_f, a_h) \\
&= f\mathrm{d}^\mathcal{A}\eta(a_g, a_h) + g\mathrm{d}^\mathcal{A}\eta(a_f, a_h) \\
&= f\{g, h\} + g\{f, h\},
\end{aligned}$$

i.e., $\{\,,\}$ satisfies the product rule.
Furthermore,

$$\begin{aligned}
i_{[a_f, a_g]_\mathcal{A}}\mathrm{d}^\mathcal{A}\eta &= [\mathcal{L}_{a_f}, i_{a_g}]_\mathcal{A}\mathrm{d}^\mathcal{A}\eta \\
&= \mathrm{d}^\mathcal{A}i_{a_f}i_{a_g}\mathrm{d}^\mathcal{A}\eta + i_{a_f}\mathrm{d}^\mathcal{A}i_{a_g}\mathrm{d}^\mathcal{A}\eta - i_{a_g}\mathrm{d}^\mathcal{A}i_{a_f}\mathrm{d}^\mathcal{A}\eta - i_{a_g}i_{a_f}\mathrm{d}^\mathcal{A}\mathrm{d}^\mathcal{A}\eta \\
&= \mathrm{d}^\mathcal{A}\eta(a_f, a_g) \\
&= \{f, g\}.
\end{aligned}$$

Thus $[a_f, a_g]_\mathcal{A} - \eta([a_f, a_g]_\mathcal{A})\xi$ is the Hamiltonian section of $\{f, g\}$.
Now we can prove the Jacobi identity. We use (4.3) as follows

$$\{f, \{g, h\}\} = -\{f, \rho(a_g) \cdot h\} = \rho(a_f) \cdot \rho(a_g) \cdot h$$

$$\{g, \{h, f\}\} = \{g, \rho(a_f) \cdot h\} \quad = -\rho(a_g) \cdot \rho(a_f) \cdot h$$

$$\{h, \{f, g\}\} = \rho(a_{\{f,g\}}) \cdot h \quad = \rho([a_f, a_g]) \cdot h$$

i.e.,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

$\square$

Lemma 4.2. *If* $(\mathcal{A}, \eta)$ *is a contact Lie algebroid such that* $L_p \subseteq \mathrm{D}_p$ *for some* $p \in M$, *then* $L_p = 0$.

Proof. For $S_p \in L_p \subseteq \mathrm{D}_p$, one can extend S_p to a section S of D. Now for $T \in \Gamma(\mathrm{D})$ we have

$$\mathrm{d}^\mathcal{A}\eta_p(S_p, T_p) = \rho(S_p).\eta(T) - \rho(T_p).\eta(S) - \eta\big(([S, T]_\mathcal{A})_p\big) = 0,$$

where the last identity is a result of ideal property of ΓL. Since η is contact, $d^{\mathcal{A}}\eta_{|_D}$ nondegenerate, Thus $S_p = 0$, i.e., $L_p = 0$. $\qquad\square$

Lemma 4.3. *For an almost contact structure (φ, ξ, η) on a Lie algebroid $\mathcal{A}$ if η is contact and $\varphi(L_p) \subseteq L_p$, for some $p \in M$, then $L_p = 0$ or $\rho(\xi_p) = 0$.*

Proof. Choose $p \in M$ such that $\varphi(L_p) \subseteq L_p$. If $L_p \neq 0$ then by Lemma 4.2 there exists $S_p \in L_p$ such that $\eta_p(S_p) \neq 0$. Thus

$$\eta_p(S_p)\xi_p = \varphi^2(S_p) + S_p.$$

Since φ preserves L_p, $\varphi^2(S_p) \in L_p$, and so $\xi_p \in L_p$. $\qquad\square$

Let $(\mathcal{A}, \varphi, \xi, \eta, g)$ be a transitive contact Riemannian Lie algebroid such that φ preserves L. If $L = 0$ then $\mathcal{A} = TM$ and our Riemannian contact Lie algebroid reduces to an ordinary contact metric manifold. So let $L \neq 0$. Since $\xi \in \Gamma L$, by (4.3), we may write

$$D = (D \cap L) \oplus L^{\perp}.$$

Moreover, for $S \in \Gamma L^{\perp}$ and $T \in \Gamma L$,

$$d^{\mathcal{A}}\eta(S, T) = g\big(S, \varphi(T)\big) = 0. \qquad (4.4)$$

Let $\lambda : TM \to \mathcal{A}$ be the splitting map corresponding to $\mathcal{A} = L^{\perp} \oplus L$. Putting $\omega := \lambda^*(d^{\mathcal{A}}\eta)$, one can see that ω is closed and non-degenerate, i.e., (M, ω) is a symplectic manifold. Clearly φ preserves $L^{\perp}$. Therefore, φ and g induce an almost complex structure J_M on $\mathcal{A}$ (with the decomposition $L + L^{\perp}$) and a Riemannian metric g_M on M, respectively. Thus, (ω, J_M, g_M) is a compatible triple on M. The interesting point is that, the Poisson structure induced by Theorem 4.1 coincides with the Poisson structure induced by ω.

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