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Inequalities for the Derivatives of a Polynomial

Abdullah Mir[a,*], Q. M. Dawood[a] and Bilal Dar[b]

[a]Department of Mathematics, University of Kashmir,
Srinagar, 190006, India.
[b]Govt. Degree College (Boys), Pulwama-192301.

E-mail: mabdullah_mir@yahoo.co.in
E-mail: qdawood@gmail.com
E-mail: darbilal85@ymail.com

Abstract. The paper presents an L^r- analogue of an inequality regarding the s^{th} derivative of a polynomial having zeros outside a circle of arbitrary radius but greater or equal to one. Our result provides improvements and generalizations of some well-known polynomial inequalities.

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1. Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree at most n and $P'(z)$ be its derivative, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1.1}$$

*Corresponding Author

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and for every $r \geq 1$,

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}. \tag{1.2}$$

Inequality (1.1) is a classical result of Bernstein[6] whereas inequality (1.2) is due to Zygmund[15] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form $P(e^{i\theta})$. Arestov[1] proved that (1.2) remains true for $0 < r < 1$ as well. If $r \to \infty$ in inequality (1.2), we get (1.1).

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then both the inequalities (1.1) and (1.2) can be sharpened. In fact, If $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.3}$$

and

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq nA_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.4}$$

where $A_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}$.

Inequality (1.3) was conjectured by Erdös and later verified by Lax[11], whereas inequality (1.4) was proved by De-Bruijn[7] for $r \geq 1$. Rahman and Schemeisser[13] later proved that (1.4) holds for $0 < r < 1$ also. If $r \to \infty$ in (1.4), we get (1.3).

As a generalization of (1.3) Malik[12] proved that if $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.5}$$

whereas under the same hypothesis, Govil and Rahman[9] extended inequality (1.4) by showing that

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq nE_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.6}$$

where $E_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |k + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}$, $r \geq 1$.

In the same paper, Govil and Rahman[9, Theorem 4] extended inequality (1.5) to the s^{th} derivative of a polynomial and proved under the same hypothesis for $1 \le s < n$ that

$$\max_{|z|=1}|P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+k^s}\max_{|z|=1}|P(z)|. \tag{1.7}$$

Inequality (1.7) was refined by Aziz and Rather [3, Corollary 1] by involving the binomial coefficients $C(n,s),\ 1 \le s < n$ and coefficients of the polynomial $P(z)$. In fact they proved that if $P(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| < k,\ k \ge 1$, then for $1 \le s < n$,

$$\max_{|z|=1}|P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+\psi_{k,s}}\max_{|z|=1}|P(z)|, \tag{1.8}$$

where

$$\psi_{k,s} = k^{s+1}\left(\frac{1+\frac{1}{C(n,s)}\left|\frac{a_s}{a_0}\right|k^{s-1}}{1+\frac{1}{C(n,s)}\left|\frac{a_s}{a_0}\right|k^{s+1}}\right). \tag{1.9}$$

In the literature there exist various results regarding the estimates for polynomials and for general analytic functions and also the approximations of polynomials and their derivatives (for example see[8],[14]). In this paper, we prove the following result which refines the inequality (1.8).

Theorem 1.1. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in $|z| < k,\ k \ge 1$, and $m = \min_{|z|=k}|P(z)|$ then for $1 \le s < n$,*

$$\max_{|z|=1}|P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+\psi_{k,s}}\Big(\max_{|z|=1}|P(z)| - \frac{m\psi_{k,s}}{k^n}\Big), \tag{1.10}$$

where $\psi_{k,s}$ is defined by (1.9).

The result is best possible for $k = 1$ and equality holds for $P(z) = z^n + 1$.

Remark 1.2. For $s = 1$ and $m = 0$, Theorem 1.1 reduces to a result of Govil et. al.[10, Theorem 1] and for $k = s = 1$, inequality (1.10) reduces to a result of Aziz and Dawood[2, Theorem A].

Remark 1.3. Note by inequality (2.2) of Lemma 2.1 (stated in section 2) that $\frac{1}{C(n,s)}\left|\frac{a_s}{a_0}\right|k^s \le 1$, which can easily be shown to be equivalent to $\psi_{k,s} \ge k^s,\ 1 \le s < n$. Using this fact in inequality (1.10), we get the following improvement of inequality (1.7).

Corollary 1.4. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in $|z| < k,\ k \ge 1$, and $m = \min_{|z|=k}|P(z)|$ then for $1 \le s < n$,*

$$\max_{|z|=1}|P^{(s)}(z)| \le \frac{n(n-1)\cdots(n-s+1)}{1+k^s}\Big(\max_{|z|=1}|P(z)| - \frac{m}{k^{n-s}}\Big). \tag{1.11}$$

In order to prove the Theorem 1.1, we prove the following more general result which extends Theorem 1.1 to its corresponding L^r- analogue.

Theorem 1.5. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in $|z| < k,\ k \geq 1,$ and $m = \min_{|z|=k} |P(z)|$, then for every complex number β with $|\beta| \leq 1$ and $1 \leq s < n$, we have*

$$\left\{ \int_0^{2\pi} \left| P^{(s)}(e^{i\theta}) + \frac{\beta m n(n-1)\cdots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})} \right|^r d\theta \right\}^{\frac{1}{r}}$$
$$\leq n(n-1)\cdots(n-s+1) C_r \left\{ \int_0^{2\pi} |P(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \qquad (1.12)$$

where $C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\psi_{k,s} + e^{i\alpha}|^r d\alpha \right\}^{\frac{-1}{r}}$, $r > 0$ and $\psi_{k,s}$ is defined by (1.9).

Remark 1.6. Using the fact that $\psi_{k,s} \geq k^s$ and take $\beta = 0$ in inequality (1.12), we obtain a result of Aziz and Shah[5].

2. Lemmas

We need the following lemmas for the proofs of Theorems. Here, throughout this paper we write $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Lemma 2.1. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in $|z| < k, k \geq 1$, then for $1 \leq s < n$ and $|z| = 1$,*

$$\left| Q^{(s)}(z) \right| \geq \psi_{k,s} \left| P^{(s)}(z) \right|, \qquad (2.1)$$

and

$$\frac{1}{C(n,s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1, \qquad (2.2)$$

where $\psi_{k,s}$ is defined by (1.9).

The above lemma is due to Aziz and Rather[3].

Lemma 2.2. *If $P(z)$ is a polynomial of degree n, then for each α, $0 \leq \alpha < 2\pi$ and $r > 0$, we have*

$$\int_0^{2\pi} \int_0^{2\pi} \left| Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta}) \right|^r d\theta d\alpha \leq 2\pi n^r \int_0^{2\pi} |P(e^{i\theta})|^r d\theta. \qquad (2.3)$$

The above lemma is due to Aziz and Shah[4].

Lemma 2.3. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *which does not vanish in* $|z| < k, k \geq 1$*, then for* $1 \leq s < n$ *and* $|z| = 1$*,*

$$\left|Q^{(s)}(z)\right| \geq \psi_{k,s}\left|P^{(s)}(z)\right| + \frac{mn(n-1)\cdots(n-s+1)}{k^n}\psi_{k,s}, \tag{2.4}$$

where $m = \min_{|z|=k}|P(z)|$*.*

Proof. Since $m \leq |P(z)|$ for $|z| = k$, we have for every β with $|\beta| < 1$,

$$\left|\frac{m\beta z^n}{k^n}\right| < |P(z)| \ for \ |z| = k.$$

Therefore by Rouche's theorem $P(z) + \frac{m\beta z^n}{k^n}$ has no zero in $|z| < k, \ k \geq 1$. Applying Lemma 2.1 to the polynomial $P(z) + \frac{m\beta z^n}{k^n}$, we get for $1 \leq s < n$ and $|z| = 1$,

$$\left|Q^{(s)}(z)\right| \geq \psi_{k,s}\left|P^{(s)}(z) + \frac{mn(n-1)\cdots(n-s+1)\beta}{k^n}\right|. \tag{2.5}$$

Choose the argument of β so that

$$\left|P^{(s)}(z) + \frac{mn(n-1)\cdots(n-s+1)\beta z^{n-s}}{k^n}\right| = \left|P^{(s)}(z)\right| + \frac{mn(n-1)\cdots(n-s+1)|\beta z^{n-s}|}{k^n},$$

it follows from (2.5) that for $|z| = 1$,

$$\left|Q^{(s)}(z)\right| \geq \psi_{k,s}\left|P^{(s)}(z)\right| + \frac{mn(n-1)\cdots(n-s+1)|\beta z^{n-s}|}{k^n}\psi_{k,s}. \tag{2.6}$$

Letting $|\beta| \to 1$ in inequality (2.6), we get

$$\left|Q^{(s)}(z)\right| \geq \psi_{k,s}\left|P^{(s)}(z)\right| + \frac{mn(n-1)\cdots(n-s+1)}{k^n}\psi_{k,s}.$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *If* A, B, C *are non-negative real numbers such that* $B + C \leq A$*. Then for every real* α*,*

$$\left|(A - C) + e^{i\alpha}(B + C)\right| \leq \left|A + e^{i\alpha}B\right|. \tag{2.7}$$

The above lemma is due to Aziz and Shah[4].

3. Proofs of Theorems

Proof of the Theorem 1.5. Since $P(z)$ is a polynomial of degree n , $P(z) \neq 0$ in $|z| < k, k \geq 1$, and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$. Therefore, for each $\alpha, 0 \leq \alpha < 2\pi, F(z) = Q(z) + e^{i\alpha}P(z)$ is a polynomial of degree n and we have

$$F^{(s)}(z) = Q^{(s)}(z) + e^{i\alpha}P^{(s)}(z),$$

which is clearly a polynomial of degree $n-s, 1 \leq s < n$. By the repeated application of inequality (1.2), we have for each $r > 0$,

$$\begin{aligned}
&\int_0^{2\pi} \left|Q^{(s)}(e^{i\theta}) + e^{i\alpha}P^{(s)}(e^{i\theta})\right|^r d\theta \\
&\leq (n-s+1)^r \int_0^{2\pi} \left|Q^{(s-1)}(e^{i\theta}) + e^{i\alpha}P^{(s-1)}(e^{i\theta})\right|^r d\theta \\
&\leq (n-s+1)^r(n-s+2)^r \int_0^{2\pi} \left|Q^{(s-2)}(e^{i\theta}) + e^{i\alpha}P^{(s-2)}(e^{i\theta})\right|^r d\theta \\
&. \\
&. \\
&. \\
&\leq (n-s+1)^r(n-s+2)^r \dots (n-1)^r \int_0^{2\pi} \left|Q'(e^{i\theta}) + e^{i\alpha}P'(e^{i\theta})\right|^r d\theta.
\end{aligned} \tag{3.1}$$

Integrating inequality (3.1) with respect to α over $[0, 2\pi]$ and using inequality (2.3) of Lemma 2.2, we get

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \left|Q^{(s)}(e^{i\theta}) + e^{i\alpha}P^{(s)}(e^{i\theta})\right|^r d\theta d\alpha \\
&\leq 2\pi(n-s+1)^r(n-s+2)^r \dots (n-1)^r n^r \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta.
\end{aligned} \tag{3.2}$$

Now, from inequality (2.4) of Lemma 2.3, it easily follows that

$$\begin{aligned}
&\psi_{k,s}\left\{ \left|P^{(s)}(e^{i\theta})\right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}\right\} \\
&\leq \left|Q^{(s)}(e^{i\theta})\right| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.
\end{aligned} \tag{3.3}$$

Taking $A = \left|Q^{(s)}(e^{i\theta})\right|, B = \left|P^{(s)}(e^{i\theta})\right|, C = \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}$ and noting that $\psi_{k,s} \geq k^s \geq 1, 1 \leq s < n$, so that by (3.3),

$$B + C \leq \psi_{k,s}(B+C) \leq A - C \leq A,$$

we get from Lemma 2.4 that

$$\begin{aligned}
&\Bigg|\left\{\left|Q^{(s)}(e^{i\theta})\right| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}\right\} \\
&\qquad + e^{i\alpha}\left\{\left|P^{(s)}(e^{i\theta})\right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}\right\}\Bigg| \\
&\leq \left|\left|Q^{(s)}(e^{i\theta})\right| + e^{i\alpha}\left|P^{(s)}(e^{i\theta})\right|\right|.
\end{aligned}$$

This implies for each $r > 0$,

$$\int_0^{2\pi} \left|F(\theta) + e^{i\alpha}G(\theta)\right|^r d\alpha \leq \int_0^{2\pi} \left|\left|Q^{(s)}(e^{i\theta})\right| + e^{i\alpha}\left|P^{(s)}(e^{i\theta})\right|\right|^r d\alpha, \qquad (3.4)$$

where

$$F(\theta) = \left|Q^{(s)}(e^{i\theta})\right| - \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}$$

and

$$G(\theta) = \left|P^{(s)}(e^{i\theta})\right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}.$$

Integrating inequality (3.4) with respect to θ on $[0, 2\pi]$ and using inequality (3.2), we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|F(\theta) + e^{i\alpha}G(\theta)\right|^r d\alpha d\theta \\
&\qquad \leq \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|\left|Q^{(s)}(e^{i\theta})\right| + e^{i\alpha}\left|P^{(s)}(e^{i\theta})\right|\right|^r d\alpha d\theta \\
&\qquad = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \left|Q^{(s)}(e^{i\theta}) + e^{i\alpha}P^{(s)}(e^{i\theta})\right|^r d\alpha d\theta \\
&\qquad \leq (n-s+1)^r(n-s+2)^r\dots(n-1)^r n^r \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta.
\end{aligned} \qquad (3.5)$$

Now for every real number α and $t_1 \geq t_2 \geq 1$, we have

$$|t_1 + e^{i\alpha}| \geq |t_2 + e^{i\alpha}|,$$

which implies for every $r > 0$,

$$\int_0^{2\pi} |t_1 + e^{i\alpha}|^r d\alpha \geq \int_0^{2\pi} |t_2 + e^{i\alpha}|^r d\alpha.$$

If $G(\theta) \neq 0$, we take $t_1 = \left|\frac{F(\theta)}{G(\theta)}\right|$ and $t_2 = \psi_{k,s}$, then from (3.3) and noting that $\psi_{k,s} \geq 1$, we have $t_1 \geq t_2 \geq 1$, hence

$$\begin{aligned}\int_0^{2\pi} \left|F(\theta) + e^{i\alpha}G(\theta)\right|^r d\alpha &= |G(\theta)|^r \int_0^{2\pi} \left|\frac{F(\theta)}{G(\theta)} + e^{i\alpha}\right|^r d\alpha \\ &= |G(\theta)|^r \int_0^{2\pi} \left|\left|\frac{F(\theta)}{G(\theta)}\right| + e^{i\alpha}\right|^r d\alpha \\ &\geq |G(\theta)|^r \int_0^{2\pi} \left|\psi_{k,s} + e^{i\alpha}\right|^r d\alpha \\ &= \left\{\left|P^{(s)}(e^{i\theta})\right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}\right\}^r \\ &\quad \int_0^{2\pi} \left|\psi_{k,s} + e^{i\alpha}\right|^r d\alpha. \qquad (3.6)\end{aligned}$$

For $G(\theta) = 0$, this inequality is trivially true. Using this in (3.5), it follows for each $r > 0$,

$$\begin{aligned}\int_0^{2\pi} &\left\{\left|P^{(s)}(e^{i\theta})\right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}\right\}^r d\theta \\ &\leq \frac{(n-s+1)^r(n-s+2)^r\dots(n-1)^r n^r}{\frac{1}{2\pi}\int_0^{2\pi}\left|\psi_{k,s} + e^{i\alpha}\right|^r d\alpha} \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta. \qquad (3.7)\end{aligned}$$

Now using the fact that for every β with $|\beta| \leq 1$,

$$\left|P^{(s)}(e^{i\theta}) + \frac{\beta mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})}\right| \leq \left|P^{(s)}(e^{i\theta})\right| + \frac{mn(n-1)\dots(n-s+1)\psi_{k,s}}{k^n(1+\psi_{k,s})},$$

the desired result follows from (3.7).

Proof of the Theorem 1.1 Making $r \to \infty$ and choosing the argument of β suitably with $|\beta| = 1$ in (1.12), Theorem 1.1 follows.

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