Iranian Journal of Mathematical Sciences and Informatics
Vol. 20, No. 1 (2025), pp 175-191
DOI: 10.61186/ijmsi.20.1.175

Some Hermite-Hadamard Type Inequalities for Convex Functions Defined on Convex Bodies Via Gauss-Ostrogradsky Identity

Silvestru Sever Dragomir[a,b]

[a]Applied Mathematics Research Group, ISILC, Victoria University, PO Box 14428Melbourne City, MC 8001, Australia
[b]Department of Mathematical and Physical Sciences, La Trobe University, Melbourne, VIC, Australia

E-mail:sever.dragomir@vu.edu.au

Abstract. In this paper, by the use of Gauss-Ostrogradsky identity, we establish some integral inequalities of Hermite-Hadamard type for functions of three variables defined on closed and bounded convex bodies of the Euclidean space $\mathbb{R}^3$. Some examples for 3-dimensional balls are also provided.

Keywords: Hermite-Hadamard inequality, Triple integral inequalities, Gauss-Ostrogradsky identity.

2000 Mathematics subject classification: 26D15.

1. Introduction

Recall the following inequalities of Hermite-Hadamard's type for convex functions defined on a ball $B(C,R)$, where $C=(a,b,c)\in\mathbb{R}^3$, $R>0$ and

$$B(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\leq R^2\right\}.$$

The following theorem holds [6].

Received 27 May 2020; Accepted 05 March 2022

Theorem 1.1. *Let $f : B(C,R) \to \mathbb{R}$ be a convex mapping on the ball $B(C,R)$. Then we have the inequalities:*

$$f(a,b,c) \leq \frac{1}{V(B(C,R))} \iiint_{B(C,R)} f(x,y,z)\,dxdydz \leq \frac{1}{A(B(C,R))} \iint_{S(C,R)} f(x,y,z)\,dS, \quad (1.1)$$

where

$$S(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}$$

and

$$V(B(C,R)) = \frac{4\pi R^3}{3},\ A(B(C,R)) = 4\pi R^2.$$

Let D be a bounded convex domain from $\mathbb{R}^3$ with a piecewise smooth boundary S. We use the notations

$$A(S) := \iint_S dS,\ V(D) = \iiint_D dV,$$

$$\overline{x_D} := \frac{1}{V(D)} \iiint_D x dV,\ \overline{y_D} := \frac{1}{V(D)} \iiint_D y dV,\ \overline{z_D} := \frac{1}{V(D)} \iiint_D z dV$$

and

$$\overline{x_S} := \frac{1}{A(S)} \iint_S x dS,\ \overline{y_S} := \frac{1}{A(S)} \iint_S y dS \text{ and } \overline{z_S} := \frac{1}{A(S)} \iint_S z dS.$$

Let us assume that the surface S is oriented with the aid of the unit normal h directed to the exterior of D

$$h = (\cos\alpha, \cos\beta, \cos\gamma).$$

The following Hermite-Hadamard type inequalities for convex functions defined on general convex domains were obtained by B. Gavrea in 2000, [9]:

Theorem 1.2. *Let $f : D \to \mathbb{R}$ be a convex function on D, a bounded convex domain from $\mathbb{R}^3$ with a piecewise smooth boundary S. Then we have the inequalities*

$$f(\overline{x_D}, \overline{y_D}, \overline{z_D}) \leq \frac{1}{V(D)} \iiint_D f dV \leq \frac{1}{4A(S)} \iint_S f dS + \frac{1}{4V(D)} \iint_S \left[(x - \overline{x_S})\cos\alpha + (y - \overline{y_S})\cos\beta + (z - \overline{z_S})\cos\gamma \right] f(x,y,z)\,dS. \quad (1.2)$$

For other multivariate Hermite-Hadamard type inequalities, see [2]-[4] and [11]-[18].

Motivated by the above results, we obtain in this paper other integral inequalities of Hermite-Hadamard type for convex functions defined on convex

domains from the space. Some examples for three dimensional balls are also provided.

2. Some preliminary facts and results

Following Apostol [1], consider a surface described by the vector equation

$$r(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k} \tag{2.1}$$

where $(u,v) \in [a,b] \times [c,d]$.

If x, y, z are differentiable on $[a,b] \times [c,d]$ we consider the two vectors

$$\frac{\partial r}{\partial u} = \frac{\partial x}{\partial u}\overrightarrow{i} + \frac{\partial y}{\partial u}\overrightarrow{j} + \frac{\partial z}{\partial u}\overrightarrow{k}$$

and

$$\frac{\partial r}{\partial v} = \frac{\partial x}{\partial v}\overrightarrow{i} + \frac{\partial y}{\partial v}\overrightarrow{j} + \frac{\partial z}{\partial v}\overrightarrow{k}.$$

The *cross product* of these two vectors $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as *Jacobian determinants*. In fact, we have [1, p. 420]

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \overrightarrow{i} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \overrightarrow{k} \tag{2.2}$$

$$= \frac{\partial(y,z)}{\partial(u,v)}\overrightarrow{i} + \frac{\partial(z,x)}{\partial(u,v)}\overrightarrow{j} + \frac{\partial(x,y)}{\partial(u,v)}\overrightarrow{k}.$$

Let $S = r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T = [a,b] \times [c,d]$. The area of S denoted A_S is defined by the double integral [1, p. 424-425]

$$A_S = \int_a^b \int_c^d \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dudv \tag{2.3}$$

$$= \int_a^b \int_c^d \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\, dudv.$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $S = r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T = [a,b] \times [c,d]$ and let $f : S \to \mathbb{C}$ defined

and bounded on S. The surface integral of f over S is defined by [1, p. 430]

$$\begin{aligned}\int\int_S f dS &= \int_a^b \int_c^d f(x,y,z) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du dv \\ &= \int_a^b \int_c^d f(x(u,v), y(u,v), z(u,v)) \\ &\times \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2} du dv.\end{aligned} \tag{2.4}$$

If $S = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to S at each regular point of the surface. At each such point there are two unit normals, a unit normal n_1, which has the same direction as N, and a unit normal n_2 which has the opposite direction. Thus

$$n_1 = \frac{N}{\|N\|} \text{ and } n_2 = -n_1.$$

Let n be one of the two normals n_1 or n_2. Let also F be a vector field defined on S and assume that the surface integral,

$$\int\int_S (F \cdot n) \, dS,$$

called the flux surface integral, exists. Here $F \cdot n$ is the dot or inner product.

We can write [1, p. 434]

$$\int\int_S (F \cdot n) \, dS = \pm \int_a^b \int_c^d F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) du dv$$

where the sign " + " is used if $n = n_1$ and the " − " sign is used if $n = n_2$.

If

$$F(x,y,z) = P(x,y,z) \overrightarrow{i} + Q(x,y,z) \overrightarrow{j} + R(x,y,z) \overrightarrow{k}$$

and

$$r(u,v) = x(u,v) \overrightarrow{i} + y(u,v) \overrightarrow{j} + z(u,v) \overrightarrow{k} \text{ where } (u,v) \in [a,b] \times [c,d]$$

then the flux surface integral for $n = n_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned}\int\int_S (F \cdot n) \, dS &= \int_a^b \int_c^d P(x(u,v), y(u,v), z(u,v)) \frac{\partial(y,z)}{\partial(u,v)} du dv \\ &+ \int_a^b \int_c^d Q(x(u,v), y(u,v), z(u,v)) \frac{\partial(z,x)}{\partial(u,v)} du dv \\ &+ \int_a^b \int_c^d R(x(u,v), y(u,v), z(u,v)) \frac{\partial(x,y)}{\partial(u,v)} du dv.\end{aligned} \tag{2.5}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\int\int_S P(x,y,z)\,dy\wedge dz+\int\int_S Q(x,y,z)\,dz\wedge dx+\int\int_S R(x,y,z)\,dx\wedge dy.$$

Let $B\subset\mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface S, and let n be the unit outer normal to S. If F is a continuously differentiable vector field defined on B, we have the *Gauss-Ostrogradsky identity*

$$\iiint_B (divF)\,dV=\int\int_S (F\cdot n)\,dS. \tag{GO}$$

If we express

$$F(x,y,z)=P(x,y,z)\overrightarrow{i}+Q(x,y,z)\overrightarrow{j}+R(x,y,z)\overrightarrow{k},$$

then (GO) can be written as

$$\begin{aligned}\iiint_B\left(\frac{\partial P(x,y,z)}{\partial x}+\frac{\partial Q(x,y,z)}{\partial y}+\frac{\partial R(x,y,z)}{\partial z}\right)dxdydz\\ =\int\int_S P(x,y,z)\,dy\wedge dz+\int\int_S Q(x,y,z)\,dz\wedge dx\\ +\int\int_S R(x,y,z)\,dx\wedge dy.\end{aligned} \tag{2.6}$$

By taking the real and imaginary part, we can extend the above inequality for complex valued functions P, Q, R defined on B.

For the body B we consider the coordinates for the *center of gravity* $G(\overline{x_B},\overline{y_B},\overline{z_B})$ defined by

$$\overline{x_B}:=\frac{1}{V(B)}\iiint_B xdxdydz,\ \overline{y_B}:=\frac{1}{V(B)}\iiint_B ydxdydz$$

and

$$\overline{z_B}:=\frac{1}{V(B)}\iiint_B zdxdydz.$$

We have:

Lemma 2.1. *Let B be a solid in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S. If $f:B\to\mathbb{C}$ is a continuously differentiable*

function defined on a open set containing B, then we have the equality

$$\begin{aligned}\iiint_B f(x,y,z)\,dxdydz &= \frac{1}{3}\iiint_B\left[(\alpha-x)\frac{\partial f(x,y,z)}{\partial x}+(\beta-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\quad\left.+(\gamma-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\\ &\quad+\frac{1}{3}\left[\int\int_S(x-\alpha)f(x,y,z)\,dy\wedge dz+\int\int_S(y-\beta)f(x,y,z)\,dz\wedge dx\right.\\ &\quad\left.+\int\int_S(z-\gamma)f(x,y,z)\,dx\wedge dy\right] \quad (2.7)\end{aligned}$$

for all α, β and γ complex numbers.

In particular, we have

$$\begin{aligned}\iiint_B f(x,y,z)\,dxdydz &= \frac{1}{3}\iiint_B\left[(\overline{x_B}-x)\frac{\partial f(x,y,z)}{\partial x}+(\overline{y_B}-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\quad\left.+(\overline{z_B}-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\\ &\quad+\frac{1}{3}\left[\int\int_S(x-\overline{x_B})f(x,y,z)\,dy\wedge dz+\int\int_S(y-\overline{y_B})f(x,y,z)\,dz\wedge dx\right.\\ &\quad\left.+\int\int_S(z-\overline{z_B})f(x,y,z)\,dx\wedge dy\right]. \quad (2.8)\end{aligned}$$

Proof. We have

$$\frac{\partial[(x-\alpha)f(x,y,z)]}{\partial x}=f(x,y,z)+(x-\alpha)\frac{\partial f(x,y,z)}{\partial x},$$
$$\frac{\partial[(y-\beta)f(x,y,z)]}{\partial y}=f(x,y,z)+(y-\beta)\frac{\partial f(x,y,z)}{\partial y}$$

and

$$\frac{\partial[(z-\gamma)f(x,y,z)]}{\partial z}=f(x,y,z)+(z-\gamma)\frac{\partial f(x,y,z)}{\partial z}.$$

By adding these three equalities we get

$$\begin{aligned}&\frac{\partial[(x-\alpha)f(x,y,z)]}{\partial x}+\frac{\partial[(y-\beta)f(x,y,z)]}{\partial y}+\frac{\partial[(z-\gamma)f(x,y,z)]}{\partial z}\\ &\qquad=3f(x,y,z)\\ &\quad+(x-\alpha)\frac{\partial f(x,y,z)}{\partial x}+(y-\beta)\frac{\partial f(x,y,z)}{\partial y}+(z-\gamma)\frac{\partial f(x,y,z)}{\partial z} \quad (2.9)\end{aligned}$$

for all $(x,y,z)\in B$.

Integrating this equality on B we get

$$\begin{aligned}\iiint_B \Bigg(&\frac{\partial\left[(x-\alpha)\,f\left(x,y,z\right)\right]}{\partial x}+\frac{\partial\left[(y-\beta)\,f\left(x,y,z\right)\right]}{\partial y}\\&+\frac{\partial\left[(z-\gamma)\,f\left(x,y,z\right)\right]}{\partial z}\Bigg)\,dxdydz\\&=3\iiint_B f\left(x,y,z\right)dxdydz\\&+\iiint_B\left[(x-\alpha)\,\frac{\partial f\left(x,y,z\right)}{\partial x}+(y-\beta)\,\frac{\partial f\left(x,y,z\right)}{\partial y}\right.\\&\left.+(z-\gamma)\,\frac{\partial f\left(x,y,z\right)}{\partial z}\right]dxdydz. \quad (2.10)\end{aligned}$$

Applying the *Gauss-Ostrogradsky identity (2.6)* for the functions

$$P\left(x,y,z\right)=(x-\alpha)\,f\left(x,y,z\right),\ Q\left(x,y,z\right)=(y-\beta)\,f\left(x,y,z\right)$$

and

$$R\left(x,y,z\right)=(z-\gamma)\,f\left(x,y,z\right)$$

we obtain

$$\begin{aligned}\iiint_B \Bigg(&\frac{\partial\left[(x-\alpha)\,f\left(x,y,z\right)\right]}{\partial x}+\frac{\partial\left[(y-\beta)\,f\left(x,y,z\right)\right]}{\partial y}\\&+\frac{\partial\left[(z-\gamma)\,f\left(x,y,z\right)\right]}{\partial z}\Bigg)\,dxdydz\\&=\int\int_S(x-\alpha)\,f\left(x,y,z\right)dy\wedge dz+\int\int_S(y-\beta)\,f\left(x,y,z\right)dz\wedge dx\\&+\int\int_S(z-\gamma)\,f\left(x,y,z\right)dx\wedge dy. \quad (2.11)\end{aligned}$$

By (2.10) and (2.11) we get

$$\begin{aligned}&3\iiint_B f\left(x,y,z\right)dxdydz\\&+\iiint_B\left[(x-\alpha)\,\frac{\partial f\left(x,y,z\right)}{\partial x}+(y-\beta)\,\frac{\partial f\left(x,y,z\right)}{\partial y}+(z-\gamma)\,\frac{\partial f\left(x,y,z\right)}{\partial z}\right]dxdydz\\&=\int\int_S(x-\alpha)\,f\left(x,y,z\right)dy\wedge dz+\int\int_S(y-\beta)\,f\left(x,y,z\right)dz\wedge dx\\&+\int\int_S(z-\gamma)\,f\left(x,y,z\right)dx\wedge dy,\end{aligned}$$

which is equivalent to the desired result (2.7). □

Remark 2.2. For a function f as in Lemma 2.1 above, we define the points

$$x_{B,\partial f}:=\frac{\iiint_B x\frac{\partial f(x,y,z)}{\partial x}dxdydz}{\iiint_B\frac{\partial f(x,y,z)}{\partial x}dxdydz},\ y_{B,\partial f}:=\frac{\iiint_B y\frac{\partial f(x,y,z)}{\partial y}dxdydz}{\iiint_B\frac{\partial f(x,y,z)}{\partial y}dxdydz},$$

and

$$z_{B,\partial f} := \frac{\iiint_B z\frac{\partial f(x,y,z)}{\partial z}dxdydz}{\iiint_B \frac{\partial f(x,y,z)}{\partial z}dxdydz}$$

provided the denominators are not zero.

If we take $\alpha = x_{B,\partial f}$, $\beta = y_{B,\partial f}$ and $\gamma = z_{B,\partial f}$ in (2.7), then we get

$$\begin{aligned}&\iiint_B f(x,y,z)\,dxdydz\\ &= \frac{1}{3}\left[\int\int_S (x - x_{B,\partial f})\, f(x,y,z)\,dy\wedge dz + \int\int_S (y - \beta y_{B,\partial f})\, f(x,y,z)\,dz\wedge dx\right.\\ &\qquad\left. + \int\int_S (z - z_{B,\partial f})\, f(x,y,z)\,dx\wedge dy\right], \quad (2.12)\end{aligned}$$

since, obviously,

$$\begin{aligned}&\iiint_B \left[(x_{B,\partial f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{B,\partial f} - y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\qquad\left. + (z_{B,\partial f} - z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz = 0.\end{aligned}$$

We also have the following dual approach:

Remark 2.3. For a function f as in Lemma 2.1 above, we define the points

$$x_{S,f} := \frac{\int\int_S xf(x,y,z)\,dy\wedge dz}{\int\int_S f(x,y,z)\,dy\wedge dz},\ y_{S,f} := \frac{\int\int_S yf(x,y,z)\,dz\wedge dx}{\int\int_S f(x,y,z)\,dz\wedge dx}$$

and

$$z_{S,f} := \frac{\int\int_S zf(x,y,z)\,dx\wedge dy}{\int\int_S f(x,y,z)\,dx\wedge dy}$$

provided the denominators are not zero.

If we take $\alpha = x_{S,f}$, $\beta = y_{S,f}$ and $\gamma = z_{S,f}$ in (2.7), then we get

$$\begin{aligned}&\iiint_B f(x,y,z)\,dxdydz\\ &\quad= \frac{1}{3}\iiint_B \left[(x_{S,f} - x)\frac{\partial f(x,y,z)}{\partial x} + (y_{S,f} - y)\frac{\partial f(x,y,z)}{\partial y}\right.\\ &\qquad\left. + (z_{S,f} - z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz \quad (2.13)\end{aligned}$$

since, obviously,

$$\begin{aligned}&\int\int_S (x - x_{S,f})\, f(x,y,z)\,dy\wedge dz + \int\int_S (y - y_{S,f})\, f(x,y,z)\,dz\wedge dx\\ &\qquad + \int\int_S (z - z_{S,f})\, f(x,y,z)\,dx\wedge dy = 0.\end{aligned}$$

3. Inequalities for convex functions

We have the following result:

Theorem 3.1. *Let B be a convex body in the three dimensional space $\mathbb{R}^3$ bounded by an orientable closed surface S and $f : B \to \mathbb{C}$ a continuously differentiable function defined on a open set containing B. If f is convex on B, then for any $(u,v,w) \in B$ we have*

$$\begin{aligned} f(u,v,w) &+ (\overline{x_B} - u)\frac{\partial f(u,v,w)}{\partial x} \\ &+ (\overline{y_B} - v)\frac{\partial f(u,v,w)}{\partial y} + (\overline{z_B} - w)\frac{\partial f(u,v,w)}{\partial z} \\ &\leq \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz \\ &\leq \frac{1}{4}f(u,v,w) + \frac{1}{4}\frac{1}{V(B)}\left[\int\int_S (x-u) f(x,y,z)\,dy\wedge dz \right. \\ &\left. + \int\int_S (y-v) f(x,y,z)\,dz\wedge dx + \int\int_S (z-w) f(x,y,z)\,dx\wedge dy\right]. \end{aligned} \tag{3.1}$$

In particular, we have

$$\begin{aligned} f(\overline{x_B},\overline{y_B},\overline{z_B}) &\leq \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz \\ &\leq \frac{1}{4}f(\overline{x_B},\overline{y_B},\overline{z_B}) + \frac{1}{4}\frac{1}{V(B)}\left[\int\int_S (x-\overline{x_B}) f(x,y,z)\,dy\wedge dz \right. \\ &\left. + \int\int_S (y-\overline{y_B}) f(x,y,z)\,dz\wedge dx + \int\int_S (z-\overline{z_B}) f(x,y,z)\,dx\wedge dy\right]. \end{aligned} \tag{3.2}$$

Proof. By the gradient inequality for the convex function f on the convex set B we have

$$\begin{aligned} (x-u)\frac{\partial f(u,v,w)}{\partial x} &+ (y-v)\frac{\partial f(u,v,w)}{\partial y} + (z-w)\frac{\partial f(u,v,w)}{\partial z} \\ &\leq f(x,y,z) - f(u,v,w) \\ &\leq (x-u)\frac{\partial f(x,y,z)}{\partial x} + (y-v)\frac{\partial f(x,y,z)}{\partial y} + (z-w)\frac{\partial f(x,y,z)}{\partial z} \end{aligned}$$

for all $(u,v,w), (x,y,z) \in B$.

If we take the integral mean over the variables $(x,y,z)\in B$, we get

$$\begin{aligned}
&\frac{1}{V(B)}\iiint_B (x-u)\frac{\partial f(u,v,w)}{\partial x}dxdydz+\frac{1}{V(B)}\iiint_B (y-v)\frac{\partial f(u,v,w)}{\partial y}dxdydz\\
&\qquad+\frac{1}{V(B)}\iiint_B (z-w)\frac{\partial f(u,v,w)}{\partial z}dxdydz\\
&\quad\leq \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-\frac{1}{V(B)}\iiint_B f(u,v,w)\,dxdydz\\
&\leq \frac{1}{V(B)}\iiint_B (x-u)\frac{\partial f(x,y,z)}{\partial x}dxdydz+\frac{1}{V(B)}\iiint_B (y-v)\frac{\partial f(x,y,z)}{\partial y}dxdydz\\
&\qquad+\frac{1}{V(B)}\iiint_B (z-w)\frac{\partial f(x,y,z)}{\partial z}dxdydz
\end{aligned}$$

namely

$$\begin{aligned}
&(\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}+(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y}+(\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}\\
&\quad\leq \frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-f(u,v,w)\\
&\leq \frac{1}{V(B)}\iiint_B (x-u)\frac{\partial f(x,y,z)}{\partial x}dxdydz+\frac{1}{V(B)}\iiint_B (y-v)\frac{\partial f(x,y,z)}{\partial y}dxdydz\\
&\qquad+\frac{1}{V(B)}\iiint_B (z-w)\frac{\partial f(x,y,z)}{\partial z}dxdydz \qquad (3.3)
\end{aligned}$$

for all $(u,v,w)\in B$, which is an inequality of interest in itself.

The first inequality in (3.3) gives now the first part of (3.1). From the identity (2.7) we get for $(\alpha,\beta,\gamma)=(u,v,w)$ that

$$\begin{aligned}
&\iiint_B f(x,y,z)\,dxdydz\\
&\qquad=\frac{1}{3}\iiint_B\left[(u-x)\frac{\partial f(x,y,z)}{\partial x}+(v-y)\frac{\partial f(x,y,z)}{\partial y}\right.\\
&\qquad\qquad\left.+(w-z)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz\\
&\quad+\frac{1}{3}\left[\int\int_S (x-u)f(x,y,z)\,dy\wedge dz+\int\int_S (y-v)f(x,y,z)\,dz\wedge dx\right.\\
&\qquad\qquad\left.+\int\int_S (z-w)f(x,y,z)\,dx\wedge dy\right],
\end{aligned}$$

namely

$$\frac{1}{V(B)}\iiint_B\left[(x-u)\frac{\partial f(x,y,z)}{\partial x}+(y-v)\frac{\partial f(x,y,z)}{\partial y}\right.$$
$$\left.+(z-w)\frac{\partial f(x,y,z)}{\partial z}\right]dxdydz=$$
$$+\frac{1}{V(B)}\left[\int\int_S(x-u)f(x,y,z)\,dy\wedge dz+\int\int_S(y-v)f(x,y,z)\,dz\wedge dx\right.$$
$$\left.+\int\int_S(z-w)f(x,y,z)\,dx\wedge dy\right]-3\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz.$$

From the second part of (3.3) we get

$$\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz-f(u,v,w)$$
$$\leq\frac{1}{V(B)}\left[\int\int_S(x-u)f(x,y,z)\,dy\wedge dz+\int\int_S(y-v)f(x,y,z)\,dz\wedge dx\right.$$
$$\left.+\int\int_S(z-w)f(x,y,z)\,dx\wedge dy\right]-3\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz,$$

namely

$$\frac{4}{V(B)}\iiint_B f(x,y,z)\,dxdydz\leq f(u,v,w)$$
$$+\frac{1}{V(B)}\left[\int\int_S(x-u)f(x,y,z)\,dy\wedge dz+\int\int_S(y-v)f(x,y,z)\,dz\wedge dx\right.$$
$$\left.+\int\int_S(z-w)f(x,y,z)\,dx\wedge dy\right],$$

which gives the second part of the inequality (3.1). □

Remark 3.2. The first inequality in (3.2) is the same as the one from inequality (1.2).

Corollary 3.3. *With the assumptions of Theorem 3.1 and if* $(x_{S,f},y_{S,f},z_{S,f})\in B$, *then*

$$f(x_{S,f},y_{S,f},z_{S,f})+(\overline{x_B}-x_{S,f})\frac{\partial f(x_{S,f},y_{S,f},z_{S,f})}{\partial x}$$
$$+(\overline{y_B}-y_{S,f})\frac{\partial f(x_{S,f},y_{S,f},z_{S,f})}{\partial y}+(\overline{z_B}-z_{S,f})\frac{\partial f(x_{S,f},y_{S,f},z_{S,f})}{\partial z}$$
$$\leq\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz\leq\frac{1}{4}f(x_{S,f},y_{S,f},z_{S,f}).\quad(3.4)$$

The proof follows by (3.1) observing that

$$\int\int_S (x - x_{S,f})\, f(x,y,z)\, dy \wedge dz + \int\int_S (y - y_{S,f})\, f(x,y,z)\, dz \wedge dx + \int\int_S (z - z_{S,f})\, f(x,y,z)\, dx \wedge dy = 0.$$

Corollary 3.4. *With the assumptions of Theorem 3.1, we have*

$$\frac{1}{V(B)} \iiint_B f(x,y,z)\, dxdydz \leq \frac{1}{3} \frac{1}{V(B)} \left[\int\int_S (x - \overline{x_B})\, f(x,y,z)\, dy \wedge dz + \int\int_S (y - \overline{y_B})\, f(x,y,z)\, dz \wedge dx + \int\int_S (z - \overline{z_B})\, f(x,y,z)\, dx \wedge dy \right]. \quad (3.5)$$

Proof. From (3.2) we get

$$\begin{aligned} &\frac{1}{V(B)} \iiint_B f(x,y,z)\, dxdydz \\ &\quad \leq \frac{1}{4} f(\overline{x_B}, \overline{y_B}, \overline{z_B}) + \frac{1}{4} \frac{1}{V(B)} \left[\int\int_S (x - \overline{x_B})\, f(x,y,z)\, dy \wedge dz \right. \\ &\quad \left. + \int\int_S (y - \overline{y_B})\, f(x,y,z)\, dz \wedge dx + \int\int_S (z - \overline{z_B})\, f(x,y,z)\, dx \wedge dy \right] \\ &\leq \frac{1}{4} \frac{1}{V(B)} \iiint_B f(x,y,z)\, dxdydz + \frac{1}{4} \frac{1}{V(B)} \left[\int\int_S (x - \overline{x_B})\, f(x,y,z)\, dy \wedge dz \right. \\ &\quad \left. + \int\int_S (y - \overline{y_B})\, f(x,y,z)\, dz \wedge dx + \int\int_S (z - \overline{z_B})\, f(x,y,z)\, dx \wedge dy \right], \end{aligned}$$

which implies that

$$\frac{3}{4V(B)} \iiint_B f(x,y,z)\, dxdydz \leq \frac{1}{4} \frac{1}{V(B)} \left[\int\int_S (x - \overline{x_B})\, f(x,y,z)\, dy \wedge dz + \int\int_S (y - \overline{y_B})\, f(x,y,z)\, dz \wedge dx + \int\int_S (z - \overline{z_B})\, f(x,y,z)\, dx \wedge dy \right]$$

that is equivalent to (3.5). □

Corollary 3.5. *With the assumptions of Theorem 3.1 we have*

$$\begin{aligned}
&\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS+\frac{1}{A(S)}\int\int_S\left[(\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}\right.\\
&\qquad\left.+(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y}+(\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}\right]dS\\
&\qquad\le\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz\\
&\le\frac{1}{4}\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS+\frac{1}{4}\frac{1}{V(B)}\left[\int\int_S(x-\overline{x_S})f(x,y,z)\,dy\wedge dz\right.\\
&\left.+\int\int_S(y-\overline{y_S})f(x,y,z)\,dz\wedge dx+\int\int_S(z-\overline{z_S})f(x,y,z)\,dx\wedge dy\right], \quad (3.6)
\end{aligned}$$

where

$$\overline{x_S}=\frac{1}{A(S)}\int\int_S u dS,\ \overline{y_S}=\frac{1}{A(S)}\int\int_S v dS,\ \overline{z_S}=\frac{1}{A(S)}\int\int_S w dS.$$

Proof. By taking the integral mean $\frac{1}{A(S)}\int\int_S(\cdot)\,dS$ over the variables (u,v,w) in the integral (3.1), we get

$$\begin{aligned}
&\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS+\frac{1}{A(S)}\int\int_S\left[(\overline{x_B}-u)\frac{\partial f(u,v,w)}{\partial x}\right.\\
&\qquad\left.+(\overline{y_B}-v)\frac{\partial f(u,v,w)}{\partial y}+(\overline{z_B}-w)\frac{\partial f(u,v,w)}{\partial z}\right]dS\\
&\qquad\le\frac{1}{V(B)}\iiint_B f(x,y,z)\,dxdydz\\
&\qquad\le\frac{1}{4}\frac{1}{A(S)}\int\int_S f(u,v,w)\,dS\\
&+\frac{1}{4}\frac{1}{V(B)}\left[\int\int_S\left(x-\frac{1}{A(S)}\int\int_S u dS\right)f(x,y,z)\,dy\wedge dz\right.\\
&\qquad+\int\int_S\left(y-\frac{1}{A(S)}\int\int_S v dS\right)f(x,y,z)\,dz\wedge dx\\
&\qquad\left.+\int\int_S\left(z-\frac{1}{A(S)}\int\int_S w dS\right)f(x,y,z)\,dx\wedge dy\right],
\end{aligned}$$

which is equivalent to (3.6). □

Remark 3.6. The second inequality in (3.6) is an equivalent formulation of the second inequality in (1.2).

4. Applications for three dimensional balls

Now, let us compute the surface integral

$$K(S(C,R),f):=\iint_{S(C,R)}f(x,y,z)\,dS,$$

where

$$S(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}.$$

If we consider the parametrization of $S(C,R)$ given by:

$$S(C,R): \begin{cases} x = R\cos\psi\cos\varphi + a \\ y = R\cos\psi\sin\varphi + b \\ z = R\sin\psi + c \end{cases} ; (\psi,\varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi]$$

and putting

$$A := \begin{vmatrix} \frac{\partial y}{\partial \psi} & \frac{\partial z}{\partial \psi} \\ \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = -R^2 \cos^2\psi\cos\varphi,$$

$$B := \begin{vmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial z}{\partial \psi} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = R^2 \cos^2\psi\sin\varphi,$$

and

$$C := \begin{vmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial y}{\partial \psi} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = -R^2 \sin\psi\cos\psi,$$

we have that

$$A^2 + B^2 + C^2 = R^4\cos^2\psi \text{ for all } (\psi,\varphi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0, 2\pi].$$

Thus,

$$\begin{aligned} K(S(C,R), f) &= \iint_{S(C,R)} f(x,y,z)\, dS \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left[f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c) \right. \\ &\qquad \left. \times \sqrt{A^2 + B^2 + C^2} \right] d\psi d\varphi \\ &= R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos\psi f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\, d\psi d\varphi. \end{aligned} \tag{4.1}$$

We also have

$$L\left(S\left(C,R\right),f\right):=\int\int_{S(C,R)}\left(x-a\right)f\left(x,y,z\right)dy\wedge dz$$
$$+\int\int_{S(C,R)}\left(y-b\right)f\left(x,y,z\right)dz\wedge dx+\int\int_{S(C,R)}\left(z-c\right)f\left(x,y,z\right)dx\wedge dy$$
$$=-R^{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\pi}\cos^{3}\psi\cos^{2}\varphi$$
$$\times f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)d\psi d\varphi$$
$$+R^{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\pi}\cos^{3}\psi\sin^{2}\varphi$$
$$\times f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)d\psi d\varphi$$
$$-R^{3}\int\int_{S}\sin^{2}\psi\cos\psi f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)d\psi d\varphi. \tag{4.2}$$

Let us consider the transformation $T_2:\mathbb{R}^3\to\mathbb{R}^3$ given by:

$$T_2\left(r,\psi,\varphi\right):=\left(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c\right).$$

It is well known that the Jacobian of T_2 is

$$J\left(T_2\right)=r^2\cos\psi$$

and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$, with values in the ball $B\left(C,R\right)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$I\left(B\left(C,R\right),f\right):=\iiint_{B(C,R)}f\left(x,y,z\right)dxdydz$$
$$=\int_{0}^{R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{2\pi}f\left(r\cos\psi\cos\varphi+a,r\cos\psi\sin\varphi+b,r\sin\psi+c\right)r^{2}\cos\psi dr d\psi d\varphi. \tag{4.3}$$

Assume that f is convex on the ball $B\left(C,R\right)$. From the inequality (3.2) we get

$$f\left(a,b,c\right)\leq\frac{1}{V\left(B\left(C,R\right)\right)}I\left(B\left(C,R\right),f\right)$$
$$\leq\frac{1}{4}f\left(a,b,c\right)+\frac{1}{4V\left(B\left(C,R\right)\right)}L\left(S\left(C,R\right),f\right), \tag{4.4}$$

where $V\left(B\left(C,R\right)\right)=\frac{4\pi R^3}{3}$, while from the inequality (3.5) we also have

$$\frac{1}{V\left(B\left(C,R\right)\right)}I\left(B\left(C,R\right),f\right)\leq\frac{1}{3V\left(B\left(C,R\right)\right)}L\left(S\left(C,R\right),f\right). \tag{4.5}$$

Further, consider

$$\begin{aligned}
J\left(S\left(C,R\right),f\right) &:= \int\int_{S(C,R)} \left[\left(\overline{x_B}-u\right)\frac{\partial f\left(u,v,w\right)}{\partial x}\right. \\
&\qquad \left. +\left(\overline{y_B}-v\right)\frac{\partial f\left(u,v,w\right)}{\partial y}+\left(\overline{z_B}-w\right)\frac{\partial f\left(u,v,w\right)}{\partial z}\right] dS \\
&= -R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi\cos\varphi\frac{\partial f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)}{\partial x}d\psi d\varphi \\
&\quad -R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi\sin\varphi\frac{\partial f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)}{\partial y}d\psi d\varphi \\
&\qquad -R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\sin\psi\frac{\partial f\left(R\cos\psi\cos\varphi+a,R\cos\psi\sin\varphi+b,R\sin\psi+c\right)}{\partial z}d\psi d\varphi.
\end{aligned}$$

Then from the inequality (3.6) we get the following inequalities of interest:

$$\begin{aligned}
&\frac{1}{A\left(S\left(C,R\right)\right)}K\left(S\left(C,R\right),f\right)+\frac{1}{A\left(S\left(C,R\right)\right)}J\left(S\left(C,R\right),f\right) \\
&\qquad\qquad \leq \frac{1}{V\left(B\left(C,R\right)\right)}I\left(B\left(C,R\right),f\right) \\
&\leq \frac{1}{4}\frac{1}{A\left(S\left(C,R\right)\right)}K\left(S\left(C,R\right),f\right)+\frac{1}{4}\frac{1}{V\left(B\left(C,R\right)\right)}L\left(S\left(C,R\right),f\right), \quad (4.6)
\end{aligned}$$

where $A\left(S\left(C,R\right)\right)=4\pi R^2$ is the area of the sphere.

Acknowledgments

The authors would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.

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