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Characteristics of Common Neighborhood Graph under Graph Operations and on Cayley Graphs

Shaban Sedghi[a], Dae-Won Lee[b,*], Nabi Shobe[c]

[a]Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran.
[b]Department of Mathematics, Yonsei University, Seoul 120-749, Republic of Korea.
[c]Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran.

E-mail: sedghi_gh@yahoo.com, sedghi.gh@qaemshahriau.ac.ir
E-mail: daewonlee@yonsei.ac.kr
E-mail: nabi_shobe@yahoo.com

Abstract. Let $G(V,E)$ be a graph. The common neighborhood graph (congraph) of G is a graph with vertex set V, in which two vertices are adjacent if and only if they have a common neighbor in G. In this paper, we obtain characteristics of congraphs under graph operations; Graph union, Graph cartesian product, Graph tensor product, and Graph join, and relations between Cayley graphs and its congraphs.

Keywords: Common Neighborhood Graph, Cayley graph, Graph operation.

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1. Introduction

The graphs considered in this paper are assumed to be connected and simple. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$

*Corresponding Author

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the complement of the graph G. As usual, C_n and K_n are cycle and complete graph with n vertices, respectively.

The neighborhood of a vertex v is the set of all vertices u such that they are the endpoints of the same edge and denoted by $N(v)$. Denote by $\overline{N(v)}$ the complement of set $N(v)$. The degree of a vertex v, denoted by $deg(v)$, is the number of neighbors of v, that is $deg(v) = |N(v)|$.

Let G be a simple graph with vertex set $V(G)$. The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with $V(con(G)) = V(G)$, in which two vertices are adjacent if they have a common neighbor in G, that is,

$$xy \in E(con(G)) \Longleftrightarrow N(x) \cap N(y) \neq \emptyset \ where\ x, y \in V(G).$$

The basic concept of congraphs came from the theory of graph energy [1, 10], and some basic properties of congraphs have been obtained [1, 3, 9].

There are several Graph operations which generate new graphs from old ones.

Definition 1.1. Let G_1 and G_2 be two graphs.

(1) Graph intersection operation of G_1 and G_2, denoted by $G_1 \cap G_2$, is a graph with $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$ [2].
(2) Graph union operation of G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ [2].
(3) Graph cartesian product of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u, v)(u', v') \in E(G_1 \times G_2)$ if $u = u'$, then $vv' \in E(G_2)$ or if $v = v'$, then $uu' \in E(G_1)$ [8].
(4) Graph tensor product of G_1 and G_2, denoted by $G_1 \otimes G_2$, is a graph with the vertex-set $V(G_1) \times V(G_2)$. For $u, v \in V(G_1)$ and $x, y \in V(G_2)$, (u, x) is adjacent to (v, y) in $G_1 \otimes G_2$ if $uv \in E(G_1)$ and $xy \in E(G_2)$ [15].
(5) If $V(G_1) \cap V(G_2) = \emptyset$, graph join operation of G_1 and G_2, denoted by $G_1 + G_2$, is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$ [8].

Let G be a non-trivial group and let S be a subset of $G - \{e\}$ with $S = S^{-1} := \{s^{-1} | s \in S\}$. The Cayley graph of G denoted by $Cay(G : S)$ is a graph with vertex set G and two vertices a and b are adjacent if $ab^{-1} \in S$. The study of Cayley graphs of the symmetric group generated by transpositions is interesting (See [7]).

Cayley graphs of finitely generated groups are a fundamental concept in group theory. They were introduced by Cayley [4] for finite groups and Dehn [5] for infinite groups. Many deep results in group theory use Cayley graphs in an essential way, see e.g. [6, 11, 12, 14]. Moreover, Cayley graphs turned out to be a link to several other fields in mathematics and theoretical computer science, e.g., automata theory, topology, and graph theory.

This paper is organized as follows: in section 2, we obtain some properties of congraphs under graph operations; Graph join, Graph union, Graph cartesian product, and Graph tensor product and in section 3, we give some relations between Cayley graphs and its congraphs.

2. Characteristics of Congraph on Graph Operations

Lemma 2.1. *Let $G(V, E)$ be a simple graph with n vertices and m edges. In the common neighborhood graph (congraph) of G, for every $v \in V$ we have:*

(1) $deg_{con(G)}(v) = |\cup_{u\in N(v)} N(u) - \{v\}| = |N_{con(G)}(v)|$. *Also, if* $N(u) \cap N(w) = \{v\}$ *then for every* $u, w \in N(v)$, *we have* $deg_{con(G)}(v) + deg_G(v) = \sum_{u_i\in N(v)} deg_G(u_i)$.

(2) *For every* $u, v \in V(G)$, *if* $deg(u) + deg(v) > n$, *then* $con(G) = K_n$.

Proof. (1)

$$\begin{aligned} u \in N_{con(G)}(v) &\iff uv \in E(con(G)) \\ &\iff N(u) \cap N(v) \neq \emptyset \textit{ hence there exists } a \in N(v) \textit{ and } a \in N(u) \\ &\iff a \in N(v) \textit{ and } u \in N(a). \end{aligned}$$

That is $N_{con(G)}(v) = \cup_{u\in N(v)} N(u) - \{v\}$.

Hence,

$$\begin{aligned} deg_{con(G)}(v) &= |\cup_{u_i\in N(v)} N(u_i) - \{v\}| \\ &= |\cup_{u_i\in N(v)} (N(u_i) - \{v\})| \\ &= \sum_{u_i\in N(v)} |N(u_i) - \{v\}| \\ &= \sum_{u_i\in N(v)} (|N(u_i)| - 1) \\ &= \sum_{u_i\in N(v)} deg_G(u_i) - |N(v)| \\ &= \sum_{u_i\in N(v)} deg_G(u_i) - deg_G(v). \end{aligned}$$

Therefore, $deg_{con(G)}(v) + deg_G(v) = \sum_{u_i\in N(v)} deg_G(u_i)$.

(2) It is enough to show that for every $u, v \in V$ we have $N(u) \cap N(v) \neq \emptyset$. Otherwise, we have

$$n \geq |N(u) \cup N(v)| = |N(u)| + |N(v)| = deg(u) + deg(v) > n,$$

which is a contradiction. Hence, it follows that $uv \in E(con(G))$, that is, $con(G) = K_n$.

□

Corollary 2.2. *Let $G(V, E)$ be a graph with n vertices and m edges and have not any cycle of order 4. Also, let $con(G)$ be a graph with n vertices and m' edges the congraph of G. Then,*

$$m' = \frac{1}{2}M_1(G) - m,$$

where $M_1(G)$ stands for the first Zagreb index, defined as $M_1(G) = \sum_{v_i \in V} deg^2(v_i)$

Proof. For every $u, w \in N(v)$ we have $v \in N(u) \cap N(w)$. Now, we show that $N(u) \cap N(w) = \{v\}$. For, if there exist $a \in N(u) \cap N(w)$ such that $a \neq v$, it follows that $au, vu, aw, vw \in E(G)$, that is, we have a cycle of order 4, which is a contradiction. Hence, by Lemma 2.1 we have $deg_{con(G)}(v_i) + deg_G(v_i) = \sum_{u_j \in N(v_i)} deg_G(u_j)$. Thus,

$$\sum_{v_i \in V} deg_{con(G)}(v_i) + \sum_{v_i \in V} deg_G(v_i) = \sum_{v_i \in V} \sum_{u_j \in N(v_i)} deg_G(u_j),$$

it follows that

$$2m' + 2m = \sum_{v_i \in V} deg^2(v_i).$$

Therefore,

$$m' = \frac{1}{2} \sum_{v_i \in V} deg^2(v_i) - m = \frac{1}{2}M_1(G) - m.$$

□

Theorem 2.3. *Let G_1 and G_2 be two graphs of order n and m respectively. If G_1 or G_2 is connected then $con(G_1 + G_2) = K_{n+m}$.*

Proof. Let $x, y \in V_1 \cup V_2$ then we show that $xy \in E(con(G_1 + G_2))$, that is, $N_{G_1+G_2}(x) \cap N_{G_1+G_2}(y) \neq \emptyset$. For, if $x, y \in V_1$ then it is easy to see that $(N_{G_1}(x) \cup V_2) \cap (N_{G_1}(y) \cup V_2) \neq \emptyset$. Similarly, if $x, y \in V_2$ then $(N_{G_2}(x) \cup V_1) \cap (N_{G_2}(y) \cup V_1) \neq \emptyset$. Now, if $x \in V_1$ and $y \in V_2$ then $(N_{G_1}(x) \cup V_2) \cap (N_{G_2}(y) \cup V_1) \neq \emptyset$, since at least one of G_1 or G_2 is connected. Therefore, for every $x, y \in V_1 \cup V_2$ we have $xy \in E(con(G_1 + G_2))$, hence $con(G_1 + G_2) = K_{n+m}$. □

Theorem 2.4. *Let G_1 and G_2 be two graphs. Then $con(G_1 \otimes G_2) = (con(G_1) \times con(G_2)) \cup (con(G_1) \otimes con(G_2))$.*

Proof. Let $(x, y), (u, v) \in V(G_1 \otimes G_2)$. If $(x, y)(u, v) \in E(con(G_1 \otimes G_2))$ then $N_{G_1 \otimes G_2}(x, y) \cap N_{G_1 \otimes G_2}(u, v) \neq \emptyset$.

$$\begin{aligned} \emptyset \neq N_{G_1 \otimes G_2}(x, y) \cap N_{G_1 \otimes G_2}(u, v) &= (N_{G_1}(x) \times N_{G_2}(y)) \cap (N_{G_1}(u) \times N_{G_2}(v)) \\ &= (N_{G_1}(x) \cap N_{G_1}(u)) \times (N_{G_2}(y) \cap N_{G_2}(v)), \end{aligned}$$

i.e.

$$\begin{aligned} & N_{G_1}(x) \cap N_{G_1}(u) \neq \emptyset \wedge y = v \ \textit{or} \ N_{G_2}(y) \cap N_{G_2}(v) \neq \emptyset \wedge x = u \\ \textit{or} \quad & N_{G_1}(x) \cap N_{G_1}(u) \neq \emptyset \wedge N_{G_2}(y) \cap N_{G_2}(v) \neq \emptyset. \end{aligned}$$

Hence,

$$xu \in E(con(G_1)), y = v \ \textit{or} \ yv \in E(con(G_2)), x = u \ \textit{or} \ xu \in E(con(G_1)), yv \in E(con(G_2))$$

which means

$$(x, y)(u, v) \in E(con(G_1) \times con(G_2)) \ \textit{or} \ (x, y)(u, v) \in E(con(G_1) \otimes con(G_2)).$$

Therefore,

$$con(G_1 \otimes G_2) = (con(G_1) \times con(G_2)) \cup (con(G_1) \otimes con(G_2)).$$

□

Theorem 2.5. *Let G_1 and G_2 be two graphs. Then $con(G_1 \times G_2) = (con(G_1) \times con(G_2)) \cup (G_1 \otimes G_2)$.*

Proof. Let $(x, y), (u, v) \in V(G_1 \times G_2)$. Then

$$\begin{aligned} & (x, y)(u, v) \in E(con(G_1 \times G_2)) \\ \Longleftrightarrow \quad & N_{G_1 \times G_2}(x, y) \cap N_{G_1 \times G_2}(u, v) \neq \emptyset \\ \Longleftrightarrow \quad & \{(x \times N_{G_2}(y)) \cup (N_{G_1}(x) \times y)\} \cap \{(u \times N_{G_2}(v)) \cup (N_{G_1}(u) \times v)\} \neq \emptyset. \end{aligned}$$

Therefore,

$$\begin{aligned} & (x \times N_{G_2}(y)) \cap (u \times N_{G_2}(v)) \cup (x \times N_{G_2}(y)) \cap (N_{G_1}(u) \times v) \\ \cup \quad & (N_{G_1}(x) \times y) \cap (u \times N_{G_2}(v)) \cup (N_{G_1}(x) \times y) \cap (N_{G_1}(u) \times v) \neq \emptyset. \end{aligned}$$

Hence,

$$\begin{aligned}
& (x \times N_{G_2}(y)) \cap (u \times N_{G_2}(v)) \neq \emptyset \\
\Longleftrightarrow \quad & x = u \wedge N_{G_2}(y) \cap N_{G_2}(v) \neq \emptyset \Longleftrightarrow x = u \wedge yv \in E(con(G_2)) \\
& or \\
\Longleftrightarrow \quad & (x \times N_{G_2}(y)) \cap (N_{G_1}(u) \times v) \neq \emptyset \\
\Longleftrightarrow \quad & x \in N_{G_1}(u) \wedge v \in N_{G_2}(v) \neq \emptyset \Longleftrightarrow xu \in E(G_1) \wedge yv \in E(G_2) \\
& or \\
\Longleftrightarrow \quad & (N_{G_1}(x) \times y) \cap (u \times N_{G_2}(v)) \neq \emptyset \\
\Longleftrightarrow \quad & u \in N_{G_1}(x) \times y \in N_{G_2}(v) \Longleftrightarrow xu \in E(G_1), yv \in E(G_2) \\
& or \\
\Longleftrightarrow \quad & (N_{G_1}(x) \times y) \cap (N_{G_1}(u) \times v) \neq \emptyset \\
\Longleftrightarrow \quad & y = v \wedge N_{G_1}(u) \cap N_{G_1}(x) \neq \emptyset \Longleftrightarrow y = v, xu \in E(con(G_1))
\end{aligned}$$

Therefore, for every condition we get:

$$\begin{aligned}
& (x,y)(u,v) \in E[con(G_1) \times con(G_2)] \ or \ (x,v)(y,v) \in E(G_1 \otimes G_2) \\
\Longleftrightarrow \quad & (x,y)(u,v) \in E[con(G_1) \times con(G_2) \cup E(G_1 \otimes G_2)].
\end{aligned}$$

□

3. Relations Between Cayley Graph and its Congraph

Let $Cay(G : S)$ be a Cayley graph. Then,

$$\begin{aligned}
N(e) &= \{x \in G \mid \{x, e\} = xe \in E\} \\
&= \{x \in G \mid xe^{-1} = x \in S\} = S.
\end{aligned}$$

Thus, $deg(e) = |N(e)| = |S|$. It is easy to see that $N(x) = N(e) \cdot x = S \cdot x$ for each $x \in G$.

Theorem 3.1. [13]*[Theorem 2.4.] If $Cay(G : S_1)$ and $Cay(G : S_2)$ are Cayley graphs. Then*
(1) $Cay(G : S_1) \cup Cay(G : S_2) = Cay(G : S_1 \cup S_2)$,
(2) $Cay(G : S_1) \cap Cay(G : S_2) = Cay(G : S_1 \cap S_2)$.

Lemma 3.2. *Let $Cay(G : S)$ be a Cayley graph. Then*

$$con(Cay(G : S)) = Cay(G : S^2 - \{e\}).$$

Proof. Let $\Gamma(V,E) = con(Cay(G:S))$, $\Gamma'(V',E') = Cay(G:S^2-\{e\})$. It is obvious that $V = V'$. Let $x, y \in V$, then

$$\begin{aligned} xy \in E &\iff N_{Cay(G:S)}(x) \cap N_{Cay(G:S)}(y) \neq \emptyset \iff (Sx) \cap (Sy) \neq \emptyset \\ &\iff \textit{ there exists } a \in (Sx) \cap (Sy) \iff a = s_1 x \textit{ and } a = s_2 y \\ &\iff s_1 x = s_2 y \iff e \neq xy^{-1} = s_1^{-1} s_2 \in S^{-1} \cdot S = S \cdot S = S^2 \\ &\iff xy \in E', \end{aligned}$$

for some $s_1, s_2 \in S$. □

By Lemma 3.2 we have the following corollary.

Corollary 3.3. *Let $Cay(G:S)$ be a Cayley graph. Then*

$$con(con(Cay(G:S))) = Cay(G:S^4-\{e\}).$$

In general, we know that $con(G_1 \cup G_2) \neq con(G_1) \cup con(G_2)$ and also $con(G_1 \cap G_2) \neq con(G_1) \cap con(G_2)$. But in the following theorem, we show that the equalities hold for a special condition.

Theorem 3.4. *Let $G_1 = Cay(G:S_1)$ and $G_2 = Cay(G:S_2)$ be two Cayley graphs such that $(S_1 \cup S_2)^2 - \{e\} = (S_1^2 - \{e\}) \cup (S_2^2 - \{e\})$. Then*

$$con(G_1 \cup G_2) = con(G_1) \cup con(G_2).$$

Also, if $(S_1 \cap S_2)^2 - \{e\} = (S_1^2 - \{e\}) \cap (S_2^2 - \{e\})$, then

$$con(G_1 \cap G_2) = con(G_1) \cap con(G_2).$$

Proof. By Lemma 3.2 and Theorem 3.1 we have:

$$\begin{aligned} con(G_1 \cup G_2) &= con(Cay(G:S_1) \cup Cay(G:S_2)) \\ &= con(Cay(G:S_1 \cup S_2)) \\ &= Cay(G:(S_1 \cup S_2)^2 - \{e\}) \\ &= Cay(G:(S_1^2 - \{e\}) \cup (S_2^2 - \{e\})) \\ &= Cay(G:S_1^2 - \{e\}) \cup Cay(G:S_2^2 - \{e\}) \\ &= con(Cay(G:S_1)) \cup con(Cay(G:S_2)) \\ &= con(G_1) \cup con(G_2). \end{aligned}$$

Also, by Lemma 3.2 and Theorem 3.1 we have:

$$\begin{aligned} con(G_1 \cap G_2) &= con(Cay(G : S_1) \cap cay(G, S_2)) \\ &= con(Cay(G : S_1 \cap S_2)) \\ &= Cay(G : (S_1 \cap S_2)^2 - \{e\}) \\ &= Cay(G : (S_1^2 - \{e\}) \cap (S_2^2 - \{e\})) \\ &= Cay(G : S_1^2 - \{e\}) \cap Cay(G : S_2^2 - \{e\}) \\ &= con(Cay(G : S_1)) \cap con(Cay(G : S_2)) \\ &= con(G_1) \cap con(G_2). \end{aligned}$$

□

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