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An Advanced Numerical Approach To Solve Viscous Flow Via Modified Generalized Laguerre Functions

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Abstract. This paper presents an advanced numerical approach that applies the quasilinearization method (QLM) and collocation method (CM) based on modified generalized Laguerre functions (MGLFs) to solve a nonlinear system of ordinary differential equations governing viscous flow with heat transfer and magnetic fields on a semi-infinite domain. We demonstrate the effectiveness and accuracy of the proposed method by comparing it with previous well-known methods. The results show that the proposed method provides an efficient and accurate solution to the problem.

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1. Introduction

Investigating phenomena in science and engineering often involves developing mathematical and computational models. Many of these models involve linear or nonlinear equations or systems of equations, which can be difficult to solve exactly, especially on infinite or semi-infinite intervals. While there are various analytical or numerical methods for solving such equations, mathematical researchers often focus on reducing errors and approximating the exact solution. Among the numerical methods available, those suitable for infinite or semi-infinite intervals are often classified into two categories: spectral methods and meshless methods. Spectral methods have seen significant success in approximating boundary value problems in semi-infinite domains [13, 14].

Oyelakin [36] used the spectral relaxation method to solve the governing equations for a problem. In this study, we propose an advanced approach that uses spectral methods to solve problems in semi-infinite domains, specifically in data analysis [50, 47].

Various spectral methods have been proposed to solve unbounded domain problems. For example, Christov [9] explained spectral methods on unbounded intervals and demonstrated how to use orthogonal systems of rational functions. A multi-domain bivariate quasi-linearization method is employed to solve the system of partial differential equations [39]. The bivariate spectral quasilinearization method is used to solve this kind of problem [37].

Another method for semi-infinite intervals is domain truncation, where the infinite domain is replaced with a finite interval, such as $[-L, L]$ or $[0, L]$, by choosing an appropriate value for L [6]. Boyd et al. [7] employed pseudospectral methods on a semi-infinite interval and compared the solutions with rational Chebyshev, Laguerre, and mapped Fourier sine methods. Parand and Babolian [58, 57] reduced the solution of semi-infinite problems to the solution of some algebraic equations using operational matrices of derivatives and products of rational Chebyshev and Legendre functions. Hojjati and Dehghan [21, 77] solved nonlinear ordinary differential equations on semi-infinite intervals using spectral methods based on the rational Tau and collocation method.

Polynomials that are orthogonal over a semi-infinite interval, such as the Hermite spectral, Laguerre spectral, and Sinc-collocation methods, are commonly used [11, 72, 74]. Oyelakin [33] applied the spectral local linearization method to solve the three-dimensional MHD stagnation point flow problem.

Conservation equations that are on semi-infinite intervals can be solved using the spectral local linearization method [38].

Several methods are for solving differential equations, and one of the approaches is machine learning. Least Squares Support Vector Regression (LS-SVR) is a machine learning method that can be used for solving differential equations. The LS-SVR method is based on the Support Vector Regression (SVR) algorithm, which is a popular machine learning algorithm for regression problems. The LS-SVR method is particularly useful for solving differential equations [31, 42, 60, 76] . Neural networks have been shown to be effective in solving differential equations, and they are becoming increasingly popular in the field of scientific computing. Neural networks can be used to approximate the solution of a differential equation by training a model on a set of input-output pairs, where the inputs correspond to the independent variables of the differential equation and the outputs correspond to the dependent variables [30, 19, 41, 79] In this paper, we presented one of the spectral methods. In the following, we briefly explained the benefits of this approach.

The use of the generalized Laguerre polynomials (GLPs) and modified generalized Laguerre functions (MGLFs) in the QLM-CM-MGLFs method for solving the viscous flow equation offers several benefits, including:

- Efficient approximation: The GLPs and MGLFs are efficient mathematical tools for approximating solutions to differential equations on semi-infinite intervals. They have been widely used in polynomials approximation and solving integral or differential equations on semi-infinite intervals.
- Orthogonality: The GLPs and MGLFs satisfy orthogonality relations, which is important in numerical methods for solving differential equations. This allows for efficient computations and accurate approximations.
- Recurrence relations: The GLPs have recurrence relations that can be used to compute their values efficiently. This is useful in numerical methods where computations need to be done quickly and accurately.
- Rodrigues' formula: Rodrigues' formula for GLPs provides a way to compute the polynomials and MGLFs efficiently. This formula allows for the computation of the polynomials and MGLFs without having to compute their derivatives, which can be time-consuming.

The use of GLPs and MGLFs in the QLM-CM-MGLFs method provides efficient and accurate approximate solutions to the viscous flow equation. The properties of these functions allow for efficient computations and accurate approximations, making them valuable tools in numerical methods for solving differential equations.

	Authors	year	Subject	Method
[80]	A.M. Wazwaz	2001	Lane-Emden	Domain Decomposition
[59]	K. Parand, M. Razzaghi	2004	Some Physical Problems on Semi-Infinite	Rational Legendre Collocation
[83]	S.A. Yousefi	2006	Lane–Emden	Legendre Wavelets Method
[82]	A. Yıldırım, T. Öziş	2007	Lane–Emden	Homotopy, Perturbation Method
[46]	K. Parand *et al.*	2009	Blasius Equation	Sinc Collocation
[65]	K. Parand, A. Taghavi	2009	Blasius Equation	Rational Scaled Generalized Laguerre Function Collocation
[62]	K. Parand, M. Shahini	2009	Thomas–Fermi	Rational Chebyshev Collocation
[61]	K. Parand *et al.*	2010	Lane–Emden	Lagrangian Collocation
[56]	K. Parand *et al.*	2012	Blasius Laminar Viscous Flow	First Kind of Bessel Functions
[5]	A.H. Bhrawy, A.S. Alofi	2012	Lane–Emden	Jacobi–Gauss Collocation Method
[22]	S. Kazem *et al.*	2012	Unsteady Flow of Gas	Radial Basis Function(RBF)
[40]	R.K. Pandey, N. Kumar	2012	Lane–Emden	Bernstein Operational Matrix of Differentiation
[4]	F.B. Babolghani, K. Parand	2013	Steady Flow of a Third Grade Fluid	Comparison Between Hermite and Sinc Collocation Methods
[48]	K. Parand *et al.*	2013	Thomas–Fermi	Sinc Collocation Method
[85]	Ş. Yüzbaşı, M. Sezer	2013	Lane–Emden	Bessel Collocation Method
[53]	K. Parand, S. Khaleqiay	2015	Lane–Emden	Rational Chebyshev of Second Kind Collocation Method
[54]	K. Parand *et al.*	2017	Eyring-Powell	Indirect Radial Basis Functions
[55]	K. Parand *et al.*	2017	Eyring-Powell	Rational Chebyshev Functions Collocation
[52]	K. Parand, M. Hemami	2017	Astrophysics Equations	Compactly Supported Radial Basis Function

Table 1. Semi-infinite problems solving via different methods.

In this study, we introduce spectral and different methods and the Laguerre polynomials/functions in subsections 1.1 and 1.2. We also describe solutions to viscous flow problems in subsection 1.3."

1.1. **Spectral and different methods.** Numerical approximations are needed to calculate the proper solution for computational models which are linear/ nonlinear and do not have an analytical solution. Some of them are Spectral Methods. Table1 proposes semi-infinite and infinite intervals problems review via Spectral Methods.

Wazwaz [80] developed a reliable algorithm for solving differential equations of the Lane-Emden type. The algorithm is designed to overcome the problem of singularities and mainly relies on the Adomian decomposition method.

Parand et al. [59] solved the Volterra model and the Lane-Emden equation using operational matrices of derivatives and products of rational Legendre functions. Yousefi [83] produced an exact solution for singular initial value problems represented by the Lane-Emden equation. He used integral operators to transform the equation into an integral equation and approximated the solution using Legendre wavelets. His homotopy perturbation method is suitable for solving nonlinear differential equations.

Yildirim and Ozis [82] used a different implementation of He's homotopy perturbation method to obtain exact approximate solutions of Lane-Emden-type equations and compared the solutions obtained by different researchers who modified the method.

Oyelakin [34] solved the nonlinear coupled equations using the spectral local linearization method. She also solved the Casson nanofluid flow past an electromagnetic stretching Riga plate problem using the spectral local linearization method[32].

Parand et al. [46] proposed the sinc-collocation method to solve the Blasius equation, which models the laminar boundary layer flow of Newtonian fluids. They showed that this method converges rapidly and provides results that match those obtained by another numerical solution method. In another study, Parand et al. [65] found analytical and numerical solutions to the nonlinear Blasius equation on a semi-infinite domain with a singularity at $x = 0$. They showed that this problem could not be transformed into a finite domain problem and used a rational Chebyshev pseudospectral method to solve it.

Other studies have used various numerical methods such as the Bessel functions collocation method [56], shifted Jacobi-Gauss collocation spectral method [5], radial basis function (RBF) collocation method [22], Bernstein polynomials [40], Hermite and Sinc functions collocation [4], and the sinc-collocation method [48] to solve various differential equations on infinite and semi-infinite domains.

1.2. **Laguerre polynomials/functions.** The Laguerre polynomials are solutions of Laguerre's equation, a second-order linear differential equation. These polynomials have been widely used for numerical solutions of differential equations on semi-infinite domains. Table 2 provides examples of studies that used Laguerre polynomials/functions to solve differential equations.

Guo et al. [17] proposed spectral methods using Laguerre functions to solve elliptic equations on regular unbounded domains. Parand and Taghavi [64] used the Laguerre method to approximate the solution of the nonlinear Lane-Emden equation on a semi-infinite domain with a singularity at $x = 0$. In [66], the authors solved Lane-Emden equations using the Laguerre method and compared their results with MGLFM and RCM, finding that the present solutions were highly accurate. Laguerre polynomials and rational Chebyshev functions were used to solve the unsteady gas equation in [63], while Parand et al. [49] obtained an approximate analytical solution to the Blasius equation using Laguerre functions.

In [45], Laguerre functions were used to study the three-order nonlinear differential equations arising from the Blasius equation. Parand and Babolghani [43] proposed a modified generalized Laguerre function to solve the problem of the steady flow of a third-grade fluid in a porous half-space. They also used the Laguerre function to investigate the flow and diffusion of chemically reactive species over a nonlinear stretching sheet and compared their results with the homotopy analysis method in [44].

	Authors	year	Method
[17]	B. Guo *et al.*	2005	Generalized Laguerre approximation
[64]	K. Parand, A. Taghavi	2008	Generalized Laguerre polynomials collocation method
[66]	K. Parand *et al.*	2009	rational Chebyshev and modified generalized Laguerre function pesudospectral methods
[63]	K. Parand *et al.*	2009	Generalized Laguerre polynomials and rational Chebyshev collocation method
[49]	K. Parand *et al.*	2010	Modified generalized Laguerre function Tau method
[45]	K. Parand *et al.*	2012	Rational Gegenbauer and Modified Generalized Laguerre functions Collocation method
[43]	K. Parand, F.B. Babolghani	2012	Modified Generalized Laguerre Functions
[44]	K. Parand, F.B. Babolghani	2012	Modified Generalized Laguerre Functions
[10]	F. Comte *et al.*	2016	Laplace deconvolution
[84]	Ş. Yüzbaşı	2016	Laguerre Approach
[51]	K. Parand, Z. Hajimohammadi	2017	Modified Generalized Laguerre functions, QLM – Collocation method
[20]	Z. Hajimohammadi, K. Parand	2017	Fractional Rational Generalized Laguerre polynomials QLM-Collocation method
[24]	CH. Lin	2017	Laguerre polynomial neural network and modified particle swarm optimization
[1]	P. Agarwal *et al.*	2017	generalized hyper geometric function and the Laguerre polynomials
[8]	O. Cherroud *et al.*	2017	Generalized Laguerre polynomials
[81]	S. Xu *et al.*	2017	Generalized Laguerre and Neural Dynamics
[27]	F.J. Liu *et al.*	2017	generalized Laguerre functions
[29]	A.M. Makarenkova *et al.*	2017	Projection Galerkin Method
[25]	Ch Lin	2017	Laguerre polynomials neural network and modified particle swarm optimization
[86]	Ch Zhang *et al.*	2017	Laguerre Spectral Method
[71]	I.I. Sharapudinov	2017	Laguerre polynomials
[73]	Y. Shi *et al.*	2017	Laguerre- Galerkin method
[87]	D. Zhang, X. Miao	2017	Weighted Laguerre Polynomials
[26]	F. Liu *et al.*	2017	generalized Laguerre functions

Table 2. Infinite/Semi-infinite problems solving via Laguerre polynomials/functions.

Other studies have used Laguerre functions for a variety of applications, including Laplace deconvolution [10], solving singular perturbated differential equations [84], and solving the Eyring-Powell problem using the modified generalized Laguerre function and the QLM-collocation method [51, 20]. Laguerre functions have also been used in control systems [24], integrals involving Laguerre and hypergeometric functions [1], constructing Wigner distribution functions [8], time-varying neural dynamics [81], elliptic equations [27], diffusion equations [29], solving the telegraph equation [87], transient electromagnetic problems [73], and second-order equations and fourth-order equations [26].

1.3. **Viscous flow.** Viscous flow is a type of fluid flow in which there is a continuous steady motion of the particles, the motion at a fixed point always remaining constant. In recent years, novel types of flow situations have been investigated, such as the boundary layer on a continuous moving surface [78]. The behavior of boundary layers on surfaces was first discussed by Sakiadis [68, 69]. Some recent research on this topic is presented in Table 3. MacCormack [28] introduced the implicit analog of the explicit method. Akyildiz *et al.* [2]

	Authors	year	Method
[28]	R.W. MacCormack	1982	Implicit method, explicit method
[2]	F.T. Akyildiz *et al.*	2006	Exact-solution
[23]	S.A. Kechil , I. Hashim	2007	Adomian Decomposition Method (ADM)
[89]	Z. Ziabakhsh *et al.*	2010	Homotopy Analysis Method (HAM) and Runge-Kutta (RK)
[70]	K.M. Sanni *et al.*	2016	Runge-Kutta (RK)
[67]	J.A. Rad *et al.*	2016	Radial Basis Function (RBF)

Table 3. The recent researches for viscous flow.

found the exact solution for a chemically reactive problem over a stretching sheet. Kechil and Hashim [23] investigated the boundary-layer equation of flow over a nonlinearly stretching sheet in the presence of a chemical reaction and a magnetic field by employing the Adomian decomposition method (ADM). Ziabakhsh *et al.* [89] analyzed the flow and diffusion of chemically reactive species over a nonlinear stretching sheet immersed in a porous medium using the Homotopy analysis method (HAM). They attempted to demonstrate the capabilities and wide-ranging applications of the HAM method in comparison with numerical methods for solving this problem. Sanni *et al.* [70] focused on the flow of a viscous fluid over a curved surface stretching with nonlinear power-law velocity. The boundary layer equations were transformed into ordinary differential equations using suitable non-dimensional transformations. These equations were numerically solved using the shooting and Runge-Kutta (RK) methods. Rad *et al.* [67] proposed a new numerical scheme to solve a nonlinear system two-point boundary value problem on a semi-infinite domain, which describes the flow and diffusion of chemically reactive species over a nonlinearly stretching sheet immersed in a porous medium.

The rest of this paper is organized as follows: we introduce the mathematical formulation of the problem in section 2; preliminaries, including the generalized Laguerre polynomials, the modified generalized Laguerre functions, the QLM method, and the collocation method, are explained in section 3; the applicability of the method is presented in section 4; results and discussions are shown in section 5, and finally, the conclusion is explained in section 6.

2. Mathematical formulation of the problem

In this section, we consider two-dimensional viscous and electrically conducting fluid flow. There are other examples of equations of this type that can be solved using spectral methods [35]. The surface under consideration is stretched with a nonlinear velocity by applying equal and opposite forces along the x-direction, while the reactive species expire from the sheet and are released into the surrounding fluid with a first-order chemical reaction. We note that both the governing equations and boundary conditions are nonlinear. Under these conditions, the resulting boundary layer equations that govern the flow are given as follows [2, 23, 89, 70, 67]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u. \tag{2.2}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-k_1 C. \tag{2.3}$$

Where x and y are the distances along and perpendicular to the sheet, u and v are the respective components of the velocity in the x and y directions, k_1 is the chemical reaction parameter, D is the mass diffusion coefficient, C is the species concentration in the fluid, B_0 is the strength of the magnetic field, σ is the electrical conductivity, μ is the dynamic viscosity, ρ is the fluid density, and g is the acceleration due to gravity.

The appropriate boundary conditions for this problem are:

$$\begin{aligned}&u\,(x,0)=ax+cx^2,\quad v(x,0)=0,\quad C(x,0)=C_w,\\ &u\to 0,\quad C\to 0,\quad as,\quad y\to\infty.\end{aligned} \tag{2.4}$$

Where a and c are constants and the subscript w denotes the condition at the wall. The introduction of the similarity transformations [2, 23, 89, 70, 67]:

$$\begin{aligned}&\eta=\sqrt{\frac{a}{\upsilon}}y,\ u=axf'(\eta)+cx^2g'(\eta),\ v=-\sqrt{a\upsilon}f(\eta)-\frac{2cx}{\sqrt{a/\upsilon}}g(\eta),\\ &C=C_w[C_0(\eta)+\frac{2cx}{a}C_1(\eta)],\ K=\frac{k_1 Sc}{a},\ N=\frac{\sigma B_0^2}{a\rho},\ Sc=\frac{\upsilon}{D}.\end{aligned} \tag{2.5}$$

Reducing equations (1)-(5) to a system of dimensionless nonlinear ordinary differential equations [67]:

$$f'''+ff''-(f')^2-Nf'=0. \tag{2.6}$$

$$g'''+fg''-3f'g'+2f''g-Ng'=0. \tag{2.7}$$

$$C_0''+ScfC_0'-KC_0=0. \tag{2.8}$$

$$C_1''-Scf'C_1+ScgC_0'+ScfC_1'-KC_1=0. \tag{2.9}$$

The boundary conditions become:

$$f'(\eta)|_{\eta=0}=g'(\eta)|_{\eta=0}=1,\ f(\eta)|_{\eta=0}=g(\eta)|_{\eta=0}=0,\ C_0(\eta)|_{\eta=0}=C_1(\eta)|_{\eta=0}=1. \tag{2.10}$$

$$f'(\eta)|_{\eta\to\infty}=g'(\eta)|_{\eta\to\infty}=C_0(\eta)|_{\eta\to\infty}=C_1(\eta)|_{\eta\to\infty}=0, \tag{2.11}$$

where f and g are the functions of velocity, C_0 and C_1 are species concentrations in the fluid, N and K are magnetic and chemical reaction parameters respectively and Sc is Schmidt number. The primes denote differentiation for η.

3. Preliminaries

In this section, the generalized Laguerre polynomials, the modified generalized Laguerre functions, the QLM method and collocation method are explained.

3.1. Generalized laguerre polynomials (GLPs). The Generalized Laguerre Polynomials (GLPs) have been widely used in polynomials approximation, solving integral or differential equations on semi-infinite intervals[65, 17, 87, 61, 45, 27, 25]. They are denoted by $GL_n^{\alpha}(x)$ and are defined as follows [15, 51]:

$$GL_n^{\alpha}(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n+\alpha \\ n-k \end{array} \right) \frac{(-1)^k}{k!} x^k, \quad x \in \Lambda, \ n \in \mathrm{N} \cup \{0\}, \tag{3.1}$$

where n is the degree of polynomial and nonnegative integers, $\alpha > -1$, $\Lambda = (0, \infty)$. GLPs have the following recurrence relations[51, 75, 88]:

$$\begin{aligned} & GL_0^{\alpha}(x) = 1, \\ & GL_1^{\alpha}(x) = 1 + \alpha - x, \\ & (n+1)GL_{n+1}^{\alpha}(x) = (2n + \alpha + 1 - x)GL_n^{\alpha}(x) - (n+\alpha)GL_{n-1}^{\alpha}(x), \end{aligned} \tag{3.2}$$

where $n \geq 1, \ n \in \mathrm{N}$. Moreover, GLPs satisfy the applied following properties; for $\alpha > -1$ and $n \in \mathrm{N} \cup \{0\}$, they hold [51, 16, 75, 15, 88]:

$$GL_n^{\alpha}(x) = GL_n^{\alpha+1}(x) - GL_{n-1}^{\alpha+1}(x). \tag{3.3}$$

$$\partial_x GL_n^{\alpha}(x) = -GL_{n-1}^{\alpha+1}(x) = -\sum_{k=0}^{n-1} GL_k^{\alpha}(x). \tag{3.4}$$

$$x\partial_x GL_n^{\alpha}(x) = nGL_n^{\alpha}(x) - (n+\alpha)GL_{n-1}^{\alpha}(x). \tag{3.5}$$

$$GL_n^{\alpha}(x) = \partial_x GL_n^{\alpha}(x) - \partial_x GL_{n+1}^{\alpha}(x). \tag{3.6}$$

The Sturm-Liouville equation of GLPs as follows[51, 16]:

$$x^{-\alpha} e^x \partial_x (x^{\alpha+1} e^{-x} \partial_x GL_n^{\alpha}(x)) + \lambda_n GL_n^{\alpha}(x) = 0, \quad n \in \mathrm{N} \cup \{0\}. \tag{3.7}$$

This above formula is equivalent by,

$$x\partial_x^2 GL_n^{\alpha}(x) + (\alpha + 1 - x)\partial_x GL_n^{\alpha}(x) + \lambda_n GL_n^{\alpha}(x) = 0, \quad n \in \mathrm{N} \cup \{0\}. \tag{3.8}$$

With the corresponding eigenvalue, $\lambda_n = n$, for any $\alpha > -1$.
The Rodrigues' formula for GLPs is [51, 15]:

$$e^{-x}x^{\alpha}GL_n^{\alpha}(x) = \frac{1}{n!}\frac{\partial^n}{\partial x^n}(e^{-x}x^{n+\alpha}). \tag{3.9}$$

and for any $\alpha > -1$, GLPs satisfy the following orthogonality relation [51, 75]:

$$\int_{\Lambda} GL_n^{\alpha}(x)GL_m^{\alpha}(x)x^{\alpha}e^{-x}dx = \gamma_n^{\alpha}\delta_{n,m}, \tag{3.10}$$

Where $\gamma_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{n!}$, n ,m are nonnegative integers, $\Lambda = (0, \infty)$ and $\delta_{n,m}$ is the Kronecker function. $x^{\alpha}e^{-x}$ is named the weight function.

3.2. **Modified generalized laguerre functions (MGLFs).** The modified generalized Laguerre functions (MGLFs), $MGL_n^{\alpha}(x)$, are defined as follows[65, 61, 66, 63, 45, 51]:

$$MGL_n^{\alpha}(x) = e^{-x}GL_n^{\alpha}(x), \quad x \in (0, \infty),\ n \in \mathrm{N} \cup \{0\}. \tag{3.11}$$

Due to above formula, MGLFs satisfy the following properties:

$$MGL_n^{\alpha}(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n+\alpha \\ n-k \end{array} \right) \frac{(-1)^k}{k!} x^k e^{-x}, \quad x \in \Lambda,\ n \in \mathrm{N} \cup \{0\}. \tag{3.12}$$

$$(n+1)MGL_{n+1}^{\alpha}(x) = (2n+\alpha+1-x)MGL_n^{\alpha}(x) - (n+\alpha)MGL_{n-1}^{\alpha}(x). \tag{3.13}$$

$$MGL_n^{\alpha}(x) = MGL_n^{\alpha+1}(x) - MGL_{n-1}^{\alpha+1}(x). \tag{3.14}$$

$$\partial_x MGL_n^{\alpha}(x) = -MGL_{n-1}^{\alpha+1}(x) - MGL_n^{\alpha}(x). \tag{3.15}$$

Moreover, MGLFs have the orthogonal system with respect to $w(x) = x/2$ as weight function.

3.3. **The quasiLinearization method (QLM).** One of the accurate, computationally, stable, simple, quick and convergence methods is the quasilinearization method (QLM), which is a well-known method to solve nonlinear differential equations, ordinary or partial. This technique leads to a sequence of linear equations and is developed basically by the Newton-Raphson method. This method has the quadratic convergence, i.e. $O(h^2)$ [51, 20, 55, 54]. Let us suppose that have a nonlinear ordinary equation on the semi-infinite domain as follows:

$$\frac{\partial^3 f}{\partial x^3} = G(f'', f', f, x). \tag{3.16}$$

With the boundary conditions: $f(0) = A$, $f'(0) = B$, $f'(\infty) = C$ for A, B, C are the real constant. QLM approximates the solution iteratively, and linearized the system in the following:

$$\frac{\partial^3 f_{i+1}}{\partial x^3} = G(f_i'', f_i', f_i, x) + \sum_{j=0}^{2} (f_{i+1}^{(j)} - f_i^{(j)}) \frac{\partial G(f_i'', f_i', f_i, x)}{\partial f_i^{(j)}}. \tag{3.17}$$

Where $f^{(j)}$ is the j-th derivative of function f and i is the number of iterations. with the boundary conditions: $f_{i+1}(0) = A$, $f'_{i+1}(0) = B$, $f'_{i+1}(\infty) = C$ for every iteration. The iterations stop when $|f_{i+1} - f_i| \leq \varepsilon$ for a small positive constant.

3.4. **The collocation method (CM).** The collocation method (CM) is a numerical technique used to solve ordinary, partial, and integral equations. This method comprises two essential components: the candidate solution and the collocation points. First, the candidate solution is expanded into a finite-dimensional space, such as polynomials up to a certain degree. Then, the equation is rewritten in terms of the candidate solution, and suitable distinct points, known as collocation points, are selected. These collocation points are substituted into the equation's solution, leading to a system of linear equations. Solving this linear system yields an accurate solution for the equation [3, 18]. This approach is suitable when an exact solution is unavailable, and the accuracy of the solution obtained can be estimated by calculating the error. Suppose the equation is as follows:

$$H(f^{(n)}, f^{(n-1)}, ..., f'', f', f, x) = b(x). \tag{3.18}$$

Where H is the nonlinear function of functional derivatives of$f$ function and b is the right side of the equation such as the constant number. $\tilde{f}(x)$ is the approximate function which is obtained by using the candidate solution as follows:

$$\tilde{f}(x) = \sum_{i=0}^{N} a_i \varphi_i(x). \tag{3.19}$$

Where $\varphi_i(x), \quad i = 0, ..., N$ is sub candidate solution such as polynomials have zero degree to N degree and $a_i, \quad i = 0, ..., N$ are the unknown coefficients. Rewriting the left side of eq. (29) via $\tilde{f}(x)$. The error estimation is defined in the following:

$$\tilde{R}(x) = H(\tilde{f}^{(n)}(x), \tilde{f}^{(n-1)}(x), ..., \tilde{f}^2(x), \tilde{f}^1(x), \tilde{f}(x), x) - b(x). \tag{3.20}$$

The above formula is called the residual function and put on it, the collocation points and solve the linear system equations. Ideally, if the residual function is zero, then gets the exact solution. When the residual function approaches to zero more and more, achieve a more accurate solution.

4. Application of the method

In this section, we explain to apply QLM-CM-MGLFs for solving the viscous flow over a nonlinearly stretching sheet with chemical reaction, heat transfer, and magnetic field. As we explain the mathematical problem in section two, we should solve the system of ordinary differential equations. We focus on eq.(2.6); this equation is the nonlinear ordinary equation and is not dependent on the others. For this reason, we first solve it to achieve a more accurate solution and apply this to other equations. So we expand the function f , in the form of truncated series of MGLFs as follows:

$$f(\eta) = \sum_{i=0}^{m_1} a_i MGL_i^{\alpha}(\eta) = A.MGL^T = MGL.A^T. \tag{4.1}$$

Where $a_i,\ i = 0, ..., m_1$ are the unknown coefficients, $MGL_i^{\alpha}(\eta),\ i = 0, ..., m_1$ are basic functions of different degree of MGLFs.
$A = [a_0, a_1, a_2, ..., a_{m_1}]^T$ and $MGL = [MGL_0^{\alpha}, MGL_1^{\alpha}, MGL_2^{\alpha}, ..., MGL_{m_1}^{\alpha}]^T$ are the vector forms.
Moreover, using the QLM on Eq.(2.6), we have the following relations:

$$\frac{\partial^3 f}{\partial \eta^3} = G(f'', f', f, \eta) = (f')^2 + Nf' - ff''. \tag{4.2}$$

Subject to the boundary conditions:

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) \to 0. \tag{4.3}$$

For satisfying the two boundary conditions at zero, we have:

$$\psi_1 = \eta + \eta^2 f. \tag{4.4}$$

This approximation provides the more flexibility and is not unique. For satisfying the boundary condition at infinity, we put the last collocation point to derivative function ψ_1, so we choose the suitable intervals for collocation points to convince the conditions of problem. Now we can write each order of derivative of function $\psi_1(\eta)$as follows:

$$\frac{\partial \psi_1}{\partial \eta} = 1 + 2\eta \sum_{i=0}^{m_1} a_i MGL_i^{\alpha} + \eta^2 \sum_{i=0}^{m_1} a_i \frac{\partial MGL_i^{\alpha}}{\partial \eta}. \tag{4.5}$$

$$\frac{\partial^2 \psi_1}{\partial \eta^2} = 2\sum_{i=0}^{m_1} a_i MGL_i^{\alpha} + 4\eta \sum_{i=0}^{m_1} a_i \frac{\partial MGL_i^{\alpha}}{\partial \eta} + \eta^2 \sum_{i=0}^{m_1} a_i \frac{\partial^2 MGL_i^{\alpha}}{\partial \eta^2}. \tag{4.6}$$

$$\frac{\partial^3 \psi_1}{\partial \eta^3} = 6\sum_{i=0}^{m_1} a_i \frac{\partial MGL_i^{\alpha}}{\partial \eta} + 6\eta \sum_{i=0}^{m_1} a_i \frac{\partial^2 MGL_i^{\alpha}}{\partial \eta^2} + \eta^2 \sum_{i=0}^{m_1} a_i \frac{\partial^3 MGL_i^{\alpha}}{\partial \eta^3}. \tag{4.7}$$

According to the QLM, the residual function obtains to the following relation:

$$(\tilde{R}_1)_i(\eta) = -\frac{\partial^3(\psi_1)_{i+1}}{\partial\eta^3} + G((\psi_1^{''})_i, (\psi_1^{'})_i, (\psi_1)_i, \eta) + ((\psi_1)_{i+1} - (\psi_1)_i)\frac{\partial G}{\partial(\psi_1)_i}$$
$$+((\psi_1^{'})_{i+1} - (\psi_1^{'})_i)\frac{\partial G}{\partial(\psi_1^{'})_i} + ((\psi_1^{''})_{i+1} - (\psi_1^{''})_i)\frac{\partial G}{\partial(\psi_1^{''})_i}. \tag{4.8}$$

Where $G((\psi_1^{''})_i, (\psi_1^{'})_i, (\psi_1)_i, x) = ((\psi_1^{'})_i)^2 + N(\psi_1^{'})_i - (\psi_1)_i(\psi_1^{''})_i$.
By using roots of shifted Laguerre polynomials on the certain domain as the collocation points to the residual function $(\tilde{R}_1)_i(\eta)$, the set of equations for each iteration can be solved, consequently, the unknown coefficients $a_i, i = 0, ..., m_1$ for each iteration will be obtained. We can continue iterations to achieve the arbitrary amount of $(\tilde{R}_1)_i(\eta)$ such as 10^{-9} or the certain iterations such as 25. The obtained results of last iteration for the function $(\psi_1)_i, i = last\_itr$, consequently, the function $f(\eta)$ is used to another equations. Now considering to Eq.(2.7) and according to obtained solution for $f(\eta)$ and applying it to this equation, we have a linear ordinary equation then we can solve it by using CM. Therefore we can expand the function g, in the form of truncated series of MGLFs as follows:

$$g(\eta) = \sum_{i=0}^{m_2} b_i MGL_i^{\alpha}(\eta) = B.MGL_2^T = MGL_2.B^T, \tag{4.9}$$

where $b_i \quad , i = 0, ..., m_2$ are the unknown coefficients, $MGL_i^{\alpha}(\eta), \quad i = 0, ..., m_2$ are basic functions of different degree of MGLFs.
$B = [b_0, b_1, b_2, ..., b_{m_2}]^T$ and $MGL_2 = [MGL_0^{\alpha}, MGL_1^{\alpha}, MGL_2^{\alpha}, ..., MGL_{m_2}^{\alpha}]^T$ are the vector forms.
Subject to the boundary conditions:

$$g(0) = 0, g'(0) = 1, g'(\infty) \to 0. \tag{4.10}$$

for satisfying the two boundary conditions at zero, we have:

$$\psi_2 = \eta + \eta^2 g. \tag{4.11}$$

Now we can write each order of derivative of function $\psi_2(\eta)$as follows:

$$\frac{\partial\psi_2}{\partial\eta} = 1 + 2\eta\sum_{i=0}^{m_2} b_i MGL_i^{\alpha} + \eta^2\sum_{i=0}^{m_2} b_i\frac{\partial MGL_i^{\alpha}}{\partial\eta}. \tag{4.12}$$

$$\frac{\partial^2\psi_2}{\partial\eta^2} = 2\sum_{i=0}^{m_2} b_i MGL_i^{\alpha} + 4\eta\sum_{i=0}^{m_2} b_i\frac{\partial MGL_i^{\alpha}}{\partial\eta} + \eta^2\sum_{i=0}^{m_2} b_i\frac{\partial^2 MGL_i^{\alpha}}{\partial\eta^2}. \tag{4.13}$$

$$\frac{\partial^3\psi_2}{\partial\eta^3} = 6\sum_{i=0}^{m_2} b_i\frac{\partial MGL_i^{\alpha}}{\partial\eta} + 6\eta\sum_{i=0}^{m_2} b_i\frac{\partial^2 MGL_i^{\alpha}}{\partial\eta^2} + \eta^2\sum_{i=0}^{m_2} b_i\frac{\partial^3 MGL_i^{\alpha}}{\partial\eta^3}. \tag{4.14}$$

So the residual function,$\tilde{R}_2(\eta)$ obtains to the following relation:

$$\tilde{R}_2(\eta) = \psi_2''' + f_{new}\psi_2'' - 3f'_{new}\psi_2' + 2f''_{new}\psi_2 - N\psi_2', \tag{4.15}$$

where f_{new} is obtained as the execution of QLM. Now we have a linear algebraic equation basis on unknown coefficients $b_i, i = 0, ..., m_2$. For achieving this equation to a linear system of algebraic equations, we put roots of Laguerre polynomials as the collocation points to the residual function $\tilde{R}_2(\eta)$ and for satisfying the boundary condition at infinity; we put the last collocation point to derivative functionψ_2, consequently, the unknown coefficients $b_i, i = 0, ..., m_2$ and g_{new} will be obtained and used to another equations. Moreover, we focus to eq. (2.8), use the expanding form of the functionC_0, in the form of truncated series of MGLFs as follows:

$$C_0(\eta) = \sum_{i=0}^{m_3} e_i MGL_i^{\alpha}(\eta) = E.MGL_3^T = MGL_3.E^T. \tag{4.16}$$

Where $e_i, i = 0, ..., m_3$ are the unknown coefficients, $MGL_i^{\alpha}(\eta), i = 0, ..., m_3$ are basic functions of different degree of MGLFs.
$E = [e_0, e_1, e_2, ..., e_{m_3}]^T$ and $MGL_3 = [MGL_0^{\alpha}, MGL_1^{\alpha}, MGL_2^{\alpha}, ..., MGL_{m_3}^{\alpha}]^T$ are the vector forms.
Subject to the boundary conditions:

$$C_0(0) = 1, C_0(\infty) \to 0. \tag{4.17}$$

For satisfying the boundary condition at zero, we have:

$$\psi_3 = 1 + \eta C_0. \tag{4.18}$$

Now we can write each order of derivative of function $\psi_3(\eta)$as follows:

$$\frac{\partial \psi_3}{\partial \eta} = \sum_{i=0}^{m_3} e_i MGL_i^{\alpha} + \eta \sum_{i=0}^{m_3} e_i \frac{\partial MGL_i^{\alpha}}{\partial \eta}. \tag{4.19}$$

$$\frac{\partial^2 \psi_3}{\partial \eta^2} = 2\sum_{i=0}^{m_3} e_i \frac{\partial MGL_i^{\alpha}}{\partial \eta} + \eta \sum_{i=0}^{m_3} e_i \frac{\partial^2 MGL_i^{\alpha}}{\partial \eta^2}. \tag{4.20}$$

There for the residual function,$\tilde{R}_3(\eta)$ defines to the following relation:

$$\tilde{R}_3(\eta) = \psi_3'' + Scf_{new}\psi_3' - K\psi_3. \tag{4.21}$$

Where f_{new} is obtained as the execution of QLM. Using roots of Laguerre polynomials as the collocation points to the residual function $\tilde{R}_3(\eta)$ lead to obtaining a linear system of equations. By solving this system, we obtain the unknown coefficients $e_i, i = 0, ..., m_3$. Furthermore, the boundary condition at infinity is satisfied by using the last collocation point to the functionψ_3 . Finally, we focus to eq. (2.9). We define the function C_1 by using the truncated series of MGLFs form as follows:

$$C_1(\eta) = \sum_{i=0}^{m_4} z_i MGL_i^{\alpha}(\eta) = Z.MGL_4^T = MGL_4.Z^T. \tag{4.22}$$

where $z_i, i = 0, ..., m_4$ are the unknown coefficients, $MGL_i^{\alpha}(\eta), i = 0, ..., m_4$ are basic functions of different degree of MGLFs.
$Z = [z_0, z_1, z_2, ..., z_{m_4}]^T$ and $MGL_4 = [MGL_0^{\alpha}, MGL_1^{\alpha}, MGL_2^{\alpha}, ..., MGL_{m_4}^{\alpha}]^T$ are the vector forms.
with the boundary conditions:

$$C_1(0) = 0, C_1(\infty) \to 0. \tag{4.23}$$

The boundary condition at zero satisfies as follows:

$$\psi_4 = \eta C_1. \tag{4.24}$$

Each order of derivative of function $\psi_4(\eta)$as follows:

$$\frac{\partial \psi_4}{\partial \eta} = \sum_{i=0}^{m_4} z_i MGL_i^{\alpha} + \eta \sum_{i=0}^{m_4} z_i \frac{\partial MGL_i^{\alpha}}{\partial \eta}. \tag{4.25}$$

$$\frac{\partial^2 \psi_4}{\partial \eta^2} = 2 \sum_{i=0}^{m_4} z_i \frac{\partial MGL_i^{\alpha}}{\partial \eta} + \eta \sum_{i=0}^{m_4} z_i \frac{\partial^2 MGL_i^{\alpha}}{\partial \eta^2}. \tag{4.26}$$

Moreover the residual function,$\tilde{R}_4(\eta)$ defines to the following relation:

$$\tilde{R}_4(\eta) = \psi_4'' - Scf_{new}'\psi_4 + Scg_{new}(C_0)'_{new} + Scf_{new}\psi_4' - K\psi_4, \tag{4.27}$$

where f_{new} , g_{new} , $(C_0)_{new}$ are obtained on prior steps. A linear system of equations achieve from the function $\tilde{R}_4(\eta)$ when roots of Laguerre polynomials which are the collocation points are put to it. We get the unknown coefficients $z_i, i = 0, ..., m_4$.by solving this system. The infinite boundary condition is obtained by placing the last collocation point in the function $\psi_4(\eta)$.

5. Results and discussions

In this section, we present the numerical values by implementing the QLM-CM-MGLFs on viscous flow in Maple. We obtain numerical results and propose graphical results for the velocity profiles and tables. The numerical results are validated by making a comparison with the published works [2, 23, 89, 70, 67]. Fixed number of collocation points, $m_i = 20, i = 1, 2, 3, 4$ for this experiment are used. We select zeros of shifted Laguerre functions as collocation points. The effects of three parameters *N*, *K* and *Sc* on the boundary layers are shown in Figs 1-8. We present the numerical values of $f''(0)$, C_0' and C_1'. Moreover graphical results for $f'(\eta)$, $g'(\eta)$, $C_0(\eta)$ and $C_1(\eta)$ with the different values of parameters *N*, *K* and *Sc* are presented.

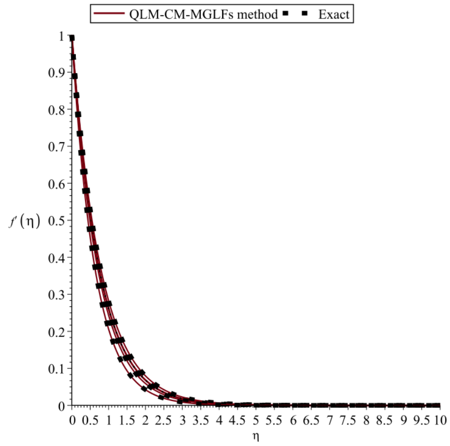


FIGURE 1. Velocity profile $f'(\eta)$ for some N with Sc=0.24, K=0.2

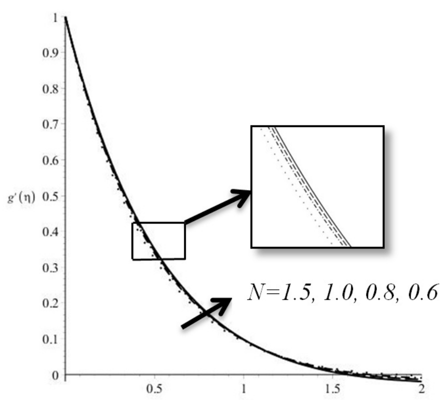


FIGURE 2. Velocity profile $g'(\eta)$ for some N with Sc=0.24, K=0.2.

The velocity profiles for different N and fixed Sc=0.24 and K=0.2 are plotted in Fig1. This reveals that the velocity profiles boundary layer reduces with the increment of the magnetic parameter. As shown in Fig1, QLM-CM-MGLFs method was successful. Integration of Eq.(6) with boundary conditions Eqs. (10, 11) has an exact solution [2] as:

$$f(\eta) = \frac{1}{\sqrt{1+N}}(1 - e^{-\sqrt{1+N}\eta}). \tag{5.1}$$

For N=0, $f(\eta) = 1 - e^{-\eta}$ is the unique solution of the problem [12].
The behavior of velocity profile $g'(\eta)$for increasing the magnetic parameter

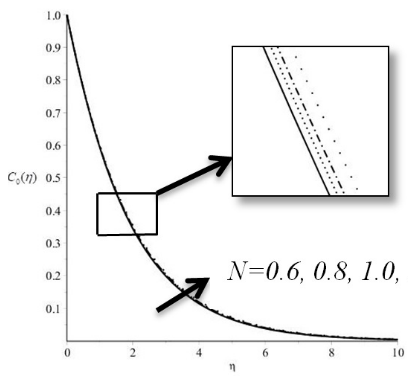


FIGURE 3. Concentration profile$C_0(\eta)$ for some N with Sc=0.24, K=0.2.

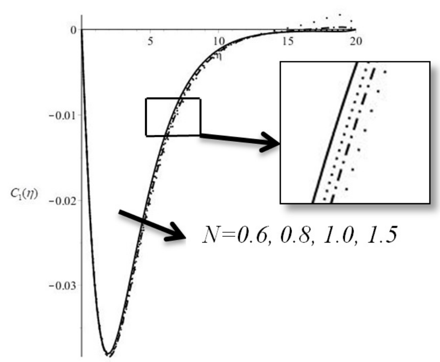


FIGURE 4. Concentration profile $C_1(\eta)$ for some N with Sc=0.24, K=0.2.

N is also seen from Fig2. Increasing the magnetic parameter N makes the velocity profile reduces. Influences of the increment parameter N for the species concentrations $C_0(\eta)$ and $C_1(\eta)$ are shown in Figs 3 and 4. Furthermore, as shown in Figs 5 and 6, for the effects of Schmidt number Sc, the concentrations profiles $C_0(\eta)$ and $C_1(\eta)$ reduce with different value of the Schmidt number Sc. The concentrations profile $C_1(\eta)$ approaches to the bottom and reaches its minimum at the same distance of η from the wall before oscillates back to zero.

We also included in Figs 7 and 8 the graphs of concentration profile. The effect of the destructive chemical parameter K is shown. These illustrate that the destructive chemical reaction parameter reduces as the concentration of $C_0(\eta)$ and $C_1(\eta)$ increases. Table 4 shows the comparison between QLM-CM-MGLFs

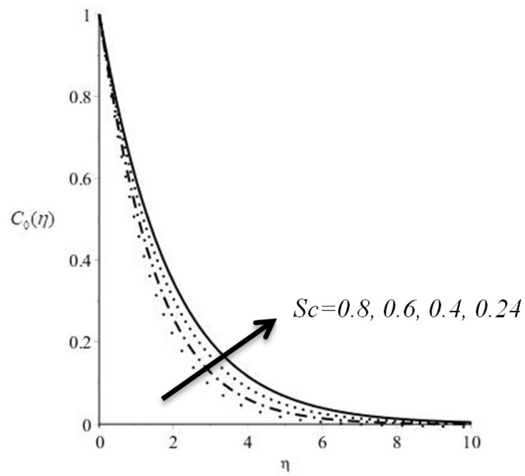


FIGURE 5. Concentration profile $C_0(\eta)$ for some Sc with N=0.8, K=0.2.

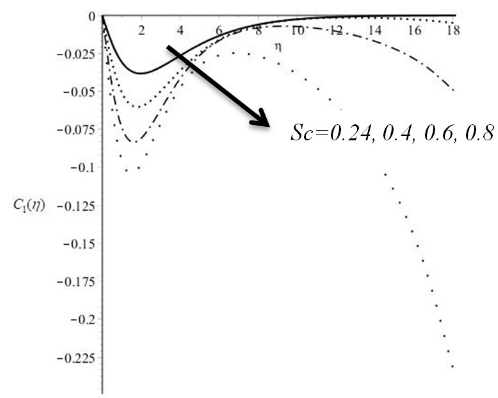


FIGURE 6. Concentration profile $C_1(\eta)$ for some Sc with N=0.8, K=0.2.

method and other methods of velocity profile $-f''(0)$ for various magnetic parameter N when K=0.2 and Sc=0.24.

The residual function for QLM-CM-MGLFs method at K=0.2, N=0.8 and Sc=0.24 are plotted in Fig 9. In Tables 5,6 are observed that the concentrations profiles $C_0(\eta)$ and $C_1(\eta)$ with various N when K=0.2 and Sc=0.24. The concentrations profiles $C_0(\eta)$ and $C_1(\eta)$ with various Sc when K=0.2 and N=0.8 are shown in Tables 7,8. The concentrations profiles $C_0(\eta)$ and $C_1(\eta)$ evaluated at Sc=0.2 and N=0.8 with various K are shown in Tables 9, 10.

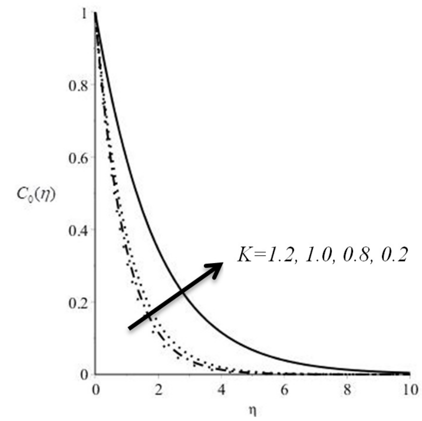


FIGURE 7. Concentration profile $C_0(\eta)$ for some K with N=0.8, Sc=0.24.

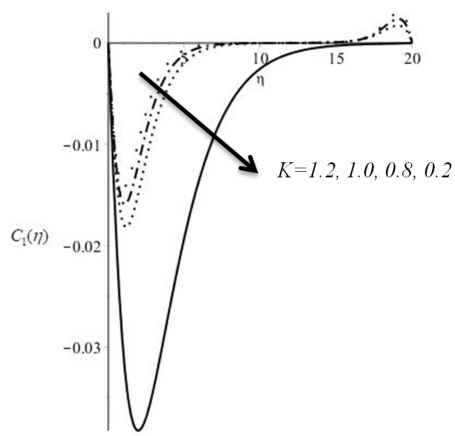
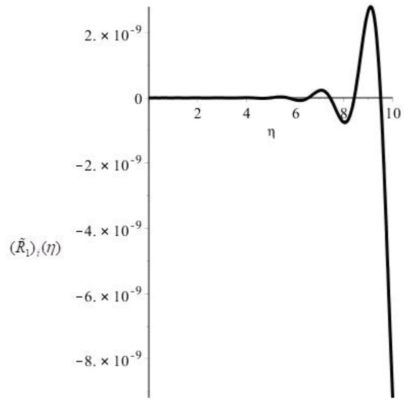


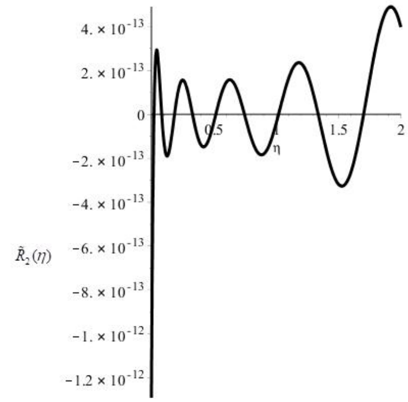
FIGURE 8. Concentration profile $C_1(\eta)$ for some K with N=0.8, Sc=0.24.

6. Conclusions

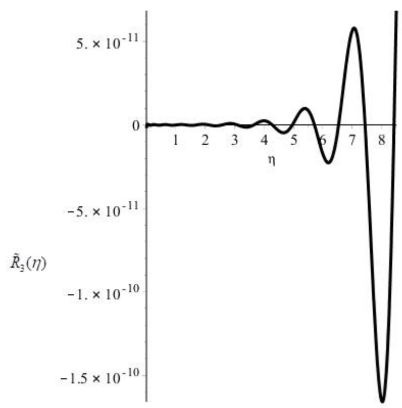
The purpose of this paper is to introduce the accurate numerical method which is named the QLM-CM-MGLFs method. This method is the combination of quasilinearization and collocation methods based on modified generalized Laguerre functions and are applied to problems in the semi-infinite domain. We used this method to solve the system's ordinary equations which included the nonlinear ordinary equations. We choose the viscose flow problem over a non-linearly semi-infinite stretching sheet under the influence of a magnetic field and chemical reactions and introduce an efficient and accurate solution.



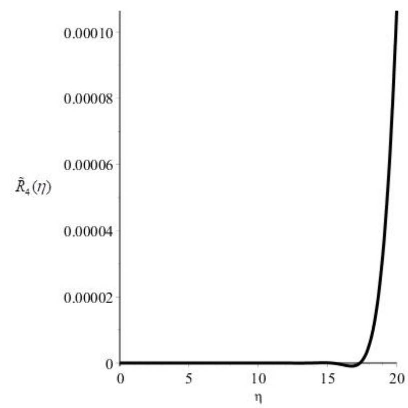
(A) Residual function$(\tilde{R}_1)_i(\eta), i = 20$.



(B) Residual function$\tilde{R}_2(\eta)$.



(C) residual function $\tilde{R}_3(\eta)$.



(D) Residual function $\tilde{R}_4(\eta)$.

FIGURE 9. The residual functions for QLM-CM-MGLFs method at $K = 0.2, N = 0.8$ and $Sc = 0.24$.

The effects of the magnetic parameter N, the Schmidt number Sc, destructive chemical reaction parameter K over the velocity profile $-f''(0)$ and concentration profiles $-C_0'(0), -C_1'(0)$ are presented. Furthermore, a comparison of the numerical results of the method with other methods like RBF (radial basis functions), HAM (homotopy analysis method), and RK (runge-kutta) methods are reported. The results show that QLM-CM-MGLFs have high accuracy and high rate convergence. The figures of residual functions show that the paper's method has a proper convergence rate. Therefore results explained QLM-CM-MGLFs is the proper accurate numerical method that can be applied to solve the nonlinear system of problems on the semi-infinite domain and has high reliability and efficiency.

N	HAM[89]	ADM[23]	RK[89]	RBF[67]	Exact solution[2]	Present method
0.0	–	1.0000000	–	1.0000000	1.0000000	1.00062373
0.6	1.264934	–	1.264911	1.265029	1.264911	1.26491121
0.8	1.341624	–	1.341640	1.341668	1.341640	1.34164080
1.0	1.414017	–	1.414213	1.414221	1.414213	1.41421356
1.5	1.580231	1.581139	1.581113	1.581139	1.581139	1.58113883
9.0	–	3.162278	–	3.162278	3.162278	3.16227766
25	–	5.099019	–	5.099019	5.099019	5.099019510

TABLE 4. Variation of the velocity profile $-f''(0)$ for various values N at K=0.2 and Sc=0.24 with different methods.

N	HAM[89]	RK[89]	RBF[67]	Present method
0.6	0.505910	0.50590	0.505874	0.5059083704
0.8	0.503601	0.50380	0.503807	0.5038173855
1.0	0.501943	0.50191	0.501965	0.501969266
1.5	0.498112	0.49813	0.498136	0.498136478

TABLE 5. Variation of the concentrations profile $-C_0'(0)$ for various values N at K=0.2 and Sc=0.24 with different methods.

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N	HAM[89]	RK[89]	RBF[67]	Present method
0.6	0.040063	0.04035	0.040300	0.040368295
0.8	0.040320	0.04022	0.040249	0.040272071
1.0	0.040070	0.04004	0.040073	0.040080567
1.5	0.039380	0.03937	0.039358	0.039357178

TABLE 6. Variation of the concentrations profile $-C_1'(0)$ for various values N at K=0.2 and Sc=0.24 with different methods.

Sc	HAM[89]	RK[89]	RBF[67]	Present method
0.24	0.503601	0.50380	0.503807	0.50381738559
0.40	0.544218	0.54429	0.544198	0.54421803445
0.60	0.596620	0.59659	0.596580	0.59666880808
0.80	0.650062	0.65001	0.65005	0.65056739908

TABLE 7. Variation of the concentrations profile $-C_0'(0)$ for various values Sc at K=0.2 and N=0.8 with different methods.

Sc	HAM[89]	RK[89]	RBF[67]	Present method
0.24	0.0403201	0.04022	0.0402495	0.040272071
0.40	0.0673822	0.06795	0.0680524	0.068096680
0.60	0.1014366	0.10221	0.1022899	0.102613246
0.80	0.1338458	0.13469	0.1348760	0.137458586

TABLE 8. Variation of the concentrations profile $-C_1'(0)$ for various values Sc at K=0.2 and N=0.8 with different methods.

K	HAM[89]	RK[89]	RBF[67]	Present method
0.2	0.503601	0.50380	0.503807	0.50381738559
0.8	0.933541	0.93358	0.933537	0.93353807310
1.0	1.036527	1.03652	1.036523	1.036524488
1.2	1.129936	1.12992	1.129918	1.12991859168

TABLE 9. Variation of the concentrations profile $-C_0'(0)$ for various values K at Sc=0.24 and N=0.8 with different methods.

K	HAM[89]	RK[89]	RBF[67]	Present method
0.2	0.040320	0.040220	0.040250	0.040272071
0.8	0.030445	0.030667	0.030669	0.030673082
1.0	0.028971	0.029101	0.029102	0.029105341
1.2	0.027672	0.027822	0.027823	0.027825153

TABLE 10. Variation of the concentrations profile $-C_1'(0)$ for various values K at Sc=0.24 and N=0.8 with different methods.

References

1. P. Agarwal, F. Qi, M. Chand, Sh. Jain, Certain Integrals Involving the Generalized Hypergeometric Function and the Laguerre Polynomials, *Journal of Computational and Applied Mathematics*, **313**, (2017), 307-317.
2. F. T. Akyildiz, H. Bellout, K. Vajravelu, Diffusion of Chemically Reactive Species in a Porous Medium Over a Stretching Sheet, *Journal of mathematical analysis and applications*, **320**(1), (2006), 322-339.
3. U. M. Ascher, L. R. Petzold, *Computer Methods for Ordinary Differential Equations and Differential-algebraic Equations*, Siam, 1998.
4. F. B. Babolghani, K. Parand, Comparison Between Hermite and Sinc Collocation Methods for Solving Steady Slow of a Third Grade Fluid in a Porous Half Space, *Proceedings of the International Conference on Scientific Computing*, (2013), 124.
5. A. H. Bhrawy, A. S. Alofi, A Jacobi–Gauss Collocation Method for Solving Nonlinear Lane–Emden Type Equations, *Communications in Nonlinear Science and Numerical Simulation*, **17**(1), (2012), 62-70.
6. J. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover Publications Inc., New York, 2000.
7. J. Boyd, C. Rangan, P. Bucksbaum, Pseudospectral Methods on a Semi-Infinite Interval With Application to the Hydrogen Atom: a Comparison of the Mapped Fourier-Sine Method With Laguerre Series and Rational Chebyshev Expansions, *Journal of Computational Physics*, **188**(1), (2003), 56-74.
8. O. Cherroud, S. A. Yahiaoui, M. Bentaiba, Generalized Laguerre Polynomials With Position-Dependent Effective Mass Visualized via Wigner's Distribution Functions, *Journal of Mathematical Physics*, **58**(6), (2017), 063503.
9. C. Christov, A Complete Orthonormal System of Functions in L^2(-∞,∞) Space, *SIAM Journal on Applied Mathematics*, **42**(6), (1982), 1337-1344.
10. F. Comte, Ch. Cuenod, M. Pensky, Y. Rozenholc, Laplace Deconvolution on the Basis of Time Domain Data and Its Application to Dynamic Contrast-Enhanced Imaging, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **79**(1), (2017), 69-94.
11. O. R. Coulaud, D. Funaro, O. Kavian, Laguerre Spectral Approximation of Elliptic Problems in Exterior Domains, *Computer Methods in Applied Mechanics and Engineering*, **80**(1-3), (1990), 451-458.
12. L. J. Crane, Flow Past a Stretching Plate, *Zeitschrift für angewandte Mathematik und Physik ZAMP*, **21**(4), (1970), 645-647.
13. M. Dehghan, F. Fakhar-Izadi, The Spectral Collocation Method With Three Different Bases for Solving a Nonlinear Partial Differential Equation Arising in Modeling of Nonlinear Waves, *Mathematical and Computer Modelling*, **53**(9-10), (2011), 1865-1877.
14. F. Fakhar-Izadi, M. Dehghan, The Spectral Methods for Parabolic Volterra Integro-Differential Equations, *Journal of Computational and Applied Mathematics*, **235**(14), (2011), 4032-4046.
15. D. Funaro, *Polynomial Approximation of Differential Equations*, Springer Science & Business Media, 2008.
16. B. Guo, J. Shen, Zh Wang, A Rational Approximation and Its Applications to Differential Equations on the Half Line, *Journal of scientific computing*, **15**(2), (2000), 117-147.
17. B. Guo, J. Shen, C. Xu, Generalized Laguerre Approximation and Its Applications to Exterior Problems, *Journal of computational Mathematics*, (2005), 113-130.
18. E. Hairer, S. P. Nørsett, G. Wanner, Solving Ordinary Differential Equations I. Nonstiff Problems, *Springer Series in Computational Mathematics*, **8**, (1993).

19. Z. Hajimohammadi, F. Baharifard, A. Ghodsi, K. Parand, Fractional Chebyshev Deep Neural Network (FCDNN) for Solving Differential Models, *Chaos, Solitons & Fractals*, **153**, (2021), 111530.
20. Z. Hajimohammadi, K. Parand, A New Numerical Solution for Solving the Flow of Eyring-Powell Fluid Problem via Fractional Rational Generalized Laguerre Polynomials, *48th Annual Iranian Mathematics Conference*, 2017.
21. G. Hojjati, K. Parand, An Efficient Computational Algorithm for Solving the Nonlinear Lane–Emden Type Equations, *Int. J. Math. Comp. Sci*, **7**(4), (2011), 182-187.
22. S. Kazem, J. A. Rad, K. Parand, M. Shaban, H. Saberi, The Numerical Study on the Unsteady Flow of Gas in a Semi-Infinite Porous Medium Using an RBF Collocation Method, *International Journal of Computer Mathematics*, **89**(16), (2012), 2240-2258.
23. S. A. Kechil, I. Hashim, Series Solution of Flow Over Nonlinearly Stretching Sheet With Chemical Reaction and Magnetic Field, *Physics Letters A*, **372**(13), (2008), 2258-2263.
24. Ch. Lin, Application of a V-belt Continuously Variable Transmission System by Using a Composite Recurrent Laguerre Orthogonal Polynomial Neural Network Control System and Modified Particle Swarm Optimization, *Journal of Vibration and Control*, **23**(9), (2017), 1437-1462.
25. Ch. Lin, Hybrid Recurrent Laguerre-Orthogonal-Polynomials Neural Network Control With Modified Particle Swarm Optimization Application for V-belt Continuously Variable Transmission System, *Neural Computing and Applications*, **28**(2), (2017), 245-264.
26. F. Liu, H. Li, Zh. Wang, Spectral Methods Using Generalized Laguerre Functions for Second and Fourth Order Problems, *Numerical Algorithms*, **75**(4), (2017), 1005-1040.
27. F. J. Liu, Z. Q. Wang, H. Y. Li, A Fully Diagonalized Spectral Method Using Generalized Laguerre Functions on the Half Line, *Advances in Computational Mathematics*, **43**(6), (2017), 1227-1259.
28. R. W. MacCormack, A Numerical Method for Solving the Equations of Compressible Viscous Flow, *AIAA journal*, **20**(9), (1982), 1275-1281.
29. A. M. Makarenkov, E. V. Seregina, M. A. Stepovich, The Projection Galerkin Method for Solving the Time-Independent Differential Diffusion Equation in a Semi-Infinite Domain, *Computational Mathematics and Mathematical Physics*, **57**(5), (2017), 802-814.
30. S. Mall, S. Chakraverty, Application of Legendre Neural Network for Solving Ordinary Differential Equations, *Applied Soft Computing*, **43**, (2016), 347-356.
31. S. Mehrkanoon, T. Falck, J. A. K. Suykens, Approximate Solutions to Ordinary Differential Equations Using Least Squares Support Vector Machines, *IEEE Trans. Neural Netw. Learn. Syst.*, **23**(9), (2012), 1356-1367.
32. I. S. Oyelakin, R. Ghosh, S. Mondal, P. Sibanda, Entropy Generation in Casson Nanofluid Flow Past an Electromagnetic Stretching Riga Plate, *Malaysian Journal of Mathematical Sciences*, **15**(3), (2021), 425-445.
33. I. S. Oyelakin, P. C. Lalramneihmawii, S. Mondal, P. Sibanda, Analysis of Double-Diffusion Convection on Three-Dimensional MHD Stagnation Point Flow of a Tangent Hyperbolic Casson Nanofluid, *International Journal of Ambient Energy*, **43**(1), (2020), 1854-1865.
34. I. S. Oyelakin, P. Mondal, S. Mondal, T. R. Mahapatra, P. Sibanda, Rheological Analysis of Suspended Single-Walled Carbon Nanotubes in a Walters' B Fluid, *Nanoscience & Nanotechnology-Asia*, **11**(6), (2021), 87-95.
35. I. S. Oyelakin, PC. Lalramneihmawii, S. Mondal, S.K. Nandy, P. Sibanda, Thermophysical Analysis of Three-Dimensional Magnetohydrodynamic Flow of a Tangent Hyperbolic Nanofluid, *Engineering Reports*, **2**(4), (2017), e12144.

36. I. S. Oyelakin, S. Mondal, P. Sibanda, Unsteady Casson Nanofluid Flow Over a Stretching Sheet With Thermal Radiation, Convective and Slip Boundary Conditions, *Alexandria Engineering Journal*, **55**(2), (2016), 1025-1035.
37. I. S. Oyelakin, S. Mondal, P. Sibanda, Unsteady Mixed Convection in Nanofluid Flow Through a Porous Medium with Thermal Radiation Using the Bivariate Spectral Quasilinearization Method, *Journal of Nanofluids*, 6(2), (2020), 273-281.
38. I. S. Oyelakin, S. Mondal, P. Sibanda, Nonlinear Radiation in Bioconvective Casson Nanofluid Flow, *International Journal of Applied and Computational Mathematics*, **5**(5), (2019), 1-20.
39. I. S. Oyelakin, S. Mondal, P. Sibanda, S. S. MOTSA, A Multi-Domain Bivariate Approach for Mixed Convection in a Casson Nanofluid with Heat Generation, *Walailak Journal of Science and Technology*, **16**(9), (2019), 681-699.
40. R. K. Pandey, N. Kumar, Solution of Lane–Emden Type Equations Using Bernstein Operational Matrix of Differentiation, *New Astronomy*, **17**(3), (2012), 303-308.
41. K. Parand, A. A. Aghaei, S. Kiani, T. IlkhasZadeh, Z. Khosravi, A Neural Network Approach for Solving Nonlinear Differential Equations of Lane–Emden Type, *Engineering with Computers*, **40**, (2024), 953–969.
42. K. Parand, A. A. Aghaei, M. Jani, A. Ghodsi, A New Approach to the Numerical Solution of Fredholm Integral Equations Using Least Squares-Support Vector Regression, *Math. Comput. Simul.*, **180**, (2021), 114-128.
43. K. Parand, F. B. Babolghani, Applying the Modified Generalized Laguerre Functions for Solving Steady Flow of a Third Grade Fluid in a Porous Half Space, *World Applied Science Journal*, **17**(4), (2012), 467-472.
44. K. Parand, F. B. Babolghani, Modified Generalized Laguerre Functions for a Numerical Investigation of Flow and Diffusion of Chemically Reactive Species Over a Nonlinearly Stretching Sheet, *World Applied Sciences Journal*, **17**(12), (2012), 1578-1587.
45. K. Parand, F. Baharifard, F. B. Babolghani, Comparison Between Rational Gegenbauer and Modified Generalized Laguerre Functions Collocation Methods for Solving the Case of Heat Transfer Equations Arising in Porous Medium, *International Journal of Industrial Mathematics*, **4**(2), (2012), 107-122.
46. K. Parand, M. Dehghan, A. Pirkhedri, Sinc-Collocation Method for Solving the Blasius Equation, *Physics Letters A*, **373**(44), (2009), 4060-4065.
47. K. Parand, M. Dehghan, A. Pirkhedri, The Use of Sinc-Collocation Method for Solving Falkner–Skan Boundary-Layer Equation, *International journal for numerical methods in fluids*,**68**(1), (2012), 36-47.
48. K. Parand, M. Dehghan, A. Pirkhedri, The Sinc-Collocation Method for Solving the Thomas–Fermi Equation, *Journal of Computational and Applied Mathematics*, **237**(1), (2013), 244-252.
49. K. Parand, M. Dehghan, A. Taghavi, Modified Generalized Laguerre Function Tau Method for Solving Laminar Viscous Flow, *International Journal of Numerical Methods for Heat & Fluid Flow*, **20**(7), (2010), 728-743.
50. K. Parand, Z. Delafkar, N. Pakniat, A. Pirkhedri, M. K. Haji, Collocation Method Using Sinc and Rational Legendre Functions for Solving Volterra's Population Model, *Communications in Nonlinear Science and Numerical Simulation*, **16**(4), (2011), 1811-1819.
51. K. Parand, Z. Hajimohammadi, Using Modified Generalized Laguerre Functions, QLM and Collocation Method for Solving an Eyring–Powell Problem, *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, **40**(4), (2018), 182.

52. K. Parand, M. Hemami, Application of Meshfree Method Based on Compactly Supported Radial Basis Function for Solving Unsteady Isothermal Gas Through a Micro–Nano Porous Medium, *Iranian Journal of Science and Technology, Transactions A: Science*, **41**(3), (2017), 677-684.
53. K. Parand, S. Khaleqi, The Rational Chebyshev of Second Kind Collocation Method for Solving a Class of Astrophysics Problems, *The European Physical Journal Plus*, **131**(2), (2016), 1-24.
54. K. Parand, Y. Lotfi, J. A. Rad, An Accurate Numerical Analysis of the Laminar Two-Dimensional Flow of an Incompressible Eyring-Powell Fluid Over a Linear Stretching Sheet, *The European Physical Journal Plus*, **132**(9), (2017), 397.
55. K. Parand, M. M. Moayeri, S. Latifi, M. Delkhosh, A Numerical Investigation of the Boundary Layer Flow of an Eyring-Powell Fluid Over a Stretching Sheet via Rational Chebyshev Functions, *The European Physical Journal Plus*, **132**(7), (2017), 325.
56. K. Parand, M. Nikarya, J. A. Rad, F. Baharifard, A New Reliable Numerical Algorithm Based on the First Kind of Bessel Functions to Solve Prandtl–Blasius Laminar Viscous Flow Over a Semi-Infinite Flat Plate, *Zeitschrift für Naturforschung A*, **67**(12), (2012), 665-673.
57. K. Parand, M. Razzaghi, Rational Chebyshev Tau Method for Solving Higher-Order Ordinary Differential Equations, *International Journal of Computer Mathematics*, **81**(1), (2004), 73-80.
58. K. Parand, M. Razzaghi, Rational Chebyshev Tau Method for Solving Volterra's Population Model, *Applied Mathematics and Computation*, **149**(3), (2004), 893-900.
59. K. Parand, M. Razzaghi, Rational Legendre Approximation for Solving Some Physical Problems on Semi-Infinite Intervals, *Physica Scripta*,**69**(5), (2004), 353.
60. K. Parand, M. Razzaghi, R. Sahleh, M. Jani, Least Squares Support Vector Regression for Solving Volterra Integral Equations, *Engineering with Computers*, **38**, (2022), 789-796.
61. K. Parand, A. R. Rezaei, A. Taghavi, Lagrangian Method for Solving Lane–Emden Type Equation Arising in Astrophysics on Semi-Infinite Domains, *Acta Astronautica*, **67**(7-8), (2010), 673-680.
62. K. Parand, M. Shahini, Rational Chebyshev Pseudospectral Approach for Solving Thomas–Fermi Equation, *Physics Letters A*, **373**(2), (2009), 210-213.
63. K. Parand, M. Shahini, A. Taghavi, Generalized Laguerre Polynomials and Rational Chebyshev Collocation Method for Solving Unsteady Gas Equation, *Int. J. Contemp. Math. Sci*, **4**(21),(2009), 1005-1011.
64. K. Parand, A. Taghavi, Generalized Laguerre Polynomials Collocation Method for Solving Lane-Emden Equation, *Applied Mathematical Sciences*, **2**(60), (2008), 2955-2961.
65. K. Parand, A. Taghavi, Rational Scaled Generalized Laguerre Function Collocation Method for Solving the Blasius Equation, *Journal of Computational and Applied Mathematics*, **233**(4), (2009), 980-989.
66. K. Parand, A. Taghavi, M. Shahini, Comparison Between Rational Chebyshev and Modified Generalized Laguerre Functions Pseudospactral Methods for Solving Lane-Emden and Unsteady Gas Equations, *Acta physica polonica B*, **40**(6), (2009).
67. J. A. Rad, K. Parand, S. Kazem, A Numerical Investigation to Viscous Flow Over Nonlinearly Stretching Sheet With Chemical Reaction, Heat Transfer and Magnetic Field, *International Journal of Applied and Computational Mathematics*, **3**(2), (2017), 919-935.
68. B. C. Sakiadis, Boundary-Layer Behavior on Continuous Solid Surfaces: I. Boundary-Layer Equations for Two-Dimensional and Axisymmetric Flow, *AIChE Journal*, **7**(1), (1961), 26-28.
69. B. C. Sakiadis, Boundary-Layer Behavior on Continuous Solid Surfaces: II. The Boundary Layer on a Continuous Flat Surface", *AiChE journal*, **7**(2), (1961), 221-225.

70. K. M. Sanni, S. Asghar, M. Jalil, N. F. Okechi, Flow of Viscous Fluid Along a Nonlinearly Stretching Curved Surface, *Results in Physics*, **7**, (2017), 1-4.
71. I. I. Sharapudinov, Special Series in Laguerre Polynomials and Their Approximation Properties, *Siberian Mathematical Journal*, **58**(2), (2017), 338-362.
72. J. Shen, Stable and Efficient Spectral Methods in Unbounded Domains Using Laguerre Functions, *SIAM Journal on Numerical Analysis*, **38**(4), (2000), 1113–1133.
73. Y. Shi, Ch. Tian, Zh. Liu, Ch Liang, A New Laguerre-Based Discontinuous Galerkin Time-Domain Method, *Microwave and Optical Technology Letters*, **59**(7), (2017), 1499-1503.
74. H. Siyyam, Laguerre Tau Methods for Solving Higher-Order Ordinary Differential Equations", *Journal of Computational Analysis and Applications*, **3**(2), (2001), 173–182.
75. J. Stoer, R. Bulirsch, *Introduction to numerical analysis*, Springer Science & Business Media, **12**, (2013).
76. J. A. K. Suykens, J. Vandewalle, Least Squares Support Vector Machine Classifiers, *Neural Process. Lett.*,**9**(3), (1999), 293-300.
77. T. Tajvidi, M. Razzaghi, M. Dehghan, Modified Rational Legendre Approach to Laminar Viscous Flow Over a Semi-Infinite Flat Plate, *Chaos, Solitons & Fractals*, **35**(1), (2008), 59-66.
78. F. K. Tsou, E. M. Sparrow, R. Jh. Goldstein, Flow and Heat Transfer in the Boundary Layer on a Continuous Moving Surface, *International Journal of Heat and Mass Transfer*, **10**(2), (1967), 219-235.
79. N. Vukovi, M. Petrovi, Z. Miljkovi, A Comprehensive Experimental Evaluation of Orthogonal Polynomial Expanded Random Vector Functional Link Neural Networks for Regression, *Applied Soft Computing*, **70**, (2018), 1083-1096.
80. A. W. Wazwaz, A New Algorithm for Solving Differential Equations of Lane–Emden Type, *Applied Mathematics and Computation*, **118**(2-3), (2001), 287-310.
81. S. Xu, Y. Li, T. Huang, R. Chan, A Sparse Multiwavelet-Based Generalized Laguerre–Volterra Model for Identifying Time-Varying Neural Dynamics from Spiking Activities, *Entropy*, **19**(8), (2017), 425-446.
82. A. Yıldırım, T. Öziş , Solutions of Singular IVPs of Lane–Emden Type by Homotopy Perturbation Method, *Physics Letters A*, **369**(1-2), (2007), 70-76.
83. S. A. Yousefi, Legendre Wavelets Method for Solving Differential Equations of Lane–Emden Type, *Applied Mathematics and Computation*, **181**(2), (2006), 1417-1422.
84. Ş. Yüzbaşı, A Numerical Method for Solving Second-Order Linear Partial Differential Equations Under Dirichlet, Neumann and Robin Boundary Conditions, *International Journal of Computational Methods*, **14**(02), (2017), 1750015.
85. Ş. Yüzbaşı, M. Sezer, An Improved Bessel Collocation Method With a Residual Error Function to Solve a Class of Lane–Emden Differential Equations, *Mathematical and Computer Modelling*, **57**(5-6), (2013), 1298-1311.
86. Ch. Zhang, D. Gu, Zh. Wang, H. Li, Efficient Space-Time Spectral Methods for Second-Order Problems on Unbounded Domains, *Journal of Scientific Computing*, **72**(2), (2017), 679-699.
87. Di. Zhang, X. Miao, New Unconditionally Stable Scheme for Telegraph Equation Based on Weighted Laguerre Polynomials, *Numerical Methods for Partial Differential Equations*, 33(5), (2017), 113-130.
88. R. Zhang, Zh. Wang, B. Guo, Mixed Fourier-Laguerre Spectral and Pseudospectral Methods for Exterior Problems Using Generalized Laguerre Functions, *Journal of Scientific Computing*, **36**(2), (2008), 263-283.
89. Z. Ziabakhsh, G. Domairry, H. Bararnia, H. Babazadeh, Analytical Solution of Flow and Diffusion of Chemically Reactive Species Over a Nonlinearly Stretching Sheet Immersed

in a Porous Medium, *Journal of the Taiwan Institute of Chemical Engineers*, **41**(1), (2010), 22-28.