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A Geometric Numerical Integration of Lie-Poisson System for Ideal Compressible Isentropic Fluid

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Abstract. In this paper we apply a geometric integrator to the problem of Lie-Poisson system for ideal compressible isentropic fluids (ICIF) numerically. Our work is based on the decomposition of the phase space, as the semidirect product of two infinite dimensional Lie groups. We have shown that the solution of (ICIF) stays in coadjoint orbit and this result extends a similar result for matrix group discussed in [6]. By using the coadjoint action of the Lie group on the dual of its Lie algebra to advance the numerical flow, we (as in [2]) devise methods that automatically stay on the coadjoint orbit. The paper concludes with a concrete example.

Keywords: Ideal compressible isentropic fluid, Lie-Poisson system, Semidirect product, Geometric integration, Coadjoint orbit.

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1. Introduction

Hamiltonian systems have a major role in physics, engineering and chemistry (see e.g., [7, 11]). Lie-Poisson formalism is a well-known generalization of Hamiltonian systems which occurs in many dynamical problems, e.g. the Euler equations for the rigid body, the Vlasov-Poisson equations, the equation for isentropic compressible fluids, ideal magnohydrodynamics, and others (see [1, 11, 14, 15]). The Lie-Poisson equations are normally formulated in the dual space of a Lie algebra $\mathfrak{g}$ and their solutions are settled in the so-called coadjoint orbit which is a nonlinear submanifold of $\mathfrak{g}^*$ (see [7] Ch. 14).
More precisely the Hamiltonian equation on $\mathfrak{g}^*$ is

$$\frac{d\mu}{dt} = ad^*_{\frac{\delta H}{\delta \mu}}(\mu), \quad \mu(0) = \mu_0 \tag{1.1}$$

where $H : \mathfrak{g}^* \longrightarrow \mathbb{R}$ is the Hamiltonian and $\frac{\delta H}{\delta \mu}$ is the functional derivative defined in [7], p.11. Engø and Faltinsen [2] have introduced a numerical integration method of arbitrary order for any Lie-Poisson system in which preserves coadjoint orbits, Casimires, and energy. Their algorithm uses the Runge-Kutta Munthe-Kaas type [12] that render correctly several of the analytic features of the exact solution of Lie-Poisson systems. The equation(1.1) in heavy top problem has been investigated by Engø and Marthinsen (see [3, 4]), who apply Munthe-Kaas and Crouch-Grossman methods for integrating it.
Some dynamical systems which are associated to semidirect product have been introduced in [10], these include the heavy top, compressible fluids, magnohydrodynamics and elasticity problems. In this paper we focus on equation (1.1) for ideal compressible isentropic fluids problem. This problem is formulated on an infinite dimensional semidirect product group $S = \mathcal{D}(\Omega) \circledS_\lambda \mathcal{F}(\Omega)$ where $\mathcal{D}(\Omega)$ and $\mathcal{F}(\Omega)$ are respectively the group of all diffeomorphisms and the group of all real valued smooth functions on a suitable 3-dimensional submanifold of $\mathbb{R}^3$. On the other hand, we have presented an explicit formula computing of the exponential map on semidirect product Lie groups (see[13], proposition 3.3). In this paper we apply this formula and the proposed algorithm [2] for integrating (1.1) on ideal compressible isentropic fluids problem. Our algorithm which is introduced in section 5, preserves the coadjoint orbit and therefore preserves the Casimires invariants. This algorithm can be applied in similar situation, i.e., problems with semidirect product phase spaces. For basic mathematical background we refer the reader to [7]; and for some geometric numerical approaches to [8, 9]. In Section 2 we have discussed some preliminaries and in section 3 the formulation of ideal compressible isentropic fluid problem is presented in detail. The main technical results, used by the algorithm, are presented in Section 4. Numerical algorithm and experiments are included in Section 5.

2. Preliminaries and Notations

2.1. **Ideal compressible isentropic fluid(ICIF).** For the sake of self sufficiency and using the already introduced and known notations and definitions we bring some related paragraphs from [10]. Let Ω be a compact submanifold of $\mathbb{R}^3$ with smooth boundary, filled with a moving fluid free of exterior forces. Let $\eta(X,t)$, where $X \in \Omega$, be the trajectory of fluid particle which at time $t=0$ is at X. Let $\eta(.,t) : \Omega \longrightarrow \Omega$, be a time dependent diffeomorphism of Ω and denote the spatial velocity field of the fluid with $\upsilon(x,t)$, i.e.,

$$\upsilon(\eta(X,t),t) = \frac{\partial \eta(X,t)}{\partial t}.$$

Let $\rho(x,t)$ denote the mass density function of the fluid at time t, and $\rho_0(x) = \rho(x,0)$ the mass density in the reference configuration. Therefore the physical problem of fluid motion has as configuration space the group $\mathcal{D}(\Omega)$ of all diffeomorphisms of Ω, and $\rho(.,t)$ is determined by the configuration when ρ_0 is known. The equations of motion are derived from three fundamental principles: conservation of mass, momentum and energy. By these principles we will have the following equations: (see [10])

$$\partial\rho/\partial t + \mathrm{div}(\rho(.,t)\upsilon) = 0, \tag{2.1}$$

and

$$\frac{\partial \upsilon}{\partial t} + (\upsilon \cdot \nabla)\upsilon = -\frac{1}{\rho}\nabla p, \tag{2.2}$$

with the boundary condition $\upsilon \parallel \partial\Omega$ (no friction exists between fluid and boundary) and the initial condition $\upsilon(x,0) = \upsilon_0(x)$ on Ω. Here p is the pressure function, and also we assume that the fluid is ideal and isentropic. The former means that the force of stress per unit area exerted across a surface element at x, with outward unit normal $\mathbf{n}$ at time t, is $-p(x,t)\mathbf{n}$, and the latter means that the internal energy of the fluid is $\int_\Omega \rho w(\rho)dx$ and $p = \rho^2 w'(\rho)$, where $p'(\rho) > 0$. These hypotheses imply that the total energy, which should be the Hamiltonian of the system, is conserved.

2.2. **Some results in Lie groups.** We bring here some useful results from [7] or [16]. Let G be a Lie group with the associated Lie algebra $\mathfrak{g}$. For any $h \in G$ consider the inner automorphism $I_h : G \longrightarrow G$, $I_h(x) = hxh^{-1}(\forall x \in G)$ and its differential map at unit element $e \in G$, the adjoint operator $Ad_h : \mathfrak{g} \longrightarrow \mathfrak{g}$, i.e., $Ad_h = d_e(I_h)$. If $\mathfrak{g}^*$ be the dual of the space $\mathfrak{g}$, then we have the coadjoint operator $Ad_h^* : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ by the usual definition

$$< Ad_h^*\alpha, \xi >:= < \alpha, Ad_h\xi > \ , \ \forall h \in G, \forall \alpha \in \mathfrak{g}^*, \forall \xi \in \mathfrak{g}.$$

It is well known (see[7]), that $\mathfrak{g}^*$ is foliated by coadjoint orbits $\mathcal{O}_y$, where

$$\mathcal{O}_y = \{Ad_{g^{-1}}^* y | g \in G\} \subset \mathfrak{g}^*, \quad y \in \mathfrak{g}^*.$$

Using the Lie bracket in $\mathfrak{g}$, for any $\xi \in \mathfrak{g}$ we have the operator $ad_\xi : \mathfrak{g} \longrightarrow \mathfrak{g}$, by definition $ad_\xi \eta = [\xi, \eta]$, for all $\eta \in \mathfrak{g}$, and its dual $ad^*_\xi : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is given by

$$\langle ad^*_\xi(\alpha), \eta\rangle = \langle \alpha, ad_\xi \eta\rangle = \langle \alpha, [\xi, \eta]\rangle, \quad \forall\ \xi, \eta \in \mathfrak{g}, \forall \alpha \in \mathfrak{g}^*$$

Also, we need the so-called dexp operator and its inverse. For this, let $\sigma(t) \in \mathfrak{g}$ be a differentiable curve then we have

$$\frac{d}{dt}\exp(\sigma(t)) = \mathrm{dexp}_\sigma(\sigma'(t))\exp(\sigma(t)) = \exp(\sigma(t))\mathrm{dexp}_{-\sigma}(\sigma'(t)), \tag{2.3}$$

where

$$\mathrm{dexp}_\sigma = \frac{\exp(ad_\sigma) - I}{ad_\sigma} = \sum_{j=0}^{\infty} \frac{1}{(j+1)!} ad^j_\sigma. \tag{2.4}$$

It can be shown that for sufficiently small $\sigma(t)$ there exists an inverse operator given by

$$\mathrm{dexp}^{-1}_u(v) = \sum_{k=0}^{\infty} \frac{B_k}{k!} ad^k_u(v), \tag{2.5}$$

where B_k are the Bernoulli numbers, $ad_u(v)$ is the commutator $[u, v]$, and $ad^k_u(v)$ is defined iteratively as $ad^k_u(v) = [u, ad^{k-1}_u(v)]$, (see [2, 12]).

2.3. **Semidirect products.** In this paper, we only need the semidirect product between two Lie groups in which one of them is a real vector space. More precisely let $(V, +)$ be a real vector space which as we know can be regarded as a commutative Lie group which itself is its Lie algebra with vanishing Lie bracket, G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\lambda : G \longrightarrow Aut(V)$ be a (left) smooth representation of the Lie group G and its derivation at identity $\lambda' : \mathfrak{g} \longrightarrow End(V)$ be its induced representation of Lie algebra $\mathfrak{g}$ on vector space V. Denote by $S = G \rtimes_\lambda V$ the semidirect product group of G and V by λ multiplication:

$$(g_1, u_1) \cdot (g_2, u_2) = (g_1 \cdot g_2, u_1 + \lambda(g_1)u_2), \tag{2.6}$$

for all $(g_1, u_1), (g_2, u_2) \in S$, the identity element is $(1_G, 0)$ and the inverse element is given by:

$$(g, u)^{-1} = (g^{-1}, -\lambda(g^{-1})u). \tag{2.7}$$

The corresponding Lie algebra is $\mathfrak{s} = \mathfrak{g} \rtimes_{\lambda'} V$ where $\mathfrak{g}$ is the Lie algebra of G and its Lie bracket is given by:

$$[(\xi_1, \upsilon_1), (\xi_2, \upsilon_2)] = ([\xi_1, \xi_2], \lambda'(\xi_1)\upsilon_2 - \lambda'(\xi_2)\upsilon_1), \tag{2.8}$$

for all $(\xi_1, \upsilon_1), (\xi_2, \upsilon_2) \in \mathfrak{s}$. The adjoint action of S on $\mathfrak{s}$ is given by:

$$Ad_{(g,u)}(\xi, \upsilon) = (Ad_g\xi, \lambda(g)\upsilon - \lambda'(Ad_g\xi)u), \tag{2.9}$$

for all $(g, u) \in S$ and all $(\xi, \upsilon) \in \mathfrak{s}$. Also, we have the following formula for the exponential map on $\mathfrak{s}$.

Proposition 2.1. *([13], Proposition 3.3) Let $(\xi, \upsilon) = (\xi(t), \upsilon(t))$ be a curve in $\mathfrak{s}$ such that emanates from the origin and is continuous at $t = 0$, then for sufficiently small $t > 0$ we have:*

$$\exp(\xi, \upsilon) = (\exp \xi, \lambda(\exp \xi)z)$$

where $z = z(t)$ comes from the so-called Baker-Campbell-Hausdorff formula and its first few terms are:

$$z = \upsilon - \frac{1}{2}\lambda'(\xi)\upsilon + \frac{1}{6}\lambda'^2(\xi)\upsilon + \frac{1}{12}[\upsilon, \lambda'(\xi)\upsilon] + \cdots . \tag{2.10}$$

For the Baker-Campbell-Hausdorff formula e.g., see [16].

3. Formulation of the Governing Equations

For the formulation of the ICIF problem [10] , consider the equation (1.1), let $S = \mathcal{D}(\Omega) \rtimes_\lambda \mathcal{F}(\Omega)$ where $\mathcal{D}(\Omega)$ is the Lie group of all diffeomorphisms of Ω, $\mathcal{F}(\Omega)$ is the vector space of all smooth real-valued functions on Ω and $\lambda : \mathcal{D}(\Omega) \longrightarrow Aut(\mathcal{F}(\Omega))$ is the action of $\mathcal{D}(\Omega)$ on $\mathcal{F}(\Omega)$ with definition:

$$\lambda(\eta)(f) = fo\eta^{-1}, \tag{3.1}$$

for all $\eta \in \mathcal{D}(\Omega)$ and $f \in \mathcal{F}(\Omega)$. Therefore the group action is:

$$(\eta_1, f_1).(\eta_2, f_2) = (\eta_1 o \eta_2, f_1 + f_2 o \eta_1^{-1}) \tag{3.2}$$

for all $(\eta_i, f_i) \in S, i = 1, 2$, and the related Lie algebra is $\mathfrak{s} = \mathfrak{X}(\Omega) \rtimes_{\lambda'} \mathcal{F}(\Omega)$ where $\mathfrak{X}(\Omega)$ is the Lie algebra of all smooth vector fields on Ω and $\lambda' : \mathfrak{X}(\Omega) \longrightarrow End(\mathcal{F}(\Omega))$, and the differential map of λ at id_Ω is given by (see [10]):

$$\lambda'(X)(f) = -L_X f = -\nabla f \cdot X \tag{3.3}$$

for all $(X, f) \in \mathfrak{s}$. Since the Lie algebra bracket on $\mathcal{D}(\Omega)$ is minus the Jacobi-Lie bracket of vector fields(see [7], p. 324), the Lie bracket in $\mathfrak{s}$ is given by:

$$[(X_1, f_1), (X_2, f_2)] = ([X_1, X_2], \nabla f_1 \cdot X_2 - \nabla f_2 \cdot X_1) \tag{3.4}$$

for all $(X_i, f_i) \in \mathfrak{s}, i = 1, 2$. The exponential map is defined as in Proposition 2.1 and for the exponential map in the first component, i.e., $\exp : \mathfrak{X}(\Omega) \longrightarrow \mathcal{D}(\Omega)$ we can use the solution of the following first order differential equation

$$\frac{d\phi(x,t)}{dt} = X(\phi(x,t)), \quad \phi(x,0) = x, \tag{3.5}$$

since (e.g. see [11, p.17])

$$\exp X(x) := \phi(x, 1), \quad x \in \Omega. \tag{3.6}$$

Moreover the topological dual of $\mathfrak{s}$ as a topological vector space is $\mathfrak{s}^* = \mathfrak{X}(\Omega)^* \times \mathcal{F}(\Omega)^*$ where $\mathfrak{X}(\Omega)^* = \Lambda^1(\Omega)$ is the space of all one-forms and $\mathcal{F}(\Omega)^*$ is the

space of all densities on Ω (see[10]) and $\mathfrak{s}^*$ acts naturally on $\mathfrak{s}$ by the Lebesgue integration pairing, i.e.,

$$\begin{aligned}\langle (M,\rho)\,,\,(X,f)\rangle_{(\mathfrak{s}^*,\mathfrak{s})} &:= \langle M, X\rangle_{(\mathfrak{X}(\Omega)^*,\mathfrak{X}(\Omega))} + \langle \rho, f\rangle_{(\mathcal{F}(\Omega)^*,\mathcal{F}(\Omega))} \\ &= \int_\Omega M(x)\cdot X(x)\,dx + \int_\Omega f(x)\rho(x)\,dx\end{aligned} \tag{3.7}$$

for all $(M,\rho)\in\mathfrak{s}^*$ and for all $(X,f)\in\mathfrak{s}$. Notice that the members of $\mathfrak{X}(\Omega)$ and $\mathfrak{X}(\Omega)^*=\Lambda^1(\Omega)$ can be considered as 3-dimensional vector fields on Ω and also the members of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Omega)^*$ can be considered as real valued functions on Ω. On the other hand the Hamiltonian function $H:\mathfrak{s}^*\longrightarrow\mathbb{R}$ is given by(see[10]):

$$H(M,\rho)=\frac{1}{2}\int_\Omega \frac{1}{\rho(x)}\|M(x)\|^2dx+\int_\Omega \rho(x)w(\rho(x))dx$$

for all $(M,\rho)\in\mathfrak{s}^*$, where $M(x)=\rho(x)\upsilon(x)$ here represents the momentum density (see[10] for more details). Now, we can compute the functional derivation $\frac{\delta H}{\delta\mu}\in\mathfrak{s}$, where $\mu=(M,\rho)\in\mathfrak{s}^*$, that it appears in the Lie-Poisson equation (1.1).

Lemma 3.1. *With the above notations we have:*

$$\frac{\delta H}{\delta\mu}=\Big(\frac{M(x)}{\rho(x)}\ ,\ -\frac{\|M(x)\|^2}{2\rho^2(x)}+w(\rho(x))+\rho(x)w'(\rho(x))\Big).$$

Proof. Since each component of $\mathfrak{s}^*=\mathfrak{X}(\Omega)^*\times\mathcal{F}(\Omega)^*$ acts on $\mathfrak{s}=\mathfrak{X}(\Omega)\rtimes_{\lambda'}\mathcal{F}(\Omega)$ separately, we can compute $\frac{\delta H}{\delta M}$ and $\frac{\delta H}{\delta\rho}$ where $\mu=(M,\rho)\in\mathfrak{s}^*$, and then we will have $\frac{\delta H}{\delta\mu}=(\frac{\delta H}{\delta M},\frac{\delta H}{\delta\rho})$. In the first component:

$$\begin{aligned}\langle\frac{\delta H}{\delta M},\delta M\rangle &= \lim_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon}[H(M+\varepsilon\ \delta M,\rho)-H(M,\rho)]\\ &= \lim_{\varepsilon\longrightarrow 0}\frac{1}{2\varepsilon}\int_\Omega\big(\frac{1}{\rho(x)}\|(M+\varepsilon\ \delta M)(x)\|^2-\frac{1}{\rho(x)}\|M(x)\|^2\big)dx\\ &= \int_\Omega\frac{1}{\rho(x)}M(x)\cdot\delta M(x)dx\\ &= \langle\frac{M}{\rho},\delta M\rangle.\end{aligned}$$

Hence $\frac{\delta H}{\delta M}=\frac{M}{\rho}\in\mathfrak{X}(\Omega)$ and in the second component:

$$\begin{aligned}\langle\frac{\delta H}{\delta\rho},\delta\rho\rangle &= \lim_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon}[H(M,\rho+\varepsilon\ \delta\rho)-H(M,\rho)]\\ &= -\frac{1}{2}\int_\Omega\frac{\|M(x)\|^2}{\rho^2(x)}\ \delta\rho(x)\,dx\\ &+\lim_{\varepsilon\longrightarrow 0}\frac{1}{\varepsilon}\int_\Omega\big((\rho+\varepsilon\ \delta\rho)(x)w(\rho(x)+\varepsilon\ \delta\rho(x))-\rho(x)w(\rho(x))\big)dx\\ &= -\frac{1}{2}\int_\Omega\frac{\|M(x)\|^2}{\rho^2(x)}\delta\rho(x)\,dx+\int_\Omega(w(\rho(x))+\rho(x)w'(\rho(x)))\,\delta\rho(x)dx.\end{aligned}$$

Using suitable choice of $\delta\rho$, and by a convergence theorem (e.g. Lebesgue dominated convergence theorem), we have interchanged the limit and the integral and then in the last equality, for computing the limit which is placed into integral, l'Hôpital's rule has been applied. Finally, we have

$$\frac{\delta H}{\delta \rho} = -\frac{\|M(x)\|^2}{2\rho^2(x)} + w(\rho(x)) + \rho(x)w'(\rho(x)),$$

which completes the proof. □

4. Main Results

In this section, we state and prove our main results which make it possible to obtain Algorithm 1 in the next section. The following theorem asserts that the solution of the equation (1.1) of ICIF under the hypothesis explained in the Section 3, belongs to the coadjoint orbit of the initial value. The proof is similar to the proof of the Theorem 1 in [2] and hence is omitted.

Theorem 4.1. *Let $\mu(x,t) = (M(x,t), \rho(x,t)) \in \mathfrak{s}^*$, the solution of the equation (1.1) with the initial value $\mu_0 = (M(x,0), \rho(x,0)) \in \mathfrak{s}^*$ on the group $S = \mathcal{D}(\Omega) \rtimes_\lambda \mathcal{F}(\Omega)$ be given by*

$$(M(x,t), \rho(x,t)) = Ad^*_{\exp(-X(t), -f(t))}(\mu_0) \tag{4.1}$$

then

$$\sigma^{'}(t) = \mathrm{dexp}^{-1}_{\sigma(t)}(\frac{\delta H}{\delta \mu}) \tag{4.2}$$

where $\sigma(t) = (X(t), f(t))$.

One can rewrite (4.1) as follows:

$$\mu(x,t) = Ad^*_{(\exp(X(t),f(t)))^{-1}}(\mu_0) \ \in \ \mathcal{O}_{\mu_o} = \{Ad^*_{s^{-1}}\mu_0 | s \in S\}, \tag{4.3}$$

hence the solution $\mu(x,t)$ of (1.1) lies on the coadjoint orbit $\mathcal{O}_{\mu_o}$. Therefore this theorem generalizes the Theorem 5.6 in [6], p. 287, and Theorem 1 in [2]. In order to use the formula (4.1), we need an explicit formula for Ad^* operator as follows.

Lemma 4.2. *Let $(\eta, f) \in S$ and $(M, \rho) \in \mathfrak{s}^*$, then*

$$Ad^*_{(\eta,f)}(M,\rho) = (\ \mathrm{J}\eta\,[D\eta]^T\Big(M\circ\eta + (\rho\circ\eta)(\nabla f\ \circ\eta)\Big)\ ,\ (\rho\circ\eta)\mathrm{J}\eta\ ). \tag{4.4}$$

where $D\eta$, $\mathrm{J}\eta$ and ∇f are derivative matrix of η, the Jacobian of η and the gradient of f, respectively.

Proof. From the pairing (3.7) between $\mathfrak{s}^*$ and $\mathfrak{s}$, (2.9) , (3.1) and the following formula (see [7], p. 324)

$$Ad_\eta X = D\eta \circ X \circ \eta^{-1},$$

we have:

$$\begin{aligned}\langle Ad^*_{(\eta,f)}(M,\rho)\,,\,(X,g)\rangle &= \langle (M,\rho)\,,\,Ad_{(\eta,f)}(X,g)\rangle\\ &= \langle (M,\rho)\,,\,(D\eta\circ X\circ\eta^{-1}\,,\,g\circ\eta^{-1}+\nabla f\cdot(D\eta\circ X\circ\eta^{-1}))\rangle\\ &= \int_\Omega M(x)\cdot[D_{\eta^{-1}(x)}\eta]\,X(\eta^{-1}(x))\,dx\\ &+ \int_\Omega \rho(x)\big(g(\eta^{-1}(x))+\nabla f(x)\,\cdot\,[D_{\eta^{-1}(x)}\eta]X(\eta^{-1}(x))\big)\,dx,\end{aligned}$$

for any $(X,g)\in\mathfrak{s}$, where " $\cdot$ " denotes the inner product between vectors in $\mathbb{R}^3$. Now, by letting $\eta^{-1}(x)=t$, we obtain

$$\begin{aligned}\langle Ad^*_{(\eta,f)}(M,\rho)\,,\,(X,g)\rangle &= \int_\Omega M(\eta(t))\cdot[D_t\eta]\,X(t)\,\mathrm{J}_t\eta\,dt\\ &+ \int_\Omega \rho(\eta(t))\big(g(t)+\nabla f(\eta(t))\,\cdot\,[D_t\eta]X(t)\big)\,\mathrm{J}_t\eta\,dt\\ &= \int_\Omega \mathrm{J}_t\eta\,[D_t\eta]^T\Big(M(\eta(t))\,+\,\rho(\eta(t))\nabla f(\eta(t)\Big)\,\cdot\,X(t)\,dt\\ &+ \int_\Omega \mathrm{J}_t\eta\rho(\eta(t))g(t)\,dt\\ &= \langle(\,\mathrm{J}\eta\,[D\eta]^T\Big(M\circ\eta+(\rho\circ\eta)(\nabla f\,\circ\eta)\Big)\,,\,(\rho\circ\eta)\mathrm{J}\eta\,)\,,\,(X,g)\rangle\end{aligned}$$

which completes the proof. □

Now, by applying the above Lemma we will have the following corollary.

Corollary 4.3. *For all $(M,\rho)\in\mathfrak{s}^*$ and for all $(X,f)\in\mathfrak{s}$ sufficiently near $(0,0)$, we have:*

$$\begin{aligned}Ad^*_{\exp(-X,-f)}(M,\rho) = \Big(&\mathrm{J}\exp(-X)[D\exp(-X)]^T\big(M\circ\exp(-X)-(\rho\circ\exp(-X))(\nabla Z\circ\exp(-X))\big)\,,\\ &\rho\circ\exp(-X)\mathrm{J}\exp(-X)\Big),\end{aligned}$$

where Z comes from Baker-Campbell-Hausdorff formula and its first few terms are

$$Z = f+\frac{1}{2}\nabla f\cdot X+\frac{1}{6}\nabla(\nabla f\cdot X)\cdot X+\cdots \tag{4.5}$$

Proof. From (2.7), (3.1) and Proposition 2.1

$$\begin{aligned}\exp(-X,-f) &= \exp(X,f)^{-1} = (\exp X\,,\,Z\circ\exp(-X))^{-1}\\ &= (\exp(-X)\,,\,-Z\circ\exp(-X)\circ\exp X)\\ &= (\exp(-X),-Z)\end{aligned} \tag{4.6}$$

where Z is similar as (2.10). Now the conclusion follows from Lemma 4.2. □

5. Numerical Algorithm and an Example

Consider the equation (1.1) under those circumstances which explained in the Section 3. Now, using the results of the Section 4, we can present the following numerical algorithm for solving the equation (1.1).

Algorithm 1

1. Given $\Omega\,, w\,, \mu_0 = (M_0, \rho_0)$ as in the Subsection 2.1 and Section 3, also let $X_0 = 0$ and $f_0 = 0$. Fix $x \in \Omega$ and $t > 0$ sufficiently small (by virtue of Proposition 2.1) we are going to compute $\mu(x,t) = (M(x,t), \rho(x,t))$, so let h be an appropriate stepsize.

2. Using Lemma 3.1, calculate $\dfrac{\delta H}{\delta \mu}$.

3. Truncate the series (2.5) for dexp^{-1}, using one of the known methods in [5, 12], e.g. Lie forward Euler, Lie trapezoidal rule, Runge-Kutta, etc, solve the equation (4.2) to obtain $\sigma(t) = (X(t), f(t))$.

4. Calculate $\exp(-X(t), -f(t)) = (\exp(-X), -Z)$; see (4.6). For this aim, solve the differential equation (3.5) and use (3.6) to obtain $\exp(-X(t))$ by an appropriate numerical method and truncate the series (4.5) to approximate Z.

5. Caculate $Ad^*_{\exp(-X(t),-f(t))}(M, \rho)$, using the Corollary 4.3.

More concretely, let $0 < t < T$ and $N = [\dfrac{T}{h}]$, now the Algorithm's steps can be summarized as follows.

Algorithm 1

Input: $\Omega\,, w\,, \mu_0$
Output: μ_N

for n=0 to N-1

$\dfrac{\delta H_{n+1}}{\delta \mu_{n+1}} = \Big(\frac{M_n(x)}{\rho_n(x)} \ , \ -\frac{\|M_n(x)\|^2}{2\rho_n^2(x)} + w(\rho_n(x)) + \rho_n(x) w'(\rho_n(x)) \Big)$

$(X_{n+1}, f_{n+1}) = \sigma_{n+1}$ form (4.2) by a Runge-Kutta method

$\exp(-X_{n+1})$ from (3.5) by a numerical or analytic method

$$Z_{n+1} = f_{n+1} + \tfrac{1}{2}\nabla f_{n+1}\cdot X_{n+1} + \tfrac{1}{6}\nabla(\nabla f_{n+1}\cdot X_n)\cdot X_{n+1} + \cdots \text{ (truncating)}$$

$$(M_{n+1}, \rho_{n+1}) = \Big(\mathrm{J}\exp(-X_{n+1})[D\exp(-X_{n+1})]^T\big(M_n\circ\exp(-X_{n+1}) - (\rho_n\circ\exp(-X_{n+1}))$$
$$(\nabla Z_{n+1}\circ\exp(-X_{n+1}))\big), \mathrm{J}\exp(-X_{n+1})\rho_n\circ\exp(-X_{n+1})\Big)$$

end for

Now, let us illustrate our approach to a concret example.

Example 1. Let Ω be the cylinder $0 \le x \le L$ and $y^2 + z^2 \le r^2$ where $0 < r \ll L$. Let $\upsilon_0(x,y,z) = (L-x,0,0)$, $\rho_0(x,y,z) = x + L$ and $M_0(x,y,z) = (L^2 - x^2, 0, 0)$ for $(x,y,z) \in \Omega$, be initial spatial velocity, initial mass density and initial momentum density of the fluid, respectively. Notice that $M = \rho\upsilon$. As usual we assume that the fluid is ideal and isentropic. Let $w(\rho) = \rho$, so $p = \rho^2 w'(\rho) = \rho^2$ is the pressure function. For a numerical experiment let $L = 1$. Now, we implement the Algorithm 1, input valus are $M_0 = 1 - x^2$ and $\rho_0 = 1 + x$. More precisely, we present the first iteration as below

$$\frac{\delta H_1}{\delta \mu_1} = \frac{\delta H}{\delta \mu}(\mu_0) = \big(\begin{bmatrix}1-x & 0 & 0\end{bmatrix}^T, \frac{1}{2}(-x^2+6x+3)\big).$$

We truncate the series (2.5) to its first term, then the equation (4.2)becomes $\sigma(t) = h\frac{\delta H}{\delta \mu}(\mu_0)$, we are assuming $h = 0.1$, this leads to

$$(X_1, f_1) = \sigma_1 = \big(\begin{bmatrix}0.1-0.1x & 0 & 0\end{bmatrix}^T, \frac{1}{20}(-x^2+6x+3)\big).$$

Solving the system (3.5) that is

$$\frac{d\phi\big((x,y,z),t\big)}{dt} = \big(\frac{d\phi_1}{dt}, \frac{d\phi_2}{dt}, \frac{d\phi_3}{dt}\big) = (0.1 - 0.1\phi_1, 0, 0),,$$

with initial value $\phi((x,y,z),0) = (x,y,z)$ and considering (3.6), we have

$$\exp(-X_1) = \begin{bmatrix}1 + (x-1)e^{0.1} & y & z\end{bmatrix}^T.$$

Truncating the series (4.5) to its second term and calculating these terms of Z_1, we have

$$Z_1 \simeq f_1 + \frac{1}{2}\nabla f_1.X_1 = \frac{1}{200}(-9x^2+56x+33) \text{ and } \nabla Z_1 = \begin{bmatrix}\frac{1}{100}(28-9x) & 0 & 0\end{bmatrix}^T,$$

moreover

$$[D\exp(-X_1)]^T = \mathrm{diag}(e^{0.1}, 1, 1), \quad \mathrm{J}\exp(-X_1) = e^{0.1},$$

and

$$M_0 \circ \exp(-X_1) = \begin{bmatrix}1 - (1 + (x-1)e^{0.1})^2 & 0 & 0\end{bmatrix}^T.$$

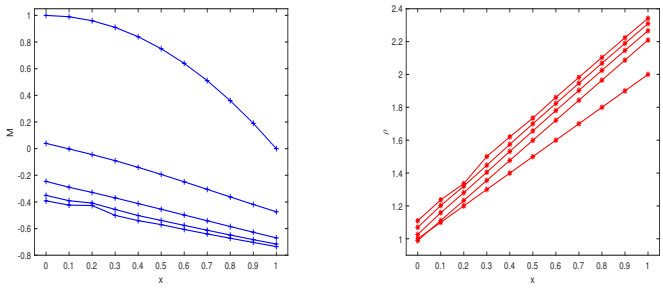
Therefore, for $0 \le x \le 1$ we have

$$M_1 = \begin{bmatrix}-0.01e^{0.2}((x-1)^2e^{0.2} + (x-1)e^{0.1} + 2)(19 - 9(x-1)e^{0.1}) & 0 & 0\end{bmatrix}^T$$

and

$$\rho_1 = e^{0.1}(2 + (x-1)e^{0.1}).$$

The approximation of other iterations up to $n = 4$ are computing with a



(A) The solution values of M_n. (B) The solution values of ρ_n.

FIGURE 1. Five iterations of M_n and ρ_n in Example 1 obtained from Algorithm 1.

Matlab program and are given in the Figure 1. We assume that the stepsize is $\Delta t = 0.1$ and we divide the interval of $0 \leqslant x \leqslant 1$ to 10 subdivisions. For computing of ϕ, we use of Euler method with $t = 0, 0.1, \cdots, 1$ and for initial values $x = 0, 0.1, \cdots, 1$. Here, we use linear interpolation for approximating of functions, say Z_i, M_i and ρ_i for $i = 0, \cdots, 4$.
The left side figure in Figure 1, shows five iterations of M_i, $i = 0, \cdots, 4$ in which the uppermost curve is M_0; and the right side figure of Figure 1 is associated to ρ_i, $i = 0, \cdots, 4$, in which the lowest curve is ρ_0. As we see, for sufficiently large n the iterations reach to a satisfactory solution for μ_n.

Error growth. In order to estimate the error growth of the Algorithm 1, recall that we have used the following approximations. The truncation of the series (2.5) to its first q terms induces an error of size $O(h^{q+1})$, see the proof of the Theorem 8.5, in [6], p.126. Solving (4.2) via the Euler method induces an error of size $O(h^2)$ as well. Also, if we apply the Simpson's rule for the numerical integrating of the separable differential equation (3.5), we have an error of size $O(h^4)$. Finally, truncating (2.5) to its first term, i.e., our algorithm displays a quadratic growth.

Conclusion. Using the decomposition of the phase space to semidirect product of two Lie groups, we have proposed a numerical method for solving ideal compressible isentropic fluids problem. The most important property of our proposed method is that the obtained approximated solutions stay in the coadjoint orbit and hence conserve the Casimires. Finally, we showed that our method

display a quadratic growth error. Investigating a method that simultaneously preserves energy and Casimires is a topic for future works.

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