

Pointwise Inner and Center Actors of a Lie Crossed Module

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ABSTRACT. Let $\mathcal{L}$ be a Lie crossed module and $\text{Act}_{pi}(\mathcal{L})$ and $\text{Act}_z(\mathcal{L})$ be the pointwise inner actor and center actor of $\mathcal{L}$, respectively. We will give a necessary and sufficient condition under which $\text{Act}_{pi}(\mathcal{L})$ and $\text{Act}_z(\mathcal{L})$ are equal.

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1. INTRODUCTION

Crossed modules of groups are introduced by Whitehead [11] to study homotopy relation among groups. Lie crossed modules are also introduced and used by Lavendhomme and Rosin [8] as a sufficient coefficient of a nonabelian cohomology of T -algebras.

A crossed module $\mathcal{L}$ in Lie algebras is a homomorphism $d : L_1 \longrightarrow L_0$ with an action of L_0 on L_1 satisfying special conditions (see Casas [3], Casas and Ladra [4, 5] for details).

In [9], Norrie extended the definition of actor to the 2-dimensional case by giving a description of the corresponding object in the category of crossed modules of groups. The analogue construction for the category of crossed modules of Lie algebras is given in [5].

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Actor of crossed module of Leibniz algebras also introduced by Casas et al. in [6].

Allahyari and Saeedi in [1] and [2] introduced a chain of subcrossed modules of $\text{Act}(\mathcal{L})$, and showed that for two Lie crossed module $\mathcal{L}$ and $\mathcal{M}$, $\text{ID}^*\text{Act}(\mathcal{L}) \cong \text{ID}^*\text{Act}(\mathcal{M})$ if $\mathcal{L}$ and $\mathcal{M}$ are isoclinic. Sheikh-Mohseni et al. [10] gives a necessary and sufficient condition for $\text{Der}_c(L)$ and $\text{Der}_z(L)$ of a Lie algebra L to be equal.

In this paper, we shall introduce a new subcrossed module of $\text{Act}(\mathcal{L})$, denoted by $\text{Act}_z(\mathcal{L})$, and study its relationships with subcrossed modules of $\text{Act}(\mathcal{L})$, say $\text{InnAct}(\mathcal{L})$ and $\text{Act}_{pi}(\mathcal{L})$. In section 2, definitions and primary notations used for Lie crossed module and $\text{Act}(\mathcal{L})$ are presented. In section 3, $\text{Act}_z(\mathcal{L})$ is defined and some of its elementary properties are proved. In section 4, we prove the main theorem, which gives a necessary and sufficient condition for the equality of $\text{Act}_{pi}(\mathcal{L})$ and $\text{Act}_z(\mathcal{L})$.

2. PRELIMINARIES ON CROSSED MODULES

Definition 2.1. A Lie crossed module is a Lie homomorphism $d : L_1 \rightarrow L_0$ together with an action of L_0 on L_1 , denoted as $(l_0, l_1) \mapsto^{l_0} l_1$ for all $l_0 \in L_0$ and $l_1 \in L_1$, such that

- (1) $d({}^{l_0}l_1) = [l_0, d(l_1)]$;
- (2) $d({}^{l_1}l'_1) = [l_1, l'_1]$,

for all $l_0 \in L_0$ and $l_1, l'_1 \in L_1$. The crossed module $\mathcal{L}$ is denoted by $\mathcal{L} : (L_1, L_0, d)$.

The crossed module $\mathcal{L}' : (L'_1, L'_0, d')$ is a subcrossed module of $\mathcal{L} : (L_1, L_0, d)$, and denoted by $\mathcal{L}' \leq \mathcal{L}$, if L'_0 and L'_1 are subalgebras of L_0 and L_1 , respectively, and d' is the restriction of d on L'_1 , and the action of L'_0 on L'_1 is induced from the action of L_0 on L_1 .

The subcrossed module $\mathcal{L}' : (L'_1, L'_0, d')$ of $\mathcal{L} : (L_1, L_0, d)$ is an ideal of $\mathcal{L}$, denoted by $\mathcal{L}' \triangleleft \mathcal{L}$, if L'_0 and L'_1 are ideals of L_0 and L_1 , respectively, and that we have ${}^{l_0}l'_1 \in L'_1$ and ${}^{l'_0}l_1 \in L'_1$ for all $l_0 \in L_0$, $l'_0 \in L'_0$, $l_1 \in L_1$, and $l'_1 \in L'_1$.

Definition 2.2. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. The center $Z(\mathcal{L})$ of $\mathcal{L}$, that is an ideal of $\mathcal{L}$, is defined as

$$Z(\mathcal{L}) : ({}^{L_0}L_1, \text{st}_{L_0}(L_1) \cap Z(L_0), d|),$$

where

$${}^{L_0}L_1 = \{l_1 \in L_1 \mid {}^{l_0}l_1 = 0, \forall l_0 \in L_0\}$$

and

$$\text{st}_{L_0}(L_1) = \{l_0 \in L_0 \mid {}^{l_0}l_1 = 0, \forall l_1 \in L_1\}.$$

and $d|_1$ is restriction of d to ${}^{L_0}L_1$.

The crossed module $\mathcal{L}$ is abelian if it coincides with its center, i.e.

$$L_1 = {}^{L_0}L_1 \quad \text{and} \quad L_0 = \text{st}_{L_0}(L_1) \cap Z(L_0).$$

The derived subcrossed module of $\mathcal{L}$, denoted as $\mathcal{L}^2$, is defined as follows:

$$\mathcal{L}^2 : (D_{L_0}(L_1), L_0^2, d|_1),$$

where

$$D_{L_0}(L_1) = \langle {}^{l_0}l_1 \mid l_0 \in L_0, l_1 \in L_1 \rangle.$$

and $d|_1$ is restriction of d to ${}^{L_0}L_1$.

A homomorphism between two Lie crossed modules $\mathcal{L} : (L_1, L_0, d)$ and $\mathcal{L}' : (L'_1, L'_0, d')$ is a pair (f, g) of Lie algebra homomorphisms $f : L_1 \rightarrow L'_1$ and $g : L_0 \rightarrow L'_0$ satisfying

- (1) $d'f = gd$;
- (2) $f({}^{l_0}l_1) = {}^{g(l_0)}f(l_1)$,

for all $l_0 \in L_0$ and $l_1 \in L_1$.

Definition 2.3. Assume $\mathcal{L} : (L_1, L_0, d)$ is a crossed module. A derivation of $\mathcal{L}$ is a pair $(\psi, \phi) : \mathcal{L} \rightarrow \mathcal{L}$ satisfying the following conditions:

- (1) $\psi \in \text{Der}(L_1)$,
- (2) $\phi \in \text{Der}(L_0)$,
- (3) $d\psi = \phi d$,
- (4) $\psi({}^{l_0}l_1) = {}^{l_0}\psi(l_1) + \phi(l_0)(l_1)$,

for all $l_0 \in L_0$ and $l_1 \in L_1$.

The set of all derivations of $\mathcal{L}$ is denoted by $\text{Der}(\mathcal{L})$, which is a Lie algebra with bracket as in the following:

$$[(\psi, \phi), (\psi', \phi')] = ([\psi, \psi'], [\phi, \phi']) = (\psi\psi' - \psi'\psi, \phi\phi' - \phi'\phi).$$

Definition 2.4. Assume $\mathcal{L} : (L_1, L_0, d)$ is a Lie algebra crossed module. The map $\delta : L_0 \rightarrow L_1$ is called crossed derivation if

$$\delta([l_0, l'_0]) = {}^{l_0}\delta(l'_0) - {}^{l'_0}\delta(l_0)$$

for all $l_0, l'_0 \in L_0$. The set of all crossed derivations from L_0 to L_1 is denoted by $\text{Der}(L_0, L_1)$, which turns into a Lie algebra via the following bracket:

$$[\delta_1, \delta_2] = \delta_1 d \delta_2 - \delta_2 d \delta_1$$

for all $\delta_1, \delta_2 \in \text{Der}(L_0, L_1)$.

Definition 2.5. To each Lie crossed module $\mathcal{L} : (L_1, L_0, d)$, there corresponds a crossed module $\text{Act}(\mathcal{L}) : (\text{Der}(L_0, L_1), \text{Der}(\mathcal{L}), \Delta)$ such that

$$\text{hom } \Delta \text{Der}(L_0, L_1) \text{Der}(\mathcal{L}) \delta(\delta d, d\delta)$$

and the action of $\text{Der}(\mathcal{L})$ on $\text{Der}(L_0, L_1)$ is defined as

$$(\alpha, \beta)\delta = \alpha\delta - \delta\beta$$

for all $(\alpha, \beta) \in \text{Der}(\mathcal{L})$ and $\delta \in \text{Der}(L_0, L_1)$, and it is called the actor of $\mathcal{L}$ (see Casas and Ladra, [5]).

Proposition 2.6. *There exists a canonical homomorphism of crossed modules as*

$$(\varepsilon, \eta) : \mathcal{L} \longrightarrow \text{Act}(\mathcal{L}),$$

where

$$\text{hom } \varepsilon L_1 \text{Der}(L_0, L_1) l_1 \delta_{l_1} \quad \text{and} \quad \text{hom } \eta L_0 \text{Der}(\mathcal{L}) l_0 (\alpha_{l_0}, \beta_{l_0}),$$

in which $\delta_{l_1}(l_0) = {}^{l_0}l_1$, $\alpha_{l_0}(l_1) = {}^{l_0}l_1$, and $\beta_{l_0}(l'_0) = [l_0, l'_0]$ for all $l_0 \in L_0$, $l'_0 \in L_0$, and $l_1 \in L_1$.

The image of (ε, η) is an ideal of $\text{Act}(\mathcal{L})$ and it is denoted as $\text{InnAct}(\mathcal{L})$. We have

$$\text{InnAct}(\mathcal{L}) : (\varepsilon(L_1), \eta(L_0), \Delta|).$$

On can easily see that $\ker(\varepsilon, \eta) = Z(\mathcal{L})$. (See allahyary and saeedi [1])

Definition 2.7. Let $\mathcal{L}$ be a Lie crossed module. Then the pointwise inner actor of $\mathcal{L}$ is defined as follows:

$$\text{Act}_{pi}(\mathcal{L}) : (\text{Der}_{pi}(L_0, L_1), \text{Der}_{pi}(\mathcal{L}), \Delta|),$$

where

$$\text{Der}_{pi}(L_0, L_1) = \{\delta \in \text{Der}(L_0, L_1) \mid \forall l_0 \in L_0, \exists l_1 \in L_1 : \delta(l_0) = {}^{l_0}l_1\}$$

and

$$\text{Der}_{pi}(\mathcal{L}) = \left\{ (\alpha, \beta) \in \text{Der}(\mathcal{L}) \mid \begin{array}{l} \forall l_1 \in L_1, \exists l_0 \in L_0 : \alpha(l_1) = {}^{l_0}l_1, \\ \forall l_0 \in L_0, \exists l'_0 \in L_0 : \beta(l_0) = [l'_0, l_0] \end{array} \right\}.$$

One can easily verify that $\text{Act}_{pi}(\mathcal{L})$ is a subcrossed module of $\text{Act}(\mathcal{L})$ and contains $\text{InnAct}(\mathcal{L})$ (see Allahyari and Saeedi [1]).

Definition 2.8. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. Then $\text{ID}^*\text{Act}(\mathcal{L})$ is defined as

$$\text{ID}^*\text{Act}(\mathcal{L}) : (\text{ID}^*(L_0, L_1), \text{ID}^*(\mathcal{L}), \Delta|),$$

where

$$\text{ID}^*(L_0, L_1) = \left\{ \delta \in \text{Der}(L_0, L_1) \mid \begin{array}{l} \delta(x_0) \in D_{L_0}(L_1), \forall x_0 \in L_0, \\ \delta(x_0) = 0, \forall x_0 \in \text{st}_{L_0}(L_1) \cap Z(L_0), \end{array} \right\}$$

and

$$\text{ID}^*(\mathcal{L}) = \left\{ (\alpha, \beta) \in \text{Der}(\mathcal{L}) \mid \begin{array}{l} \alpha(x_1) \in D_{L_0}(L_1), \forall x_1 \in L_1, \\ \alpha(x_1) = 0, \forall x_1 \in {}^{L_0}L_1, \\ \beta(x_0) \in L_0^2, \forall x_0 \in L_0, \\ \beta(x_0) = 0, \forall x_0 \in \text{st}_{L_0}(L_1) \cap Z(L_0) \end{array} \right\}.$$

One can easily show that $\text{ID}^*\text{Act}(\mathcal{L})$ is a subcrossed module of $\text{Act}(\mathcal{L})$ and contains $\text{Act}_{pi}(\mathcal{L})$ (see Allahyari and Saeedi [1]).

3. CENTER ACTOR OF LIE CROSSED MODULES

In this section we define subcrossed module of $\text{Act}(\mathcal{L})$ namely $\text{Act}_z(\mathcal{L})$ and we prove some of its elementary properties.

Definition 3.1. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. The $\text{Act}_z(\mathcal{L})$ is defined as follows:

$$\text{Act}_z(\mathcal{L}) : (\text{Der}_z(L_0, L_1), \text{Der}_z(\mathcal{L}), \Delta|),$$

where

$$\text{Der}_z(L_0, L_1) = \{\delta \in \text{Der}(L_0, L_1) \mid \delta(l_0) \in {}^{L_0}L_1, \forall l_0 \in L_0\}$$

and

$$\text{Der}_z(\mathcal{L}) = \left\{ (\alpha, \beta) \in \text{Der}(\mathcal{L}) \mid \begin{array}{l} \alpha(l_1) \in {}^{L_0}L_1, \forall l_1 \in L_1, \\ \beta(l_0) \in \text{st}_{L_0}(L_1) \cap Z(L_0), \forall l_0 \in L_0. \end{array} \right\}$$

Note that $\Delta|$ is the restriction of Δ to $\text{Der}_z(L_0, L_1)$.

Proposition 3.2. $\text{Act}_z(\mathcal{L})$ is a subcrossed module of $\text{Act}(\mathcal{L})$.

Proof. We have to show that

- (1) $\text{Der}_z(L_0, L_1) \leq \text{Der}(L_0, L_1)$;
- (2) $\text{Der}_z(\mathcal{L}) \leq \text{Der}(\mathcal{L})$;
- (3) $\Delta|_{\text{Der}_z(L_0, L_1)} \subseteq \text{Der}_z(\mathcal{L})$.

(1) Assume δ, δ' are two arbitrary elements of $\text{Der}_z(L_0, L_1)$. Then

$$\delta(x_0) \in {}^{L_0}L_1 \quad \text{and} \quad \delta'(x_0) \in {}^{L_0}L_1$$

for all $x_0 \in L_0$. Now since $[\delta, \delta'](x_0) = \delta d\delta'(x_0) - \delta' d\delta(x_0)$, one can easily verify that

$$[\delta, \delta'](x_0) \in {}^{L_0}L_1$$

for all $x_0 \in L_0$. Hence $\text{Der}_z(L_0, L_1) \leq \text{Der}(L_0, L_1)$.

(2) Let (α, β) and (α', β') be two elements of $\text{Der}_z(\mathcal{L})$. Then

$$\begin{aligned} \alpha(x_1) &\in {}^{L_0}L_1 \quad \text{and} \quad \alpha'(x_1) \in {}^{L_0}L_1, \\ \beta(x_0) &\in \text{st}_{L_0}(L_1) \cap Z(L_0) \quad \text{and} \quad \beta'(x_0) \in \text{st}_{L_0}(L_1) \cap Z(L_0) \end{aligned}$$

for all $x_0 \in L_0$ and $x_1 \in L_1$. Since

$$[(\alpha, \beta), (\alpha', \beta')] = ([\alpha, \alpha'], [\beta, \beta']) = (\alpha\alpha' - \alpha'\alpha, \beta\beta' - \beta'\beta),$$

one can see that

$$\begin{aligned} (\alpha\alpha' - \alpha'\alpha)(x_1) &= \alpha\alpha'(x_1) - \alpha'\alpha(x_1) \in {}^{L_0}L_1, \\ (\beta\beta' - \beta'\beta)(x_0) &= \beta\beta'(x_0) - \beta'\beta(x_0) \in \text{st}_{L_0}(L_1) \cap Z(L_0) \end{aligned}$$

for all $x_0 \in L_0$ and $x_1 \in L_1$. Therefore $[(\alpha, \beta), (\alpha', \beta')] \in \text{Der}_z(\mathcal{L})$ so that $\text{Der}_z(\mathcal{L}) \leq \text{Der}(\mathcal{L})$.

(3) Assume $\delta \in \text{Der}_z(L_0, L_1)$. From the definition of Δ , we have

$$\Delta(\delta) = (\delta d, d\delta).$$

One can easily check that

$$\begin{aligned} \delta d(x_1) &\in^{L_0} L_1, \\ d\delta(x_0) &\in \text{st}_{L_0}(L_1) \cap Z(L_0) \end{aligned}$$

for all $x_0 \in L_0$ and $x_1 \in L_1$. Thus $\Delta(\delta) = (\delta d, d\delta) \in \text{Der}_z(\mathcal{L})$, and so $\Delta|_{\text{Der}_z(L_0, L_1)} \subseteq \text{Der}_z(\mathcal{L})$. Therefore $\text{Act}_z(\mathcal{L}) \leq \text{Act}(\mathcal{L})$, and the proof is complete. $\square$

Definition 3.3. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and $\mathcal{M} : (M_1, M_0, d)$ be an ideal of $\mathcal{L}$. Then the centralizer of $\mathcal{M}$ in $\mathcal{L}$, denoted as $\mathcal{C}_{\mathcal{L}}(\mathcal{M})$, is defined as

$$\mathcal{C}_{\mathcal{L}}(\mathcal{M}) : ({}^{M_0}L_1, C_{L_0}(M_0) \cap \text{st}_{L_0}(M_1), d|),$$

where

$$\begin{aligned} {}^{M_0}L_1 &= \{x_1 \in L_1 \mid {}^{x_0}x_1 = 0, \forall x_0 \in M_0\}, \\ C_{L_0}(M_0) &= \{x_0 \in L_0 \mid [x_0, y_0] = 0, \forall y_0 \in M_0\}, \\ \text{st}_{L_0}(M_1) &= \{x_0 \in L_0 \mid {}^{x_0}x_1 = 0, \forall x_1 \in M_1\}. \end{aligned}$$

Let $\mathcal{M} : (M_1, M_0, d_1)$ and $\mathcal{N} : (N_1, N_0, d_1)$ be two ideals of the crossed module $\mathcal{L} : (L_1, L_0, d)$. Then the ideal $\mathcal{M} \cap \mathcal{N}$ of $\mathcal{L}$ is defined as

$$\mathcal{M} \cap \mathcal{N} : (M_1 \cap N_1, M_0 \cap N_0, d|).$$

Lemma 3.4. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and $\mathcal{M} : (M_1, M_0, d)$ be an ideal of $\mathcal{L}$. Then $\mathcal{M} \cap \mathcal{C}_{\mathcal{L}}(\mathcal{M}) = Z(\mathcal{M})$.

Proof. It is obvious. $\square$

Lemma 3.5. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and $\text{InnAct}(\mathcal{L}) \leq \mathcal{H} \leq \text{ID}^* \text{Act}(\mathcal{L})$. Then

$$C_{\text{Act}(\mathcal{L})}(\mathcal{H}) = \text{Act}_z(\mathcal{L}).$$

Proof. Assume $\mathcal{H} : (H_1, H_0, \Delta)$. We need to show that

- (1) ${}^{H_0}\text{Der}(L_0, L_1) = \text{Der}_z(L_0, L_1)$;
- (2) $C_{\text{Der}(\mathcal{L})}(H_0) \cap \text{st}_{\text{Der}(\mathcal{L})}(H_1) = \text{Der}_z(\mathcal{L})$.

(1) Let $\delta \in \text{Der}_z(L_0, L_1)$. Then $\delta(l_0) \in^{L_0} L_1$ for all $l_0 \in L_0$. Now if $(\alpha, \beta) \in H_0$, then we observe that

$$({}^{\alpha, \beta})\delta(l_0) = (\alpha\delta - \delta\beta)(l_0) = \alpha(\delta(l_0)) - \delta(\beta(l_0)) = -\delta(\beta(l_0)).$$

Since $\beta(l_0) \in L_0^2$, there exist $x_0, y_0 \in L_0$ such that $\beta(l_0) = [x_0, y_0]$. Then

$$({}^{\alpha, \beta})\delta(l_0) = \delta([x_0, y_0]) = {}^{y_0}\delta(x_0) - {}^{x_0}\delta(y_0) = 0.$$

Thus $\delta \in {}^{H_0} \text{Der}(L_0, L_1)$ and consequently $\text{Der}_z(L_0, L_1) \subseteq {}^{H_0} \text{Der}(L_0, L_1)$.

Conversely, assume $\delta \in {}^{H_0} \text{Der}(L_0, L_1)$. Then $(\alpha, \beta)\delta(x_0) = 0$ for all $x_0 \in L_0$ and $(\alpha, \beta) \in H_0$. Now since $\mathcal{H}$ contains $\text{InnAct}(\mathcal{L})$, we can write $(\alpha, \beta) = (\alpha_{l_0}, \beta_{l_0})$ for some $l_0 \in L_0$. Then

$$\begin{aligned} (\alpha_{l_0}, \beta_{l_0})\delta(x_0) = 0 &\Rightarrow (\alpha_{l_0}\delta - \delta\beta_{l_0})(x_0) = 0, \\ &\Rightarrow \alpha_{l_0}(\delta(x_0)) - \delta(\beta_{l_0}(x_0)) = 0, \\ &\Rightarrow^{l_0} \delta(x_0) - \delta([l_0, x_0]) = 0, \\ &\Rightarrow^{l_0} \delta(x_0) - {}^{l_0} \delta(x_0) + {}^{x_0} \delta(l_0) = 0, \\ &\Rightarrow {}^{x_0} \delta(l_0) = 0 \end{aligned}$$

for all $x_0, l_0 \in L_0$. Therefore $\delta \in \text{Der}_z(L_0, L_1)$ so that ${}^{H_0} \text{Der}(L_0, L_1) \subseteq \text{Der}_z(L_0, L_1)$.

(2) Let $(\alpha, \beta) \in \text{Der}_z(\mathcal{L})$. Then

$$\alpha(l_1) \in {}^{L_0} L_1 \quad \text{and} \quad \beta(l_0) \in \text{st}_{L_0}(L_1) \cap Z(L_0)$$

for all $l_0 \in L_0$ and $l_1 \in L_1$. Now assume $(\alpha', \beta') \in H_0$ is any element. Then

$$\begin{aligned} [(\alpha, \beta), (\alpha', \beta')] &= ([\alpha, \alpha'], [\beta, \beta']), \\ [\alpha, \alpha'](l_1) &= (\alpha\alpha' - \alpha'\alpha)(l_1) = \alpha(\alpha'(l_1)) - \alpha'(\alpha(l_1)) = \alpha(\alpha'(l_1)). \end{aligned}$$

Since $\alpha'(l_1) \in D_{L_0}(L_1)$, there exist $x_0 \in L_0$ and $x_1 \in L_1$ such that

$$[\alpha, \alpha'](l_1) = \alpha(\alpha'(l_1)) = \alpha({}^{x_0} x_1) = {}^{x_0} \alpha(x_1) + {}^{\beta(x_0)} x_1 = 0.$$

Similarly, we can show that

$$\begin{aligned} [\beta, \beta'](l_0) &= (\beta\beta' - \beta'\beta)(l_0) = \beta(\beta'(l_0)) - \beta'(\beta(l_0)) \\ &= \beta([x_0, y_0]) = [\beta(x_0), y_0] + [x_0, \beta(y_0)] = 0 \end{aligned}$$

for some $x_0, y_0 \in L_0$. Hence, we conclude that $[(\alpha, \beta), (\alpha', \beta')] = 0$ and so

$$\text{Der}_z(\mathcal{L}) \subseteq C_{\text{Der}(\mathcal{L})}(H_0). \quad (3.1)$$

Now suppose that $\delta \in H_1$. Then

$$(\alpha, \beta)\delta(x_0) = \alpha(\delta(x_0)) - \delta(\beta(x_0)) = \alpha(\delta(x_0)).$$

Since $H_1 \subseteq \text{ID}^*(L_0, L_1)$, there exist elements $y_0 \in L_0$ and $y_1 \in L_1$ such that $\delta(x_0) = {}^{y_0} y_1$. Then we have

$$(\alpha, \beta)\delta(x_0) = \alpha(\delta(x_0)) = \alpha({}^{y_0} y_1) = {}^{y_0} \alpha(y_1) + {}^{\beta(y_0)} y_1 = 0.$$

Thus

$$\text{Der}_z(\mathcal{L}) \subseteq \text{st}_{\text{Der}(\mathcal{L})}(H_1). \quad (3.2)$$

From (3.1) and (3.2) it follows that

$$\text{Der}_z(\mathcal{L}) \subseteq C_{\text{Der}(\mathcal{L})}(H_0) \cap \text{st}_{\text{Der}(\mathcal{L})}(H_1).$$

Conversely, assume $(\alpha, \beta) \in C_{\text{Der}(\mathcal{L})}(H_0) \cap \text{st}_{\text{Der}(\mathcal{L})}(H_1)$. Then

$$^{(\alpha, \beta)}\delta = 0 \quad \text{and} \quad [(\alpha, \beta), (\alpha', \beta')] = 0$$

for all $\delta \in H_1$ and $(\alpha', \beta') \in H_0$. Now since $\text{InnAct}(\mathcal{L}) \subseteq \mathcal{H}$, we can write $\delta = \delta_{l_1}$ for some $l_1 \in L_1$. Then

$$\begin{aligned} ^{(\alpha, \beta)}\delta_{l_1}(x_0) = 0 &\Rightarrow \alpha(\delta_{l_1}(x_0)) - \delta_{l_1}(\beta(x_0)) = 0, \\ &\Rightarrow \alpha(^{x_0}l_1) - ^{\beta(x_0)}l_1 = 0, \\ &\Rightarrow ^{x_0}\alpha(l_1) + ^{\beta(x_0)}l_1 - ^{\beta(x_0)}l_1 = 0, \\ &\Rightarrow ^{x_0}\alpha(l_1) = 0 \end{aligned}$$

for all $x_0 \in L_0$ and $l_1 \in L_1$. This shows that

$$\alpha(l_1) \in {}^{L_0}L_1 \tag{3.3}$$

for all $l_1 \in L_1$.

On the other hand, for all $l_0 \in L_0$, we have

$$\begin{aligned} [(\alpha, \beta), (\alpha_{l_0}, \beta_{l_0})] = 0 &\Rightarrow [\alpha, \alpha_{l_0}](x_1) = 0, \\ &\Rightarrow \alpha(\alpha_{l_0}(x_1)) - \alpha_{l_0}(\alpha(x_1)) = 0, \\ &\Rightarrow \alpha(^{l_0}x_1) - ^{l_0}\alpha(x_1) = ^{l_0}\alpha(x_1) + ^{\beta(l_0)}x_1 - ^{l_0}\alpha(x_1) = ^{\beta(l_0)}x_1 = 0 \end{aligned}$$

for all $x_1 \in L_1$, which implies that $\beta(l_0) \in \text{st}_{L_0}(L_1)$. Also

$$\begin{aligned} [\beta, \beta_{l_0}] = 0 &\Rightarrow [\beta, \beta_{l_0}](x_0) = 0, \\ &\Rightarrow \beta(\beta_{l_0}(x_0)) - \beta_{l_0}(\beta(x_0)) = 0, \\ &\Rightarrow \beta([l_0, x_0]) - [l_0, \beta(x_0)] = 0, \\ &\Rightarrow [\beta(l_0), x_0] + [l_0, \beta(x_0)] - [l_0, \beta(x_0)] = [\beta(l_0), x_0] = 0 \end{aligned}$$

for all $x_0 \in L_0$, which implies that $\beta(l_0) \in Z(L_0)$. Hence

$$\beta(l_0) \in \text{st}_{L_0}(L_1) \cap Z(L_0). \tag{3.4}$$

From (3.3) and (3.4), we get $(\alpha, \beta) \in \text{Der}_z(\mathcal{L})$. $\square$

Corollary 3.6. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and $\text{InnAct}(\mathcal{L}) \leq \mathcal{H} \leq \text{ID}^*\text{Act}(\mathcal{L})$. Then*

$$\mathcal{H} \cap \text{Act}_z(\mathcal{L}) = Z(\mathcal{H}).$$

Proof. The result follows by Lemmas 3.4 and 3.5. $\square$

4. MAIN THEOREM

We are now ready to prove our main theorem, which gives a necessary and sufficient condition for $\text{Act}_{pi}(\mathcal{L})$ and $\text{Act}_z(\mathcal{L})$ to be equal. To this end, we need some preliminary lemmas.

Lemma 4.1. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module and $\text{Act}_{pi}(\mathcal{L}) = \text{Act}_z(\mathcal{L})$. Then $\text{InnAct}(\mathcal{L})$ is abelian.*

Proof. The result follows from the fact that $\text{InnAct}(\mathcal{L}) \subseteq \text{Act}_{pi}(\mathcal{L})$ and $\text{Act}_{pi}(\mathcal{L}) = \text{Act}_z(\mathcal{L})$. $\square$

Definition 4.2. Let $\mathcal{L} : (L_1, L_0, d_{\mathcal{L}})$ and $\mathcal{M} : (M_1, M_0, d_{\mathcal{M}})$ be two Lie crossed modules. The set of all linear transformations from $\mathcal{L}$ to $\mathcal{M}$ is denoted by $T(\mathcal{L}, \mathcal{M})$ and it is defined as

$$T(\mathcal{L}, \mathcal{M}) : (T(L_0, M_1), (T(L_1, M_1), T(L_0, M_0))),$$

where for example $T(L_0, M_1)$ is the vector space of linear transformations from L_0 to M_1 .

Definition 4.3. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. The dimension of $\mathcal{L}$ is defined as

$$\dim \mathcal{L} = (\dim L_1, \dim L_0).$$

Lemma 4.4. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie crossed module. Then we have the following vector space isomorphisms:

- (1) $\text{Der}_z(L_0, L_1) \cong T(L_0/L_0^2, {}^{L_0}L_1)$;
- (2) $\text{Der}_z(\mathcal{L}) \cong (T(L_1/D_{L_0}(L_1), {}^{L_0}L_1), T(L_0/L_0^2, \text{st}_{L_0}(L_1) \cap Z(L_0)))$.

Proof. (1) For each $\delta \in \text{Der}_z(L_0, L_1)$, we can define the map $\psi_\delta : L_0/L_0^2 \rightarrow {}^{L_0}L_1$ by $\psi_\delta(l_0 + L_0^2) = \delta(l_0)$ for all $l_0 \in L_0$. Clearly, ψ_δ is well-defined. Also, it is easy to see that the map

$$\psi : \text{Der}_z(L_0, L_1) \rightarrow T\left(\frac{L_0}{L_0^2}, {}^{L_0}L_1\right)$$

define by $\psi(\delta) = \psi_\delta$ is an one-to-one and onto linear transformation. Thus

$$\text{Der}_z(L_0, L_1) \cong T\left(\frac{L_0}{L_0^2}, {}^{L_0}L_1\right).$$

(2) For each $(\alpha, \beta) \in \text{Der}_z(\mathcal{L})$, we may define the maps $\phi_\alpha : L_1/D_{L_0}(L_1) \rightarrow {}^{L_0}L_1$ and $\phi_\beta : L_0/L_0^2 \rightarrow \text{st}_{L_0}(L_1) \cap Z(L_0)$ by $\phi_\alpha(l_1 + D_{L_0}(L_1)) = \alpha(l_1)$ and $\phi_\beta(l_0 + L_0^2) = \beta(l_0)$, respectively. One can easily check that, the maps ϕ_α and ϕ_β are well-defined linear transformations. Now, it is easy to show that the map

$$\text{hom } \phi \text{Der}_z(\mathcal{L}) \left(T\left(\frac{L_1}{D_{L_0}(L_1)}, {}^{L_0}L_1\right), T\left(\frac{L_0}{L_0^2}, \text{st}_{L_0}(L_1) \cap Z(L_0)\right) \right) (\alpha, \beta)(\phi_\alpha, \phi_\beta)$$

is a one-to-one and onto linear transformation. Thus

$$\text{Der}_z(\mathcal{L}) \cong \left(T\left(\frac{L_1}{D_{L_0}(L_1)}, {}^{L_0}L_1\right), T\left(\frac{L_0}{L_0^2}, \text{st}_{L_0}(L_1) \cap Z(L_0)\right) \right),$$

as required $\square$

Corollary 4.5. *We have*

$$\dim \text{Act}_z(\mathcal{L}) = \left(\dim T \left(\frac{L_0}{L_0^2}, {}^{L_0}L_1 \right), \right. \\ \left. \dim \left(T \left(\frac{L_1}{D_{L_0}(L_1)}, {}^{L_0}L_1 \right), T \left(\frac{L_0}{L_0^2}, \text{st}_{L_0}(L_1) \cap Z(L_0) \right) \right) \right).$$

Theorem 4.6. *Let $\mathcal{L} : (L_1, L_0, d)$ be a nonabelian Lie crossed module of finite dimension with $Z(\mathcal{L}) \neq 0$. Then $\text{Act}_z(\mathcal{L}) = \text{Act}_{pi}(\mathcal{L})$ if and only if $Z(\mathcal{L}) = \mathcal{L}^2$ and*

$$\dim \text{Act}_{pi}(\mathcal{L}) = \left(\dim T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1) \right), \right. \\ \left. \dim \left(T \left(\frac{L_1}{L_0 L_1}, D_{L_0}(L_1) \right), T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2 \right) \right) \right).$$

Proof. First assume that $\text{Act}_z(\mathcal{L}) = \text{Act}_{pi}(\mathcal{L})$. Since $\text{InnAct}(\mathcal{L}) \subseteq \text{Act}_{pi}(\mathcal{L})$, we get $\mathcal{L}^2 \subseteq Z(\mathcal{L})$. For each $\delta \in \text{Der}_{pi}(L_0, L_1)$, we define the well-defined linear transformation $\psi_\delta : L_0/\text{st}_{L_0}(L_1) \cap Z(L_0) \longrightarrow D_{L_0}(L_1)$ by $\psi_\delta(x_0 + \text{st}_{L_0}(L_1) \cap Z(L_0)) = \delta(x_0)$. One can easily check that the map

$$\psi : \text{Der}_{pi}(L_0, L_1) \longrightarrow T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1) \right)$$

define by $\psi(\delta) = \psi_\delta$ is a one-to-one and onto linear transformation. Thus

$$\dim \text{Der}_{pi}(L_0, L_1) = \dim T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1) \right). \quad (4.1)$$

Also, for each $(\alpha, \beta) \in \text{Der}_{pi}(\mathcal{L})$, the maps $\phi_\alpha : L_1/{}^{L_0}L_1 \longrightarrow D_{L_0}(L_1)$ and $\phi_\beta : L_0/\text{st}_{L_0}(L_1) \cap Z(L_0) \longrightarrow L_0^2$ defined by $\phi_\alpha(x_1 + {}^{L_0}L_1) = \alpha(x_1)$ and $\phi_\beta(x_0 + \text{st}_{L_0}(L_1) \cap Z(L_0)) = \beta(x_0)$, respectively, are well-defined linear transformations. One can easily see that

$$\phi : \text{Der}_{pi}(\mathcal{L}) \longrightarrow \left(T \left(\frac{L_1}{L_0 L_1}, D_{L_0}(L_1) \right), T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2 \right) \right)$$

given by $\phi(\alpha, \beta) = (\phi_\alpha, \phi_\beta)$ is a one-to-one and onto linear transformation. Thus

$$\dim \text{Der}_{pi}(\mathcal{L}) = \dim \left(T \left(\frac{L_1}{L_0 L_1}, D_{L_0}(L_1) \right), T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2 \right) \right). \quad (4.2)$$

From (4.1) and (4.2), it follows that

$$\dim \text{Act}_{pi}(\mathcal{L}) = \left(\dim T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1) \right), \right. \\ \left. \dim \left(T \left(\frac{L_1}{L_0 L_1}, D_{L_0}(L_1) \right), T \left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2 \right) \right) \right).$$

Suppose on the contrary that $\mathcal{L}^2 \subset Z(\mathcal{L})$. Then

$$\dim T\left(\frac{\mathcal{L}}{Z(\mathcal{L})}, \mathcal{L}^2\right) < \dim T\left(\frac{\mathcal{L}}{\mathcal{L}^2}, Z(\mathcal{L})\right),$$

which contradicts the equality of $\text{Act}_{pi}(\mathcal{L})$ and $\text{Act}_z(\mathcal{L})$. Therefore $\mathcal{L}^2 = Z(\mathcal{L})$.

Conversely, assume that $\mathcal{L}^2 = Z(\mathcal{L})$ and

$$\begin{aligned} \dim \text{Act}_{pi}(\mathcal{L}) &= \left(\dim T\left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1)\right), \right. \\ &\quad \left. \dim\left(T\left(\frac{L_1}{L_0 L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2\right)\right) \right). \end{aligned}$$

Since $\mathcal{L}^2 \subseteq Z(\mathcal{L})$, we have

$$\text{Act}_{pi}(\mathcal{L}) \leq \text{Act}_z(\mathcal{L}). \quad (4.3)$$

On the other hand, we have

$$\begin{aligned} \dim \text{Der}_z(L_0, L_1) &= \dim T\left(\frac{L_0}{L_0^2}, {}^{L_0}L_1\right) \\ &= \dim\left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, D_{L_0}(L_1)\right) \\ &= \dim \text{Der}_{pi}(L_0, L_1) \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \dim \text{Der}_z(\mathcal{L}) &= \dim T\left(\frac{L_1}{L_0 L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{L_0^2}, \text{st}_{L_0}(L_1) \cap Z(L_0)\right) \\ &= \dim\left(T\left(\frac{L_1}{L_0 L_1}, D_{L_0}(L_1)\right), T\left(\frac{L_0}{\text{st}_{L_0}(L_1) \cap Z(L_0)}, L_0^2\right)\right) \\ &= \dim \text{Der}_{pi}(\mathcal{L}) \end{aligned} \quad (4.5)$$

From (4.4) and (4.5), we conclude that $\dim \text{Act}_z(\mathcal{L}) = \dim \text{Act}_{pi}(\mathcal{L})$. Since $\text{Act}_{pi}(\mathcal{L}) \leq \text{Act}_z(\mathcal{L})$ by (4.3), it follows that $\text{Act}_z(\mathcal{L}) = \text{Act}_{pi}(\mathcal{L})$. The proof is completed. $\square$

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