A Shorter and Simple Approach to Study Fixed Point Results via b-Simulation Functions

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\begin{abstract}
The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on b-metric spaces via b-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].
\end{abstract}

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1. Introduction and Preliminaries

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping \( \zeta : [0, +\infty)^2 \to \mathbb{R} \) is called a simulation function if it satisfies the following:

\[
(\zeta_1) \quad \zeta(0, 0) = 0;
\]

\[
(\zeta_2) \quad \zeta(t, s) < s - t \text{ for all } t, s > 0;
\]

\[
(\zeta_3) \quad \text{if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0,
\]

then \( \lim_{n \to \infty} \zeta(t_n, s_n) < 0. \)

Also, they denoted the set of all simulation functions by \( \mathcal{Z} \).

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition \( (\zeta_1) \) (also, see [7]). Also, Roldan et al. [8] revised \( (\zeta_3) \) by taking \( t_n < s_n \). Hence, we can say that a mapping \( \zeta : [0, +\infty)^2 \to \mathbb{R} \) is called a simulation function if it satisfies:

\[
(\zeta_2) \quad \zeta(t, s) < s - t \text{ for all } t, s > 0;
\]

\[
(\zeta_3) \quad \text{if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0
\]

and \( t_n < s_n \) for all \( n \in \mathbb{N} \), then \( \lim_{n \to \infty} \zeta(t_n, s_n) < 0. \)

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

**Definition 1.1.** [4] Let \((X, d)\) be a metric space and \( \zeta \in \mathcal{Z} \). Then a mapping \( T : X \to X \) is called a \( \mathcal{Z} \)-contraction with respect to \( \zeta \) if the following condition is satisfied:

\[
\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]

(1.1)

Now, it is clear that \( \zeta(t, t) < 0 \) when \( t > 0 \); further (1.1) implies that \( d(Tx, Ty) < d(x, y) \) when \( x \neq y \) for each \( x, y \in X \). This means that each \( \mathcal{Z} \)-contraction with respect to \( \zeta \) is continuous.

**Theorem 1.2.** [4] Let \((X, d)\) be a complete metric space and \( T : X \to X \) be a \( \mathcal{Z} \)-contraction with respect to \( \zeta \). Then \( T \) has a unique fixed point in \( X \) and for every \( x_0 \in X \), the Picard sequence \( \{x_n\} \), where \( x_n = Tx_{n-1} \) for all \( n \in \mathbb{N} \), converges to the fixed point of \( T \).

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely, \((X, d)\) is b-metric space if \( X \neq \emptyset \) and \( d : X \times X \to [0, +\infty) \) be a mapping such that for all \( x, y, z \in X \) hold:

\[
d(x, y) = 0 \iff x = y; d(x, y) = d(y, x) \text{ and } d(x, y) \leq b(d(x, y) + d(y, z)) \text{ for } b \geq 1.
\]

Then \( d \) is called b-metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows.

**Definition 1.3.** Let \((X, d)\) be a b-metric space. A b-simulation function is a function \( \zeta : [0, +\infty)^2 \to \mathbb{R} \) satisfying the following:
\begin{align}
(\xi_1) \quad &\xi(t, s) < s - t \text{ for all } t, s > 0; \\
(\xi_2) \quad &\text{if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that} \\
&0 < \lim_{n \to +\infty} t_n \leq \lim_{n \to +\infty} s_n \leq \lim_{n \to +\infty} b \lim_{n \to +\infty} t_n < +\infty, \quad (1.2)
\end{align}

then \( \lim_{n \to +\infty} \xi(bt_n, s_n) < 0. \)

It is clear if \( b = 1, \) then \( b \)-simulation function is in fact the simulation function in the framework of (standard) metric spaces.

**Example 1.4.** [2] Let \( \xi : [0, +\infty)^2 \to \mathbb{R} \) be defined by

(i) \( \xi(t, s) = \lambda s - t \) for all \( t, s \in [0, +\infty), \) where \( \lambda \in [0, 1). \)

(ii) \( \xi(t, s) = \psi(s) - \varphi(t) \) for all \( t, s \in [0, +\infty), \) where \( \varphi, \psi : [0, +\infty) \to [0, +\infty) \) are two continuous functions such that \( \psi(t) = \varphi(t) = 0 \) if and only if \( t = 0 \) and \( \psi(t) < t \leq \varphi(t) \) for all \( t > 0. \)

(iii) \( \xi(t, s) = s - \frac{f(t, s)}{g(t, s)}t \) for all \( t, s \in [0, +\infty), \) where \( f, g : [0, +\infty)^2 \to (0, +\infty) \) are two continuous functions with respect to each variable such that \( f(t, s) > g(t, s) \) for all \( t, s > 0. \)

(iv) \( \xi(t, s) = s - \varphi(s) - t \) for all \( t, s \in [0, +\infty), \) where \( \varphi : [0, +\infty) \to [0, +\infty) \) is a lower semi-continuous function such that \( \varphi(t) = 0 \) if and only if \( t = 0. \)

(v) \( \xi(t, s) = s\varphi(s) - t \) for all \( t, s \in [0, +\infty), \) where \( \varphi : [0, +\infty) \to [0, 1) \) is such that \( \lim_{t \to +\infty} \varphi(t) < 1 \) for all \( r > 0. \)

Each of the function considered in (i)-(v) is a \( b \)-simulation function.

The following important and very interesting results are proved in [2].

**Lemma 1.5.** Let \( (X, d) \) be a \( b \)-metric space and \( f : X \to X \) be a mapping. Suppose that there exists a \( b \)-simulation function \( \xi \) such that following condition holds.

\[ \xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X. \quad (1.3) \]

Let \( \{x_n\} \) be a sequence of Picard of initial at point \( x_0 \in X \) and \( x_{n-1} \neq x_n \) for all \( n \in \mathbb{N}. \) Then

\[ \lim_{n \to +\infty} d(x_{n-1}, x_n) = 0. \]

**Lemma 1.6.** Let \( (X, d) \) be a \( b \)-metric space and \( f : X \to X \) be a mapping. Suppose that there exists a \( b \)-simulation function \( \xi \) such that (1.3) holds. Let \( \{x_n\} \) be a sequence of Picard of initial at point \( x_0 \in X \) and \( x_{n-1} \neq x_n \) for all \( n \in \mathbb{N}. \) Then \( \{x_n\} \) is a bounded sequence.

**Lemma 1.7.** Let \( (X, d) \) be a \( b \)-metric space and \( f : X \to X \) be a mapping. Suppose that there exists a \( b \)-simulation function \( \xi \) such that (1.3) holds. Let \( \{x_n\} \) be a sequence of Picard of initial at point \( x_0 \in X \) and \( x_{n-1} \neq x_n \) for all \( n \in \mathbb{N}. \) Then \( \{x_n\} \) is a Cauchy sequence.
**Theorem 1.8.** Let \((X, d)\) be a complete \(b\)-metric space and let \(f : X \to X\) be a mapping. Suppose that there exists a \(b\)-simulation function \(\xi\) such that (1.3) holds; that is,

\[
\xi(b d(f x, f y), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]

Then \(f\) has a unique fixed point.

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

2. **Main results**

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from \(b\)-metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of \(b\)-metric spaces in [3].

**Lemma 2.1.** Let \(\{y_n\}\) be a sequence in a \(b\)-metric space \((X, d)\) such that

\[
d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)
\]

for some \(\lambda, 0 \leq \lambda < \frac{1}{b}\) and each \(n = 1, 2, \ldots\). Then \(\{y_n\}\) is a Cauchy sequence in \((X, d)\).

By utilizing Lemma 2.1, Jovanović et al. [3] proved following result.

**Theorem 2.2.** Let \((X, d)\) be a complete \(b\)-metric space and \(f : X \to X\) be a map such that

\[
d(f x, f y) \leq \lambda d(x, y)
\]

holds for all \(x, y \in X\), where \(0 \leq \lambda < \frac{1}{b}\). Then \(f\) has a unique fixed point \(z\) and for every \(x_0 \in X\), the sequence \(\{f^n x_0\}\) converges to \(z\).

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

**Theorem 2.3.** Let \((X, d)\) be a complete \(b\)-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a \(b\)-simulation function \(\xi\) such that (1.3) holds; that is,

\[
\xi(b d(f x, f y), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]

Then \(f\) has a unique fixed point.

**Proof.** It is enough clear that (2.3) implies

\[
b d(f x, f y) \leq d(x, y) \quad \forall x, y \in X.
\]

Indeed, (2.4) holds if \(x = y\). In the case that \(x \neq y\) there are two possibilities, either \(f x = f y\) or \(f x \neq f y\). In the first case we have that \(b \cdot d(f x, f y) = 0 < d(x, y)\), while in second case the result follows from \((\xi_1)\). This means that (2.3) implies (2.4) for all \(x, y \in X\). Further, obviously, (2.4) implies that

\[
d(f^2 x, f^2 y) \leq \frac{1}{b^2} d(x, y) = \lambda d(x, y).
\]
Since $\lambda = \frac{1}{\beta} \in [0, \frac{1}{b})$, then according to Theorem 2.2, $f^2$ has a unique fixed point (say $z$) in $X$. This further means that $f$ has a unique fixed point $z$ in $X$. Now, the proof of this theorem is complete. □

Obviously, our proof is much shorter than the corresponding ones from Demma et al.’s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

**Corollary 2.4.** Let $(X,d)$ be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that

(i) $\lambda \in [0, 1)$ such that $bd(fx, fy) \leq \lambda d(x, y)$;

(ii) a lower semi-continuous function $\varphi : [0, +\infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ such that $bd(fx, fy) \leq d(x, y) - \varphi(d(x, y))$;

(iii) $\varphi : [0, +\infty) \to [0, 1)$ with $\lim_{t \to r^+} \varphi(t) < 1$ for all $r > 0$ such that $bd(fx, fy) \leq \varphi(d(x, y))d(x, y)$;

(iv) $\eta : [0, +\infty) \to [0, \infty)$ with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ such that $bd(fx, fy) \leq \eta(d(x, y))$

for all $x, y \in X$. Then $f$ has a unique fixed point in each one of above condition.

**Proof.** Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate b-simulation function in Example 1.4. Hence, we obtain that $bd(fx, fy) \leq d(x, y)$ for all $x, y \in X$. The result then follows according to Theorem 2.3. □

**Example 2.5.** Now, we consider Example 4.5 from [2]. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|^2$. Then $(X, d)$ is a complete b-metric space with $b = 2$. Consider a mapping $f : X \to X$ by

$$fx = \frac{ax}{1 + x}$$

for all $x \in X$, where $a \in [0, \frac{1}{\sqrt{2}}]$. Now, we have

$$2d(fx, fy) = 2\left| \frac{ax}{1 + x} - \frac{ay}{1 + y} \right|^2 = 2a^2 \left( \frac{|x - y|^2}{(1 + x)^2 (1 + y)^2} \right) \leq |x - y|^2 = d(x, y)$$

for all $x, y \in X$. Further, (2.6) implies that

$$d(f^2x, f^2y) \leq \frac{1}{4}d(x, y);$$

that is, $f^2$ has a unique fixed point according to Theorem 2.2. This means that $f$ has a unique fixed point. Here it is $z = 0$.

The next result is probably known, but our proof is very condensed.
Theorem 2.6. Let $(X,d)$ be a complete b-metric space and let $f : X \to X$ be a mapping such that
\[
d (fx, fy) \leq \lambda d (x,y)
\] for all $x,y \in X$, where $\lambda \in [0,1)$. Then $f$ has a unique fixed point (say $z$) in $X$ and for $x_0 \in X$ the sequence $\{f^n x_0\}_{n \in \mathbb{N}}$ converges to $z$.

Proof. The condition (2.7) implies that
\[
d (f^n x, f^n y) \leq \lambda d (f^{n-1} x, f^{n-1} y) \leq \cdots \leq \lambda^n d (x,y)
\] for all $x,y \in X$ and $n \in \mathbb{N}$. Since $\lambda^n \to 0$ as $n \to \infty$, there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{b}$. Therefore, we have
\[
d (f^{k+1} x, f^{k+1} y) \leq \frac{1}{b^2} d (x,y).
\] The result now follows by Theorem 2.2.

Question 1. Does Theorem 2.3 holds if $\xi (d (fx, fy), d (x,y)) \geq 0$ for all $x,y \in X$, where $(X,d)$ is a given complete b-metric space and $f : X \to X$ be a mapping and $\xi$ a given b-simulation function?

Question 2. Can you obtain this results by considering ordered b-metric spaces or cone b-metric spaces instead of b-metric spaces?

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