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Some New Uniqueness Results of Solutions for Fractional Volterra-Fredholm Integro-Differential Equations

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Abstract. This paper establishes a study on some important latest innovations in the uniqueness of solution for Caputo fractional Volterra-Fredholm integro-differential equations. To apply this, the study uses Banach contraction principle and Bihari's inequality. A wider applicability of these techniques are based on their reliability and reduction in the size of the mathematical work.

Keywords: Caputo fractional derivative, Volterra-Fredholm integro-differential equation, Banach contraction principle, Bihari's inequality.

2010 Mathematics subject classification: 26A33, 34A12, 26D10.

1. Introduction

In the fractional calculus the various integral inequalities plays an important role in the study of qualitative and quantitative properties of solution of differential and integral equations.

In recent years, many authors focus on the development of techniques for discussing the solutions of fractional integro-differential equations. For instance, we can remember the following works:

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Ahmad and Sivasundaram [1] studied some existence and uniqueness results in a Banach space for the fractional integro-differential equation (1.1) with nonlinear condition $h(x_0) = h_0 - f(x)$. Momani et al.[2], proved the Local and global uniqueness result by using Bihari's inequality for the fractional integro-differential equation (1.1) with the initial condition $h(x_0) = h_0$, Wu and Liu [3] discussed the existence and uniqueness of solutions for fractional integro-differential equations (1.1) with conditions $h(x_0)+f(x) = h_0$. Karthikeyan and Trujillo [4], proved existence and uniqueness of solutions for fractional integro-differential equations with boundary value conditions.

$$^{c}D^{\alpha}h(x) = g(x, h(x)) + \int_{x_0}^{x} K(x, t, h(t))dt, \qquad 0 < \alpha \leq 1, \tag{1.1}$$

Recently, in [2, 5] the author's obtained the result on uniqueness of solutions for fractional integro-differential with initial condition using the Bihari's inequality.

Motivated by above work, in this paper we discuss new uniqueness results for Caputo fractional Volterra-Fredholm integro-differential equation of the form [6, 7]:

$$^{c}D^{\alpha}h(t) = f(t)h(t)+g(t, h(t))+\int_{t_0}^{t} Z_1(t, s, h(s))ds+\int_{t_0}^{b} Z_2(t, s, h(s))ds, \tag{1.2}$$

with the initial condition

$$h(t_0) = h_0, \tag{1.3}$$

where $^{c}D^{\alpha}$ is the Caputo's fractional derivative, $0 < \alpha \leq 1$ and $h : J \longrightarrow \mathbb{R}$, where $J = [t_0, b]$ is the continuous function which has to be determined, $g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $Z_i : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}, i = 1, 2$ are continuous functions.

The main objective of the present paper is to study the new uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential equation.

The rest of the paper is organized as follows: In Section 2, some preliminaries, basic definitions and Lemma related to fractional calculus are recalled. In Section 3, the new uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential equation have been proved. Finally, we will give a report on our paper and a brief conclusion is given in Section 4.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The fractional derivative and applications have been addressed extensively by several researchers. For example, we refer the reader to [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 24] and the references cited therein. In this section, we show the most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [7, 8, 19, 20, 21, 22, 23]. Let $C(J,\mathbb{R})$ is the Banach space endowed with the infinity norm $\|h\|_\infty = \sup\{|h(x)| : x \in J = [t_0, b]\}$, for any $h \in C(J,\mathbb{R})$.

Definition 2.1. [25] **(Riemann-Liouville fractional integral).** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function h is defined as

$$\begin{aligned} J^\alpha h(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} h(t)dt, \qquad x > 0, \quad \alpha \in \mathbb{R}^+, \\ J^0 h(x) &= h(x), \end{aligned}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. [25] **(Caputo fractional derivative).** The fractional derivative of $h(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^cD_x^\alpha h(x) &= J^{m-\alpha} D^m h(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m h(t)}{dt^m} dt, & m-1 < \alpha < m, \\ \frac{d^m h(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{2.1}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

(1) $J^\alpha J^v h = J^{\alpha+v} h, \quad \alpha, v > 0.$
(2) $J^\alpha h^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} h^{\beta+\alpha},$
(3) $D^\alpha h^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} h^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0.$
(4) $J^\alpha D^\alpha h(x) = h(x) - \sum_{k=0}^{m-1} h^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 2.3. [25] **(Riemann-Liouville fractional derivative).** The Riemann Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha h(x) = D^m J^{m-\alpha} h(x), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.2}$$

Lemma 2.4. [26] ***(Banach contraction principle).*** *Let (X,d) be a complete metric space, then each contraction mapping $\Psi : X \longrightarrow X$ has a unique fixed point x of Ψ in X i.e. $\Psi x = x$.*

Lemma 2.5. [2] ***(Bihari's inequality).*** *Let $g : [0,+\infty) \longrightarrow (0,+\infty)$ is continuous and monotone-increasing. h: $[a,b] \longrightarrow \mathbb{R}^+$ be a continuous function that satisfies the inequality*

$$h(t) \leq a + \int_{t_0}^t K(s) g(h(s)) ds.$$

Then the following inequality hold

$$h(t) \leq \theta^{-1}\left[\theta(a) + \int_{t_0}^{t} K(s)ds.\right]$$

where $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ *is a primitive of* $\frac{1}{g(t)}$, *ie.* $\theta(x) = \int_{u_o}^{u} \frac{ds}{g(s)}$, $x \in \mathbb{R}$.

3. Main Results

In this section, we will display and prove the uniqueness results of problem $(1.2)-(1.3)$. Before starting and proving our main results, we present the following lemma and some useful hypotheses:

(H1): $Z_1, Z_2 : J \times J \times \mathbb{R} \to \mathbb{R}$ are continuous on $D = \{(t,s) : 0 \leq t_0 \leq s \leq t \leq b\}$ such that

$$\int_{s}^{t} |Z_1(\tau, s, h_1(s)) - Z_1(\tau, s, h_2(s))|dt \leq L_{z_1}\lambda|h_1(s) - h_2(s)|$$

$$\int_{s}^{b} |Z_2(\tau, s, h_1(s)) - Z_2(\tau, s, h_2(s))|dt \leq L_{z_2}\lambda|h_1(s) - h_2(s)|$$

(H2): The function $g : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

$$|g(t, h_1) - g(t, h_2)| \leq \lambda|h_1 - h_2|$$

(H3): The function $f : J \to \mathbb{R}$ is continuous.

where $h_1, h_2 \in C(J, \mathbb{R})$, $\lambda : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is nondecreasing continuous function with $\lambda(0) = 0$ and $\int_0^R \frac{dt}{\lambda(t)} = +\infty$, $0 < t < 1$, and L_{z_1}, L_{z_2} are positive constants.

Lemma 3.1. *If* $h_0(t) \in C(J, \mathbb{R})$, *then* $h(t) \in C(J, \mathbb{R}^+)$ *is a solution of the problem* $(1.2)-(1.3)$ *iff* h *satisfying*

$$\begin{aligned} h(t) &= h_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)h(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,h(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(\int_{s}^{t} Z_1(\tau, s, h(s))d\tau + \int_{s}^{b} Z_2(\tau, s, h(s))d\tau\right)ds, \end{aligned} \tag{3.1}$$

for $t \in J$.

Proof. It can be proved easily by applying the integral operator (2.1) to both sides of (1.2) to get the integral equation (3.1). □

Our first result depends on Bihari's inequality.

Theorem 3.2. *Assume that (H1)–(H3) hold. If*

$$\frac{\|f\|_\infty b^\alpha}{\Gamma(\alpha+1)} < 1. \tag{3.2}$$

Then there exists a unique solution $h(t) \in C(J)$ *to* $(1.2)-(1.3)$.

Proof. By Lemma 3.1. we know that a function h is a solution to (1.2) – (1.3) iff h satisfies

$$\begin{aligned} h(t) &= h_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f(s)h(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} g(s,h(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\left(\int_s^t Z_1(\tau,s,h(s))d\tau + \int_s^b Z_2(\tau,s,h(s))d\tau\right)ds. \end{aligned}$$

Let $h_1, h_2 \in C(J,\mathbb{R})$ and for any $t \in J$ such that

$$\begin{aligned} h_1(t) &= h_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f(s)h_1(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} g(s,h_1(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\left(\int_s^t Z_1(\tau,s,h_1(s))d\tau + \int_s^b Z_2(\tau,s,h_1(s))d\tau\right)ds. \end{aligned}$$

and

$$\begin{aligned} h_2(t) &= h_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} f(s)h_2(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} g(s,h_2(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\left(\int_s^t Z_1(\tau,s,h_2(s))d\tau + \int_s^b Z_2(\tau,s,h_2(s))d\tau\right)ds. \end{aligned}$$

Consequently, by (H1), (H2) and (H3), then for $t \in J$, we have

$$\begin{aligned} |h_1(t)-h_2(t)| &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}|f(s)||h_1(t)-h_2(t)|ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}|g(s,h_1(s))-g(s,h_2(s))|ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\Big(\int_s^t |Z_1(\tau,s,h_1(s))-Z_1(\tau,s,h_2(s))|d\tau \\ &+\int_s^b |Z_2(\tau,s,h_1(s))-Z_2(\tau,s,h_2(s))|d\tau\Big)ds. \\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\|f\|_\infty|h_1(s)-h_2(s)|ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\lambda|h_1(s)-h_2(s)|ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\Big(L_{z_1}\lambda|h_1(s)-h_2(s)|+L_{z_2}\lambda|h_1(s)-h_2(s)|\Big)ds. \\ &\le \frac{\|f\|_\infty b^\alpha}{\Gamma(\alpha+1)}|h_1(t)-h_2(t)| + \frac{(1+L_{z_1}+L_{z_2})}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\lambda|h_1(s)-h_2(s)|ds. \end{aligned}$$

Thus

$$|h_1(t) - h_2(t)| \quad \leq \quad \sigma + \frac{(1 + L_{z_1} + L_{z_2})}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \lambda |h_1(s) - h_2(s)| ds.$$

where $\sigma > 0$, now we can apply Bihari's inequality to obtain

$$|h_1(t) - h_2(t)| \leq \theta^{-1}\Big[\theta(\sigma) + \frac{Nb^\alpha}{\alpha\Gamma(\alpha)}\Big] \tag{3.3}$$

where $\theta(h)$ is a primitive of the function $\frac{1}{\theta(h)}$, and θ^{-1} denotes the inverse of θ and $N = \frac{(1+L_{z_1}+L_{z_2})}{1-\|f\|_\infty}$. It follows that $\theta^{-1}\Big[\theta(\sigma) + \frac{Nb^\alpha}{\alpha\Gamma(\alpha)}\Big] \longrightarrow 0$. We shall prove that the right-hand side of (3.3) tends toward zero as $\sigma \longrightarrow 0$. Since $|h_1(t) - h_2(t)|$ is independent of σ, it follows that $h_1(t) = h_2(t)$. So, $h(t) \in C(J, \mathbb{R})$ is the unique solution of the initial value problem $(1.2) - (1.3)$ and the proof is completed.

□

We shall next discuss another uniqueness result for the initial value problem $(1.2) - (1.3)$ using the Banach contraction principle.

Before starting and proving we introduce the new following hypotheses:

(I): $Z_1, Z_2 : J \times J \times \mathbb{R} \to \mathbb{R}$ are continuous on $D = \{(t,s) : 0 \leq t_0 \leq s \leq t \leq b\}$ such that

$$|Z_1(\tau, s, h_1(s)) - Z_1(\tau, s, h_2(s))| dt \leq L^*_{z_1} \|h_1(s) - h_2(s)\|$$

$$|Z_2(\tau, s, h_1(s)) - Z_2(\tau, s, h_2(s))| dt \leq L^*_{z_2} \|h_1(s) - h_2(s)\|$$

(II): The function $g : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

$$|g(t, h_1) - g(t, h_2)| \leq L^*_g \|h_1 - h_2\|$$

(III): The function $f : J \to \mathbb{R}$ is continuous.

where $L^*_{z_1}, L^*_{z_2}$ and L^*_g are positive constants.

Theorem 3.3. *Assume that the hypotheses (I)–(III) are satisfied. And let β and γ be two positive real numbers such that $0 < \beta < 1$ and*

$$\Big[\frac{\|f\|_\infty + L^*_g}{\Gamma(\alpha+1)} + \frac{(L^*_{z_1} + L^*_{z_2})b}{\alpha + 1\Gamma(\alpha)}\Big] b^\alpha = \beta,$$

$$|h_0| + \Big[\frac{g_0}{\Gamma(\alpha+1)} + \frac{(z^*_1 + z^*_2)b}{\alpha + 1\Gamma(\alpha)}\Big] b^\alpha = (1-\beta)\gamma.$$

Then the initial value problem $(1.2) - (1.3)$ has a unique solution continuous on $[t_0, b]$, where $g_0 = \max\{|g(s,0)| : s \in J\}$, $z^*_1 = \max\{|Z_1(\tau, s, 0)| : (\tau, s) \in D\}$ and $z^*_2 = \max\{|Z_2(\tau, s, 0)| : (\tau, s) \in D\}$.

Proof. Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined by

$$\begin{aligned}(Th)(t) \quad = \quad & h_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)h(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,h(s))ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(\int_{s}^{t}Z_1(\tau,s,h(s))d\tau + \int_{t_0}^{b}Z_2(\tau,s,h(s))d\tau\right)ds,\end{aligned}$$

and define $\Phi_\gamma = h \in C(J,\mathbb{R}) : \|h\|_\infty \leq \gamma$ for some $\gamma > 0$. Now, we need to prove that the operator T has a fixed point on $\Phi_\gamma \subset C(J,\mathbb{R})$. This fixed point is the unique solution of the initial value problem $(1.2)-(1.3)$. In order that, we present the proof in two steps:

Step 1. We need to prove that the operator $T\Phi_\gamma \subset \Phi_\gamma$.
By the above hypotheses, then for any $h \in \Phi_\gamma$ and for $t \in J$, we have

$$\begin{aligned}|(Th)(t)| \quad \leq \quad & |h_0| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|f(s)||h(s)|ds + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|g(s,h(s))|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(\int_{s}^{t}|Z_1(\tau,s,h(s))|d\tau + \int_{t_0}^{b}|Z_2(\tau,s,h(s))|d\tau\right)ds, \\ \leq \quad & |h_0| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f\|_\infty\|h\|_\infty ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}(|g(s,h(s))-g(s,0)| + |g(s,0)|)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big(\int_{s}^{t}(|Z_1(\tau,s,h(s))-Z_1(\tau,s,0)| + |Z_1(\tau,s,0)|)d\tau \\ & + \int_{t_0}^{b}(|Z_2(\tau,s,h(s))-Z_2(\tau,s,0)| + |Z_2(\tau,s,0)|)d\tau\Big)ds, \\ \leq \quad & |h_0| + \frac{\|f\|_\infty b^\alpha\gamma}{\Gamma(\alpha+1)} + \frac{b^\alpha}{\Gamma(\alpha+1)}(L_g^*\gamma + g_0) + \frac{b^{(\alpha+1)}}{(\alpha+1)\Gamma(\alpha)}(L_{z_1}^*\gamma + z_1^*) \\ & + \frac{b^{(\alpha+1)}}{(\alpha+1)\Gamma(\alpha)}(L_{z_2}^*\gamma + z_2^*) \\ \leq \quad & |h_0| + \Big(\frac{g_0}{\Gamma(\alpha+1)} + \frac{(z_1^*+z_2^*)b}{(\alpha+1)\Gamma(\alpha)}\Big)b^\alpha + \Big(\frac{\|f\|_\infty + L_g^*}{\Gamma(\alpha+1)} + \frac{(L_{z_1}^*+L_{z_2}^*)b}{(\alpha+1)\Gamma(\alpha)}\Big)b^\alpha\gamma \\ = \quad & (1-\beta)\gamma + \beta\gamma = \gamma.\end{aligned}$$

It follows that $\|Th\| \leq \gamma$, this implise that $Th \in \Phi_\gamma$ which leads to $T\Phi_\gamma \subset \Phi_\gamma$.

Step 2. We need to prove that T is contraction mapping.

Let $h_1, h_2 \in \Phi_\gamma$ we get:

$$
\begin{aligned}
& |(Th_1)(t) - (Th_2)(t)| \\
\leq \ & \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |f(s)| \, |h_1(s) - h_2(s)| \, ds \\
& \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} |g(s,h_1(s)) - g(s,h_2(s))| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big(\int_{s}^{t} |Z_1(\tau,s,h_1(s)) - Z_1(\tau,s,h_2(s))| \, d\tau \\
& + \int_{s}^{b} |Z_2(\tau,s,h_1(s)) - Z_2(\tau,s,h_2(s))| \, d\tau \Big) ds
\end{aligned}
$$

$$
\begin{aligned}
\leq \ & \frac{\|f\|_\infty b^\alpha}{\Gamma(\alpha+1)} \|h_1 - h_2\| + \frac{L_g^* b^\alpha}{\Gamma(\alpha+1)} \|h_1 - h_2\| + \frac{L_{z_1}^* b^{\alpha+1} + L_{z_2}^* b^{\alpha+1}}{(\alpha+1)\Gamma(\alpha)} \|h_1 - h_2\| \\
= \ & \Big(\frac{\|f\|_\infty + L_g^*}{\Gamma(\alpha+1)} + \frac{(L_{z_1}^* + L_{z_2}^*) b}{(\alpha+1)\Gamma(\alpha)} \Big) b^\alpha \|h_1 - h_2\| \\
= \ & \varepsilon \|h_1 - h_2\| .
\end{aligned}
$$

Since $\varepsilon = \Big(\frac{\|f\|_\infty + L_g^*}{\Gamma(\alpha+1)} + \frac{(L_{z_1}^* + L_{z_2}^*) b}{(\alpha+1)\Gamma(\alpha)} \Big) b^\alpha < 1$, we get

$$
\|Th_1 - Th_2\| \leq \varepsilon \|h_1 - h_2\| .
$$

This implies that T is contraction mapping. As consequence of Lemma 2.2, there exists a fixed point $h \in C(J, \mathbb{R})$ such that $Th = h$ which is the unique solution of the initial value problem (1.2)−(1.3), and the proof is completed. □

4. Conclusions

The main purpose of this paper was to present new uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential. The techniques used to prove our results are a variety of tools such as Bihari's inequality, some properties of fractional calculus and Banach contraction mapping principle. Moreover, the results of references [2, 4, 5] appear as special cases of our results.

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