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On a New Reverse Hilbert's Type Inequality

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Abstract. In this paper, by using the Euler-Maclaurin expansion for the Riemann-ζ function, we establish an inequality of a weight coefficient. Using this inequality, we derive a new reverse Hilbert's type inequality. As an applications, an equivalent form is obtained.

Keywords: Hilbert's type inequality, Weight coefficien, Hölder inequality, Riemann-ζ function, Reverse.

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1. Introduction

If $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $b_n \geq 0$, for $n \geq 1, n \in N$ and $0 < \sum\limits_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{\frac{1}{q}}, \tag{1.2}$$

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where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

In [5], [10] and [9], Krnic, Pecaric and Yang gave some generalization and reinforcement of inequality (1.1). In [3], Kuang and Debnath gave a reinforcement of inequality (1.2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < \left\{\sum_{n=1}^{\infty}[pq-G(p,n)]a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}[pq-G(q,n)]b_n^q\right\}^{\frac{1}{q}} \quad (1.3)$$

where $G(r,n)=\frac{r+\frac{1}{3r}-\frac{4}{3}}{(2n+1)^{\frac{1}{r}}}>0$ $(r=p,q)$.

In [6] and [7], Xi gave a generalization and reinforcement of inequalities (1.2) and (1.3):

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} &< \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}\right]n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{3pn^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (1.4)\end{aligned}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$, $2-\min\{p,q\}<\lambda\le 2$.

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda+A,n^\lambda+B\}} &< \left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{q+\lambda-2}{q}}}\left(\frac{1}{3q}-\frac{B}{1+B}\right)\right]\right. \\ &\left.\times n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left(\frac{1}{3p}-\frac{A}{1+A}\right)\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (1.5)\end{aligned}$$

For the reverse Hilbert's type inequality, In [8] , Xi and Wang gave a reverse Hilbert's type inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2,n^2\}} > 2\left[\sum_{n=1}^{\infty}\left(1-\frac{1}{2n}\right)\frac{1}{n}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\frac{1}{n}b_n^q\right]^{\frac{1}{q}}. \quad (1.6)$$

In this paper, by introducing a parameter λ and using the Euler-Maclaurin expansion for the Riemann-ζ function, we establish an inequality of a weight coefficient. Using this inequality, we derive a reverse of the Hilbert's type inequality (1.4).

2. A Lemma

First, we need the following formula of the Riemann-ζ function (see [4], [12] and [11]):

$$\zeta(\sigma) = \sum_{k=1}^{n} \frac{1}{k^{\sigma}} - \frac{n^{1-\sigma}}{1-\sigma} - \frac{1}{2n^{\sigma}} - \sum_{k=1}^{l-1} \frac{B_{2k}}{2k} \begin{pmatrix} -\sigma \\ 2k-1 \end{pmatrix} \frac{1}{n^{\sigma+2k-1}}$$
$$- \frac{B_{2l}}{2l} \begin{pmatrix} -\sigma \\ 2l-1 \end{pmatrix} \frac{\varepsilon}{n^{\sigma+2l-1}}, \tag{2.1}$$

where $\sigma > 0$, $\sigma \neq 1$, $n,\ l \geq 1$, $n,\ l \in N$, $0 < \varepsilon = \varepsilon(\sigma, l, n) < 1$. The numbers $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $\cdots$ are Bernoulli numbers. In particular, $\zeta(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k^{\sigma}}$ $(\sigma > 1)$.

Since $\zeta(0) = -1/2$, then the formula of the Riemann-ζ function (2.1) is also true for $\sigma = 0$.

Lemma 2.1. *If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - p < \lambda \leq 2$, $n \geq 1$ and $n \in N$, then*

$$\omega(n, \lambda, q) = \sum_{k=1}^{\infty} \frac{1}{\max\{k^{\lambda}, n^{\lambda}\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} > \frac{qn^{1-\lambda}}{q+\lambda-2}, \tag{2.2}$$

and

$$\omega(n, \lambda, p) = \sum_{k=1}^{\infty} \frac{1}{\max\{k^{\lambda}, n^{\lambda}\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}}$$
$$< n^{1-\lambda} \left[\kappa(\lambda) - \frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right], \tag{2.3}$$

where $\kappa(\lambda) = \frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$.

Proof. Equalities (2.2) and (2.3) define the weight coefficient. When $2 - p < \lambda \leq 2$, taking $\sigma = \frac{2-\lambda}{p} \geq 0$, $l = 1$, in (2.1), we obtain

$$\zeta\left(\frac{2-\lambda}{p}\right) = \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2} - \frac{1}{2n^{\frac{2-\lambda}{p}}} + \frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}}\varepsilon_1, \tag{2.4}$$

where $0 < \varepsilon_1 < 1$.

Since $\frac{2}{q} + \frac{\lambda}{p} = 2 + \frac{\lambda-2}{p} = \frac{2p+\lambda-2}{p} > 0 (p + \lambda - 2 > 0)$. Taking $\sigma = \frac{\lambda}{p} + \frac{2}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{q} + \frac{\lambda}{p}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{q\lambda+2p}{12pqn^{1+\frac{2}{q}+\frac{\lambda}{p}}}\varepsilon_2, \tag{2.5}$$

where $0 < \varepsilon_2 < 1$.

Since $\frac{2}{p} + \frac{\lambda}{q} = 2 + \frac{\lambda-2}{q} = \frac{2q+\lambda-2}{q} > 0 (q + \lambda - 2 < 0, q < 0)$. Taking $\sigma = \frac{2}{p} + \frac{\lambda}{q}$, $l = 1$, we obtain

$$\zeta\left(\frac{2}{p} + \frac{\lambda}{q}\right) = \sum_{k=1}^{n-1} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2} + \frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}} + \frac{p\lambda+2q}{12pqn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\varepsilon_3, \tag{2.6}$$

where $0 < \varepsilon_3 < 1$.

In addition,

$$\begin{aligned}\omega(n,\lambda,q) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &= \sum_{k=1}^{n} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &= \sum_{k=1}^{n} \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{q}} \\ &= \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p}+\frac{2}{q}}} \\ &> \frac{1}{n^{\frac{(q+1)\lambda-2}{q}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{\lambda}{p}+\frac{2}{q}}}.\end{aligned}$$

By (2.5) and $\frac{2}{q} + \frac{\lambda}{p} = \frac{q\lambda+2p}{pq} > 0$

$$\begin{aligned}\omega(n,\lambda,q) &> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[\frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q}+\frac{\lambda}{p}}} + \frac{q\lambda+2p}{12pqn^{1+\frac{2}{q}+\frac{\lambda}{p}}}\varepsilon_2\right] \\ &> \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{q}} \left[\frac{pn^{-\frac{p+\lambda-2}{p}}}{p+\lambda-2} + \frac{1}{2n^{\frac{2}{q}+\frac{\lambda}{p}}}\right] \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{n^\lambda} + \frac{qn^{\,1-\lambda}}{q+\lambda-2} + \frac{1}{2n^\lambda} \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} - \frac{1}{2n^\lambda} + \frac{qn^{\,1-\lambda}}{q+\lambda-2} \\ &> \frac{qn^{\,1-\lambda}}{q+\lambda-2}.\end{aligned}$$

Using the last result and the inequality for $\omega(n,\lambda,q)$ above, we obtain (2.2).

$$\begin{aligned}\omega(n,\lambda,p) &= \sum_{k=1}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &= \sum_{k=1}^{n} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{\max\{k^\lambda, n^\lambda\}} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &= \sum_{k=1}^{n} \frac{1}{n^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} - \frac{1}{n^\lambda} + \sum_{k=n}^{\infty} \frac{1}{k^\lambda} \left(\frac{n}{k}\right)^{\frac{2-\lambda}{p}} \\ &= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}} \sum_{k=1}^{n} \frac{1}{k^{\frac{2-\lambda}{p}}} - \frac{1}{n^\lambda} + n^{\frac{2-\lambda}{p}} \sum_{k=n}^{\infty} \frac{1}{k^{\frac{2}{p}+\frac{\lambda}{q}}}.\end{aligned}$$

By (2.4) and (2.6)

$$\begin{aligned}
\omega(n,\lambda,p) &< \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\left[\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\frac{p+\lambda-2}{p}}}{p+\lambda-2}+\frac{1}{2n^{\frac{2-\lambda}{p}}}\right]-\frac{1}{n^{\lambda}}\\
&\quad +n^{\frac{2-\lambda}{p}}\left[\frac{qn^{-\frac{q+\lambda-2}{q}}}{q+\lambda-2}+\frac{1}{2n^{\frac{2}{p}+\frac{\lambda}{q}}}+\frac{p\lambda+2q}{12p\,qn^{1+\frac{2}{p}+\frac{\lambda}{q}}}\right]\\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pn^{\,1-\lambda}}{p+\lambda-2}+\frac{1}{2n^{\lambda}}-\frac{1}{n^{\lambda}}+\frac{qn^{\,1-\lambda}}{q+\lambda-2}\\
&\quad +\frac{1}{2n^{\lambda}}+\frac{p\lambda+2q}{12p\,qn^{1+\lambda}}\\
&= \frac{1}{n^{\frac{(p+1)\lambda-2}{p}}}\zeta\left(\frac{2-\lambda}{p}\right)+\frac{pq\,\lambda\;n^{\,1-\lambda}}{(p+\lambda-2)(q+\lambda-2)}+\frac{p\lambda+2q}{12pqn^{1+\lambda}}\\
&= n^{1-\lambda}\left\{\frac{pq\,\lambda}{(p+\lambda-2)(q+\lambda-2)}-\frac{1}{n^{\frac{p+\lambda-2}{p}}}\left[-\zeta\left(\frac{2-\lambda}{p}\right)\right.\right.\\
&\quad \left.\left.-\frac{p\lambda+2q}{12p\,qn^{\frac{p-\lambda+2}{p}}}\right]\right\}.
\end{aligned}$$

In (2.4), taking $n=1$, by $2-p<\lambda\le 2$, we obtain

$$\begin{aligned}
\zeta\left(\frac{2-\lambda}{p}\right) &= 1-\frac{p}{p+\lambda-2}-\frac{1}{2}+\frac{(2-\lambda)\varepsilon_1}{12p}\\
&< \frac{1}{2}-\frac{p}{p+\lambda-2}+\frac{2-\lambda}{12p}\\
&= -\frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}\\
&< 0.
\end{aligned}$$

So for $n\ge 1$, $n\in N$, $2-p<\lambda\le 2$, we have

$$\begin{aligned}
&-\zeta\left(\frac{2-\lambda}{p}\right)+\frac{2-\lambda}{12pn^{1+\frac{2-\lambda}{p}}}\\
&\le -\zeta\left(\frac{2-\lambda}{p}\right)+\frac{2-\lambda}{12p}\\
&= \frac{(\lambda-2-3p)(\lambda-2-2p)}{12p(p+\lambda-2)}+\frac{2-\lambda}{12p}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\lambda-2-3p)(\lambda-2-2p)+(2-\lambda)(p+\lambda-2)}{12p(p+\lambda-2)} \\
&= \frac{(\lambda-2)^2+(-5p-p-\lambda+2)(\lambda-2)+6p^2}{12p(p+\lambda-2)} \\
&= \frac{6p(2-\lambda)+6p^2}{12p(p+\lambda-2)} \\
&= \frac{(2-\lambda)+p}{2(p+\lambda-2)} \\
&\le \frac{2+p}{2(p+\lambda-2)}.
\end{aligned}
$$

Using the last result and the inequality for $\omega(n,\lambda,p)$ above, we obtain (2.3). □

3. Main Results

Theorem 3.1. *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-p<\lambda\le 2$, $a_n \ge 0$, $b_n \ge 0$, *for* $n\ge 1, n\in N$ *and* $0<\sum_{n=1}^{\infty} a_n^p<\infty$, $0<\sum_{n=1}^{\infty} b_n^q<\infty$, *then*

$$
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} > \left\{\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (3.1)
$$

where $\kappa(\lambda)=\frac{p\,q\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$.

Proof. By the reverse Hölder 's inequality [2], we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^\lambda,n^\lambda\}} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m}{\max\{m^\lambda,n^\lambda\}^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}}\right] \\
&\quad\times\left[\frac{b_n}{\max\{m^\lambda,n^\lambda\}^{\frac{1}{q}}}(\frac{n}{m})^{\frac{2-\lambda}{pq}}\right] \\
&\ge \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{a_m^p}{\max\{m^\lambda,n^\lambda\}}(\frac{m}{n})^{\frac{2-\lambda}{q}}\right]\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{b_n^q}{\max\{m^\lambda,n^\lambda\}}(\frac{n}{m})^{\frac{2-\lambda}{p}}\right]\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=1}^{\infty}\omega(m,\lambda,q)a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(n,\lambda,p)b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Since $0<p<1$ and $q<0$. By (2.2), (2.3), we obtain (3.1). Theorem 3.1 is proved. □

Theorem 3.2. *If* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $2-p<\lambda\le 2$, $a_n\ge 0$, $b_n\ge 0$, *for* $n\ge 1, n\in N$ *and* $0<\sum\limits_{n=1}^{\infty} n^{1-\lambda}b_n^q<\infty$, *then*

$$\sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda,n^\lambda\}}\right)^q$$
$$>\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q. \tag{3.2}$$

where $\kappa(\lambda)=\frac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0$.

Inequalities (3.2) *and* (3.1) *are equivalent.*

Proof. Let

$$a_n=\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left[\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda,n^\lambda\}}\right]^{q-1}, \qquad n\in N.$$

By (3.1), we have

$$\begin{aligned}\left\{\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right\}^q &=\left\{\sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\right.\\ &\qquad\left.\times\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda,n^\lambda\}}\right)^q\right\}^q\\ &=\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_nb_m}{\max\{m^\lambda,n^\lambda\}}\right\}^q\\ &>\left\{\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right\}^{q-1}\left\{\sum_{n=1}^{\infty}\left[\kappa(\lambda)\right.\right.\\ &\qquad\left.\left.-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q\right\}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p&=\sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda,n^\lambda\}}\right)^q\\ &>\sum_{n=1}^{\infty}\left[\kappa(\lambda)-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right]n^{1-\lambda}b_n^q.\end{aligned}$$

On the other-hand, by the reverse Hölder 's inequality [2], we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{\max\{m^\lambda,n^\lambda\}}=\sum_{n=1}^{\infty}\left[\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{-\frac{1}{p}}\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda,n^\lambda\}}\right]$$
$$\times\left[\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{\frac{1}{p}}a_n\right]$$
$$\geq\left[\sum_{n=1}^{\infty}\left(\frac{q}{q+\lambda-2}n^{1-\lambda}\right)^{1-q}\left(\sum_{m=1}^{\infty}\frac{b_m}{\max\{m^\lambda,n^\lambda\}}\right)^q\right]^{\frac{1}{q}}$$
$$\times\sum_{n=1}^{\infty}\left[\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right]^{\frac{1}{p}}.$$

From (3.2), it follows that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{\max\{m^\lambda,n^\lambda\}} > \sum_{n=1}^{\infty}\left[\frac{q}{q+\lambda-2}n^{1-\lambda}a_n^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{n=1}^{\infty}\left(\kappa(\lambda)-\frac{p+2}{2(p+\lambda-2)n^{\frac{p+\lambda-2}{p}}}\right)n^{1-\lambda}b_n^q\right]^{\frac{1}{q}}.$$

Then, (3.2) and (3.1) are equivalent. Theorem 3.2 is proved. □

In inequality (3.1), taking $\lambda = 2$, we obtain:

Corollary 3.3. *If* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$,$a_n \geq 0$, $b_n \geq 0$, *for* $n \geq 1, n \in N$ *and* $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2,n^2\}} > \left\{\sum_{n=1}^{\infty}\frac{1}{n}a_n^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=1}^{\infty}\left[1-\frac{p+2}{2pn}\right]\frac{1}{n}b_n^q\right\}^{\frac{1}{q}}. \qquad (3.3)$$

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