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ζ_ς-R_0 and ζ_ς-R_1 Strong Generalized Topological Spaces

Nongluk Viriyapong, Chawalit Boonpok*

Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

E-mail: nongluk.h@msu.ac.th
E-mail: chawalit.b@msu.ac.th

Abstract. The purpose of the present paper is to introduce the concepts of ζ_ς-R_0 and ζ_ς-R_1 strong generalized topological spaces are defined by utilizing the notions of (ζ, ς)-open sets and (ζ, ς)-closure operators. Moreover, several characterizations of ζ_ς-R_0 and ζ_ς-R_1 strong generalized topological spaces are investigated.

Keywords: Generalized topological space, (ζ, ς)-open set, (ζ, ς)-closure, ζ_ς-R_0 strong generalized topological space, ζ_ς-R_1 strong generalized topological space.

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1. Introduction

General topology plays an important role in many fields of pure and applied sciences such as data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, information system, particle physics and quantum physics etc. The theory of generalized topological spaces, which was founded by Á. Császár [4], is one of the most important development of general topology in recent years. In [14], the present author introduced the notion of generalized

*Corresponding Author

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submaximal spaces and discussed some characterizations of generalized submaximal spaces. Ekici and Roy [13] introduced the concepts of $\bigwedge_\mu$-sets and $\bigvee_\mu$-sets and investigated some properties of $\bigwedge_\mu$-sets and $\bigvee_\mu$-sets. Roy and Ekici [19] introduced and studied the concepts of $(\bigwedge, \mu)$-open sets and $(\bigwedge, \mu)$-closed sets via μ-open and μ-closed sets in generalized topological spaces. The notion of R_0 topological spaces was introduced by Shanin [20] in 1943. Later, Davis [9] rediscovered it and studied some properties of this weak separation axioms. Several topologists (e.g. [11], [18]) further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, Davis also introduced the notion of R_1 topological spaces which are independent of both T_0 and T_1 but strictly weaker than T_2. Murdeshwar and Naimpally [17] and Dube [12] studied some of the fundamental properties of the class of R_1 topological spaces. As natural generalizations of the separation axioms R_0 and R_1, the concepts of semi-R_0 and semi-R_1 were introduced and studied by Maheshwari and Prasad [15] and Dorsett [10]. The concept of Λ-sets was first introduced by Maki [16]. Arenas et al. [1] introduced the notion of λ-closed sets in terms of the concept of Λ-sets and used λ-closed sets to characterize some low separation axioms. In [2], the concepts of Λ_θ-sets, (Λ, θ)-closed sets and (Λ, θ)-open sets were introduced by utilizing θ-open sets and θ-closed sets due to Veličko [21]. Caldas et al. [3] introduced and studied two new weak separation axioms called Λ_θ-R_0 spaces and Λ_θ-R_1 spaces by using the notions of (Λ, θ)-open sets and (Λ, θ)-closure operators. In this paper, we introduce the concepts of ζ_ς-R_0 and ζ_ς-R_1 strong generalized topological spaces. In particular, some characterizations of ζ_ς-R_0 and ζ_ς-R_1 strong generalized topological spaces are discussed.

2. Preliminaries

We recall some basic definitions and notations. Let X be a set and denote $\mathcal{P}(X)$ the power set of X. A subset μ of $\mathcal{P}(X)$ is said to be a *generalized topology* (briefly GT) on X if $\emptyset \in \mu$ and an arbitrary union of elements of μ belongs to μ [5]. Let μ be a GT on X, the elements of μ are called *μ-open sets* and the complements of μ-open sets are called *μ-closed sets*. Let M_μ denote the union of all elements of μ. We say μ is *strong* [6] if $M_\mu = X$. If $A \subseteq X$, then $i_\mu(A)$ denotes the union of all μ-open sets contained in A and $c_\mu(A)$ is the intersections of all μ-closed sets containing A [7]. According to [8], for $A \subseteq X$ and $x \in X$, we have $x \in c_\mu(A)$ iff $x \in M \in \mu$ implies $M \cap A \neq \emptyset$.

3. (ζ, ς)-closed sets

In this section, we introduce the notion of (ζ, ς)-closed sets in strong generalized topological spaces and investigate some properties of (ζ, ς)-closed sets.

Definition 3.1. Let (X, μ) be a strong generalized topological space and let A be a subset of X. A point $x \in X$ is called a ς-*cluster point* of A if $A \cap c_\mu(U) \neq \emptyset$ for every μ-open set U containing x. The set of all ς-cluster points of A is called the ς-*closure* of A and is denoted by $c_\varsigma(A)$.

A subset A of a strong generalized topological space (X, μ) is called ς-*closed* if $A = c_\varsigma(A)$. The complement of a ς-closed set is called ς-*open*. By $\varsigma O(X, \mu)$, we denote the family of all ς-open sets of a strong generalized topological space (X, μ).

Definition 3.2. Let (X, μ) be a strong generalized topological space and let A be a subset of X. The subset $\zeta_\varsigma(A)$ is defined as follows:

$$\zeta_\varsigma(A) = \cap\{U \mid A \subseteq U, U \in \varsigma O(X, \mu)\}.$$

Lemma 3.3. *For subsets A, B, $C_\gamma(\gamma \in \Gamma)$ of a strong generalized topological space (X, μ), the following properties hold:*

(1) $A \subseteq \zeta_\varsigma(A)$;
(2) $\zeta_\varsigma(\zeta_\varsigma(A)) = \zeta_\varsigma(A)$;
(3) *if* $A \subseteq B$, *then* $\zeta_\varsigma(A) \subseteq \zeta_\varsigma(B)$;
(4) $\zeta_\varsigma(\cap\{C_\gamma \mid \gamma \in \Gamma\}) \subseteq \cap\{\zeta_\varsigma(C_\gamma) \mid \gamma \in \Gamma\}$;
(5) $\zeta_\varsigma(\cup\{C_\gamma \mid \gamma \in \Gamma\}) = \cup\{\zeta_\varsigma(C_\gamma) \mid \gamma \in \Gamma\}$.

Proof. (1) This is obvious from the definition.

(2) By (1), we have $\zeta_\varsigma(\zeta_\varsigma(A)) \supseteq \zeta_\varsigma(A)$. Suppose that $x \notin \zeta_\varsigma(A)$. Then, there exists $U \in \varsigma O(X, \mu)$ such that $A \subseteq U$ and $x \notin U$. Since $A \subseteq \zeta_\varsigma(A) \subseteq U$, we have $x \notin \zeta_\varsigma(\zeta_\varsigma(A))$ and hence $\zeta_\varsigma(\zeta_\varsigma(A)) \subseteq \zeta_\varsigma(A)$.

(3) Suppose that $x \notin \zeta_\varsigma(B)$. Then, there exists $U \in \varsigma O(X, \mu)$ such that $B \subseteq U$ and $x \notin U$. Since $A \subseteq B$, we have $x \notin \zeta_\varsigma(A)$ and hence $\zeta_\varsigma(A) \subseteq \zeta_\varsigma(B)$.

(4) Suppose that $x \notin \cap\{\zeta_\varsigma(C_\gamma) \mid \gamma \in \Gamma\}$. Then, there exists $\gamma_0 \in \Gamma$ such that $x \notin \zeta_\varsigma(C_{\gamma_0})$ and there exists a ς-open set U such that $x \notin U$ and $C_{\gamma_0} \subseteq U$. Since $\cap_{\gamma\in\Gamma} C_\gamma \subseteq C_{\gamma_0}$, we have $x \notin \zeta_\varsigma(\cap\{C_\gamma \mid \gamma \in \Gamma\})$ and hence

$$\zeta_\varsigma(\cap\{C_\gamma \mid \gamma \in \Gamma\}) \subseteq \cap\{\zeta_\varsigma(C_\gamma) \mid \gamma \in \Gamma\}.$$

(5) Since $C_\gamma \subseteq \cup_{\gamma\in\Gamma} C_\gamma$, by (3), $\zeta_\varsigma(C_\gamma) \subseteq \zeta_\varsigma(\cup_{\gamma\in\Gamma} C_\gamma)$ and

$$\cup_{\gamma\in\Gamma}\zeta_\varsigma(C_\gamma) \subseteq \zeta_\varsigma(\cup_{\gamma\in\Gamma} C_\gamma).$$

Suppose that $x \notin \cup_{\gamma\in\Gamma}\zeta_\varsigma(C_\gamma)$. Then, $x \notin \zeta_\varsigma(C_\gamma)$ for each $\gamma \in \Gamma$ and hence there exists $U_\gamma \in \varsigma O(X, \mu)$ such that $C_\gamma \subseteq U_\gamma$ and $x \notin U_\gamma$ for each $\gamma \in \Gamma$. Thus, $\cup_{\gamma\in\Gamma} C_\gamma \subseteq \cup_{\gamma\in\Gamma} U_\gamma$ and $\cup_{\gamma\in\Gamma} U_\gamma$ is a ς-open set not containing x. Therefore, $x \notin \zeta_\varsigma(\cup_{\gamma\in\Gamma} C_\gamma)$. This shows that $\cup_{\gamma\in\Gamma}\zeta_\varsigma(C_\gamma) \supseteq \zeta_\varsigma(\cup_{\gamma\in\Gamma} C_\gamma)$. □

Definition 3.4. A subset A of a strong generalized topological space (X, μ) is called a ζ_ς-*set* if $A = \zeta_\varsigma(A)$.

Lemma 3.5. *For subsets B and $C_\gamma(\gamma \in \Gamma)$ of a strong generalized topological space (X, μ), the following properties hold:*

(1) *$\zeta_\varsigma(B)$ is a ζ_ς-set;*
(2) *if B is a ς-open set, then B is a ζ_ς-set;*
(3) *if C_γ is a ζ_ς-set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} C_\gamma$ is a ζ_ς-set;*
(4) *if C_γ is a ζ_ς-set for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} C_\gamma$ is a ζ_ς-set.*

Proof. (1) and (2) are obvious.

(3) Let C_γ be a ζ_ς-set for each $\gamma \in \Gamma$, by Lemma 3.3(4),

$$\cup_{\gamma\in\Gamma} C_\gamma = \cup_{\gamma\in\Gamma} \zeta_\varsigma(C_\gamma) = \zeta_\varsigma(\cup_{\gamma\in\Gamma} C_\gamma) \supseteq \cup_{\gamma\in\Gamma} C_\gamma.$$

Thus, $\cup_{\gamma\in\Gamma} C_\gamma = \zeta_\varsigma(\cup_{\gamma\in\Gamma} C_\gamma)$ and hence $\cup_{\gamma\in\Gamma} C_\gamma$ is a ζ_ς-set.

(4) Let C_γ be a ζ_ς-set for each $\gamma \in \Gamma$, by Lemma 3.3(5),

$$\cap_{\gamma\in\Gamma} C_\gamma = \cap_{\gamma\in\Gamma} \zeta_\varsigma(C_\gamma) \supseteq \zeta_\varsigma(\cap_{\gamma\in\Gamma} C_\gamma) \supseteq \cap_{\gamma\in\Gamma} C_\gamma.$$

Thus, $\cap_{\gamma\in\Gamma} C_\gamma = \zeta_\varsigma(\cap_{\gamma\in\Gamma} C_\gamma)$. This shows that $\cap_{\gamma\in\Gamma} C_\gamma$ is a ζ_ς-set. □

Definition 3.6. Let A be a subset of a strong generalized topological space (X, μ).

(i) A is called a (ζ, ς)-*closed* set if $A = H \cap F$, where H is a ζ_ς-set and F is a ς-closed set. The complement of a (ζ, ς)-closed set is called (ζ, ς)-*open*. The family of all (ζ, ς)-open (resp. (ζ, ς)-closed) sets in a strong generalized topological space (X, μ) is denoted by $\zeta_\varsigma O(X, \mu)$ (resp. $\zeta_\varsigma C(X, \mu)$);

(ii) A point $x \in X$ is called a (ζ, ς)-*cluster point* of A if $A \cap U \neq \emptyset$ for every (ζ, ς)-open set U of X containing x. The set of all (ζ, ς)-cluster points of A is called the (ζ, ς)- *closure* of A and is denoted by $c_{(\zeta,\varsigma)}(A)$.

Lemma 3.7. *Let A and B be subsets of a strong generalized topological space (X, μ). For the (ζ, ς)-closure, the following properties hold:*

(1) *$A \subseteq c_{(\zeta,\varsigma)}(A)$;*
(2) *$c_{(\zeta,\varsigma)}(A) = \cap\{F \mid A \subseteq F, F \in \zeta_\varsigma C(X, \mu)\}$;*
(3) *if $A \subseteq B$, then $c_{(\zeta,\varsigma)}(A) \subseteq c_{(\zeta,\varsigma)}(B)$;*
(4) *A is (ζ, ς)-closed if and only if $A = c_{(\zeta,\varsigma)}(A)$;*
(5) *$c_{(\zeta,\varsigma)}(A)$ is (ζ, ς)-closed.*

Lemma 3.8. *Let A and $B_\gamma (\gamma \in \Gamma)$ be subsets of a strong generalized topological space (X, μ). Then, the following properties hold:*

(1) *if A is (ζ, ς)-closed, then $A = \zeta_\varsigma(A) \cap c_\varsigma(A)$;*
(2) *if A is ς-closed, then A is (ζ, ς)-closed;*
(3) *if B_γ is (ζ, ς)-closed for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} B_\gamma$ is (ζ, ς)-closed.*

Proof. (1) Let A be a (ζ, ς)-closed set. Then, there exist a ζ_ς-set H and a ς-closed set F such that $A = H \cap F$. Since $A \subseteq H$, $A \subseteq \zeta_\varsigma(A) \subseteq \zeta_\varsigma(H) = H$, and also by $A \subseteq F$, $A \subseteq c_\varsigma(A) \subseteq c_\varsigma(F) = F$. Now, $A \subseteq \zeta_\varsigma(A) \cap c_\varsigma(A) \subseteq H \cap F = A$. Thus, $A = \zeta_\varsigma(A) \cap c_\varsigma(A)$.

(2) If A is ς-closed, then X being a ζ_ς-set, we have $A = X \cap A$.

(3) Suppose that B_γ is a (ζ,ς)-closed set for each $\gamma \in \Gamma$. Then, for each $\gamma \in \Gamma$ there exist a ζ_ς-set H_γ and a ς-closed set F_γ such that $B_\gamma = H_\gamma \cap F_\gamma$. Now, $\cap_{\gamma\in\Gamma} B_\gamma = \cap_{\gamma\in\Gamma}(H_\gamma \cap F_\gamma) = (\cap_{\gamma\in\Gamma} H_\gamma) \cap (\cap_{\gamma\in\Gamma} F_\gamma)$. By Lemma 3.5(4), $\cap_{\gamma\in\Gamma} H_\gamma$ is a ζ_ς-set and $\cap_{\gamma\in\Gamma} F_\gamma$ is a ς-closed set. This shows that $\cap_{\gamma\in\Gamma} B_\gamma$ is (ζ,ς)-closed. □

Theorem 3.9. *For a subset A of a strong generalized topological space (X,μ), the following properties are equivalent:*

(1) *A is (ζ,ς)-closed;*
(2) *$A = U \cap c_\varsigma(A)$, where U is a ζ_ς-set;*
(3) *$A = \zeta_\varsigma(A) \cap c_\varsigma(A)$.*

Proof. (1) ⇒ (2): Suppose that $A = U \cap F$, where U is a ζ_ς-set and F is a ς-closed set. Since $A \subseteq F$, we have $c_\varsigma(A) \subseteq F$ and $A = A \cap F \supseteq U \cap c_\varsigma(A) \supseteq A$. Thus, $A = U \cap c_\varsigma(A)$.

(2) ⇒ (3): Suppose that $A = U \cap c_\varsigma(A)$, where U is a ζ_ς-set. Since $A \subseteq U$, we have $\zeta_\varsigma(A) \subseteq \zeta_\varsigma(U) = U$ and hence $A \subseteq \zeta_\varsigma(A) \cap c_\varsigma(A) \subseteq U \cap c_\varsigma(A) = A$. Therefore, $A = \zeta_\varsigma(A) \cap c_\varsigma(A)$.

(3) ⇒ (1): By Lemma 3.5, we have $\zeta_\varsigma(A)$ is a ζ_ς-set. Since $c_\varsigma(A)$ is ς-closed, by (3), $A = \zeta_\varsigma(A) \cap c_\varsigma(A)$ and hence A is (ζ,ς)-closed. □

Definition 3.10. Let (X,μ) be a strong generalized topological space, $A \subseteq X$ and $x \in X$. Then:

(i) $\zeta_{(\zeta,\varsigma)}(A) = \cap\{G \in (\zeta,\varsigma)O(X,\mu) \mid A \subseteq G\}$,
(ii) $\langle x\rangle_\varsigma = c_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{x\})$.

Lemma 3.11. *Let A and B be subsets of a strong generalized topological space (X,μ). Then, the following properties hold:*

(1) *$A \subseteq B$ implies $\zeta_{(\zeta,\varsigma)}(A) \subseteq \zeta_{(\zeta,\varsigma)}(B)$;*
(2) *$\zeta_{(\zeta,\varsigma)}(\zeta_{(\zeta,\varsigma)}(A)) = \zeta_{(\zeta,\varsigma)}(A)$.*

Lemma 3.12. *Let (X,μ) be a strong generalized topological space and let $x, y \in X$. Then, $y \in \zeta_{(\zeta,\varsigma)}(\{x\})$ if and only if $y \in c_{(\zeta,\varsigma)}(\{y\})$.*

Proof. Let $y \notin \zeta_{(\zeta,\varsigma)}(\{x\})$. Then, there exists a (ζ,ς)-open set V containing x such that $y \notin V$. Hence, $x \notin c_{(\zeta,\varsigma)}(\{y\})$. This converse is similarly shown. □

A subset N_x of a strong generalized topological space (X,μ) is called a (ζ,ς)-*neighbourhood* of a point $x \in X$ if there exists a (ζ,ς)-open set U such that $x \in U \subseteq N_x$

Lemma 3.13. *A subset A of a strong generalized topological space (X,μ) is (ζ,ς)-open if and only if it is a (ζ,ς)-neighbourhood of each point of A.*

Proposition 3.14. *Let (X,μ) be a strong generalized topological space. Then, the following properties hold:*

(1) *$\zeta_{(\zeta,\varsigma)}(A)=\{x\in X \mid c_{(\zeta,\varsigma)}(\{x\})\cap A\neq\emptyset\}$ for each subset A of X;*
(2) *for each $x\in X$, $\zeta_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)=\zeta_{(\zeta,\varsigma)}(\{x\})$;*
(3) *for each $x\in X$, $c_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)=c_{(\zeta,\varsigma)}(\{x\})$;*
(4) *if U is a (ζ,ς)-open set of X and $x\in U$, then $\langle x\rangle_\varsigma\subseteq U$;*
(5) *if F is a (ζ,ς)-closed set of X and $x\in F$, then $\langle x\rangle_\varsigma\subseteq F$.*

Proof. (1) Suppose that $c_{(\zeta,\varsigma)}(\{x\})\cap A=\emptyset$. Then, $x\notin X-c_{(\zeta,\varsigma)}(\{x\})$ which is a (ζ,ς)-open set containing A. Therefore, $x\notin\zeta_{(\zeta,\varsigma)}(A)$. Consequently, we obtain $\zeta_{(\zeta,\varsigma)}(A)\subseteq\{x\in X \mid c_{(\zeta,\varsigma)}(\{x\})\cap A\neq\emptyset\}$. Next, let $x\in X$ such that $c_{(\zeta,\varsigma)}(\{x\})\cap A\neq\emptyset$ and suppose that $x\notin\zeta_{(\zeta,\varsigma)}(A)$. Then, there exists a (ζ,ς)-open set U containing A and $x\notin U$. Let $y\in c_{(\zeta,\varsigma)}(\{x\})\cap A$. Hence, U is a (ζ,ς)-neighbourhood of y which does not contain x. By this contradiction $x\in\zeta_{(\zeta,\varsigma)}(A)$.

(2) Let $x\in X$, then we have $\{x\}\subseteq c_{(\zeta,\varsigma)}(\{x\})\cap\zeta_{(\zeta,\varsigma)}(\{x\})=\langle x\rangle_\varsigma$. By Lemma 3.11, $\zeta_{(\zeta,\varsigma)}(\{x\})\subseteq\zeta_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)$. Next, we show the opposite implication. Suppose that $y\notin\zeta_{(\zeta,\varsigma)}(\{x\})$. Then, there exists a (ζ,ς)-open set V such that $x\in V$ and $y\notin V$. Since $\langle x\rangle_\varsigma\subseteq\zeta_{(\zeta,\varsigma)}(\{x\})\subseteq\zeta_{(\zeta,\varsigma)}(V)=V$, we have $\zeta_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)\subseteq V$. Since $y\notin V, y\notin\zeta_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)$. Consequently, we obtain $\zeta_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)\subseteq\zeta_{(\zeta,\varsigma)}(\{x\})$ and hence $\zeta_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)=\zeta_{(\zeta,\varsigma)}(\{x\})$.

(3) By the definition of $\langle x\rangle_\varsigma$, we have $\{x\}\subseteq\langle x\rangle_\varsigma$ and $c_{(\zeta,\varsigma)}(\{x\})\subseteq c_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)$ by Lemma 3.7. On the other hand, we have $\langle x\rangle_\varsigma\subseteq c_{(\zeta,\varsigma)}(\{x\})$ and

$$c_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)\subseteq c_{(\zeta,\varsigma)}(c_{(\zeta,\varsigma)}(\{x\}))=c_{(\zeta,\varsigma)}(\{x\}).$$

Thus, $c_{(\zeta,\varsigma)}(\langle x\rangle_\varsigma)=c_{(\zeta,\varsigma)}(\{x\})$.

(4) Since $x\in U$ and U is a (ζ,ς)-open set, $\zeta_{(\zeta,\varsigma)}(\{x\})\subseteq U$. Thus, $\langle x\rangle_\varsigma\subseteq U$.

(5) Since $x\in F$ and F is a (ζ,ς)-closed set,

$$\langle x\rangle_\varsigma=c_{(\zeta,\varsigma)}(\{x\})\cap\zeta_{(\zeta,\varsigma)}(\{x\})\subseteq c_{(\zeta,\varsigma)}(\{x\})\subseteq F.$$

□

Lemma 3.15. *The following statements are equivalent for any points x and y in a strong generalized topological space (X,μ):*

(1) *$\zeta_{(\zeta,\varsigma)}(\{x\})\neq\zeta_{(\zeta,\varsigma)}(\{y\})$;*
(2) *$c_{(\zeta,\varsigma)}(\{x\})\neq c_{(\zeta,\varsigma)}(\{y\})$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\zeta_{(\zeta,\varsigma)}(\{x\})\neq\zeta_{(\zeta,\varsigma)}(\{y\})$. Then, there exists a point z in X such that $z\in\zeta_{(\zeta,\varsigma)}(\{x\})$ and $z\notin\zeta_{(\zeta,\varsigma)}(\{y\})$ or $z\in\zeta_{(\zeta,\varsigma)}(\{y\})$ and $z\notin\zeta_{(\zeta,\varsigma)}(\{x\})$. We prove only the first case being the second analogous. From $z\in\zeta_{(\zeta,\varsigma)}(\{x\})$ it follows that $\{x\}\cap c_{(\zeta,\varsigma)}(\{z\})\neq\emptyset$ which implies $x\in c_{(\zeta,\varsigma)}(\{z\})$. By $z\notin\zeta_{(\zeta,\varsigma)}(\{y\})$, we have $\{y\}\cap c_{(\zeta,\varsigma)}(\{z\})=\emptyset$. Since $x\in c_{(\zeta,\varsigma)}(\{z\})$,

$$c_{(\zeta,\varsigma)}(\{x\})\subseteq c_{(\zeta,\varsigma)}(\{z\})$$

and $\{y\}\cap c_{(\zeta,\varsigma)}(\{z\})=\emptyset$. Therefore, it follows that $c_{(\zeta,\varsigma)}(\{x\})\neq c_{(\zeta,\varsigma)}(\{y\})$. Thus, $\zeta_{(\zeta,\varsigma)}(\{x\})\neq\zeta_{(\zeta,\varsigma)}(\{y\})$ implies that $c_{(\zeta,\varsigma)}(\{x\})\neq c_{(\zeta,\varsigma)}(\{y\})$.

(2) $\Rightarrow$ (1): Suppose that $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$. Then, there exists a point z in X such that $z \in c_{(\zeta,\varsigma)}(\{x\})$ and $z \notin c_{(\zeta,\varsigma)}(\{y\})$ or $z \in c_{(\zeta,\varsigma)}(\{y\})$ and $z \notin c_{(\zeta,\varsigma)}(\{x\})$. We prove only the first case being the second analogous. It follows that there exists a (ζ,ς)-open set containing z and therefore x but not y, namely, $y \notin \zeta_{(\zeta,\varsigma)}(\{x\})$ and thus $\zeta_{(\zeta,\varsigma)}(\{x\}) \neq \zeta_{(\zeta,\varsigma)}(\{y\})$. $\square$

4. ζ_ς-R_0 strong generalized topological spaces

We begin this section by introducing the concept of ζ_ς-R_0 strong generalized topological spaces.

Definition 4.1. A strong generalized topological space (X,μ) is called a ζ_ς-R_0 *space* if, for each (ζ,ς)-open set U and each $x \in U$, $c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$.

Proposition 4.2. *For a strong generalized topological space* (X,μ)*, the following properties are equivalent:*

1. (X,μ) *is a* ζ_ς-R_0 *space;*
2. *for any* $F \in (\zeta,\varsigma)C(X,\mu)$*,* $x \notin F$ *implies* $F \subseteq U$ *and* $x \notin U$ *for some* $U \in (\zeta,\varsigma)O(X,\mu)$*;*
3. *for any* $F \in (\zeta,\varsigma)C(X,\mu)$*,* $x \notin F$ *implies* $F \cap c_{(\zeta,\varsigma)}(\{x\}) = \emptyset$*;*
4. *for any distinct point* x *and* y *of* X*, either* $c_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{y\})$ *or* $c_{(\zeta,\varsigma)}(\{x\}) \cap c_{(\zeta,\varsigma)}(\{y\}) = \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let $F \in (\zeta,\varsigma)C(X,\mu)$ and let $x \notin F$. By (1),

$$c_{(\zeta,\varsigma)}(\{x\}) \subseteq X - F.$$

Put $U = X - c_{(\zeta,\varsigma)}(\{x\})$, then $U \in (\zeta,\varsigma)O(X,\mu)$, $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let $F \in (\zeta,\varsigma)C(X,\mu)$ and $x \notin F$. There exists $U \in (\zeta,\varsigma)O(X,\mu)$ such that $F \subseteq U$ and $x \notin U$. Since $U \in (\zeta,\varsigma)O(X,\mu)$, $U \cap c_{(\zeta,\varsigma)}(\{x\}) = \emptyset$ and $F \cap c_{(\zeta,\varsigma)}(\{x\}) = \emptyset$.

(3) $\Rightarrow$ (4): Suppose that $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$ for distinct points $x, y \in X$. There exists $z \in c_{(\zeta,\varsigma)}(\{x\})$ such that $z \notin c_{(\zeta,\varsigma)}(\{y\})$ (or $z \in c_{(\zeta,\varsigma)}(\{y\})$ such that $z \notin c_{(\zeta,\varsigma)}(\{x\})$). There exists $V \in (\zeta,\varsigma)O(X,\mu)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Thus, $x \notin c_{(\zeta,\varsigma)}(\{y\})$, by (3), $c_{(\zeta,\varsigma)}(\{x\}) \cap c_{(\zeta,\varsigma)}(\{y\}) = \emptyset$. The proof for the other case is similarly.

(4) $\Rightarrow$ (1): Let $V \in (\zeta,\varsigma)O(X,\mu)$ and let $x \in V$. For each $y \notin V, x \neq y$ and $x \notin c_{(\zeta,\varsigma)}(\{y\})$. Then, $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$, by (4),

$$c_{(\zeta,\mu)}(\{x\}) \cap c_{(\zeta,\varsigma)}(\{y\}) = \emptyset$$

for each $y \in X - V$ and hence $c_{(\zeta,\varsigma)}(\{x\}) \cap (\cup_{y \in X-V} c_{(\zeta,\varsigma)}(\{y\})) = \emptyset$. On the other hand, since $V \in (\zeta,\varsigma)O(X,\mu)$ and $y \in X - V$, we have

$$c_{(\zeta,\varsigma)}(\{y\}) \subseteq X - V$$

and hence $X - V = \cup_{y \in X-V} c_{(\zeta,\varsigma)}(\{y\})$. Thus, $(X - V) \cap c_{(\zeta,\varsigma)}(\{x\}) = \emptyset$ and $c_{(\zeta,\varsigma)}(\{x\}) \subseteq V$. This shows that (X,μ) is a ζ_ς-R_0 space. $\square$

Corollary 4.3. *A strong generalized topological space (X,μ) is ζ_ς-R_0 if and only if, for any x and y in X, $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$ implies*

$$c_{(\zeta,\varsigma)}(\{x\}) \cap c_{(\zeta,\varsigma)}(\{y\}) = \emptyset.$$

Proof. This is an immediate consequence of Lemma 3.7. $\square$

Proposition 4.4. *A strong generalized topological space (X,μ) is ζ_μ-R_0 if and only if, for any x and y in X, $\zeta_{(\zeta,\varsigma)}(\{x\}) \neq \zeta_{(\zeta,\varsigma)}(\{y\})$ implies*

$$\zeta_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{y\}) = \emptyset.$$

Proof. Suppose that (X,μ) is ζ_ς-R_0 space. Thus by Lemma 3.15, for any points x and y in X if $\zeta_{(\zeta,\varsigma)}(\{x\}) \neq \zeta_{(\zeta,\varsigma)}(\{y\})$, then $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$. Now, we prove that $\zeta_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{y\}) = \emptyset$. Assume that

$$z \in \zeta_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{y\}).$$

By $z \in \zeta_{(\zeta,\varsigma)}(\{x\})$, it follows that $x \in c_{(\zeta,\varsigma)}(\{z\})$. Since $x \in c_{(\zeta,\varsigma)}(\{x\})$, by Corollary 4.3, $c_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{z\})$. Similarly, we have

$$c_{(\zeta,\varsigma)}(\{y\}) = c_{(\zeta,\varsigma)}(\{z\}) = c_{(\zeta,\varsigma)}(\{x\}).$$

This is a contradiction. Therefore, we have $\zeta_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{y\}) = \emptyset$.

Conversely, let (X,μ) be a generalized topological space such that for any points x and y in X, $\zeta_{(\zeta,\varsigma)}(\{x\}) \neq \zeta_{(\zeta,\varsigma)}(\{y\})$ implies

$$\zeta_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{y\}) = \emptyset.$$

If $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$, then by Lemma 3.15, $\zeta_{(\zeta,\varsigma)}(\{x\}) \neq \zeta_{(\zeta,\varsigma)}(\{y\})$. Therefore, $\zeta_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{y\}) = \emptyset$ which implies $c_{(\zeta,\varsigma)}(\{y\}) \cap c_{(\zeta,\varsigma)}(\{y\}) = \emptyset$. Because $z \in c_{(\zeta,\varsigma)}(\{x\})$ implies that $x \in \zeta_{(\zeta,\varsigma)}(\{x\})$ by Proposition 3.14 and therefore $\zeta_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{z\}) \neq \emptyset$. By the hypothesis, we have

$$\zeta_{(\zeta,\varsigma)}(\{x\}) = \zeta_{(\zeta,\varsigma)}(\{z\}).$$

Then, $z \in c_{(\zeta,\varsigma)}(\{x\}) \cap c_{(\zeta,\varsigma)}(\{y\})$ would imply that

$$c_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{z\}) = c_{(\zeta,\varsigma)}(\{y\}).$$

This is a contradiction. Thus, $c_{(\zeta,\varsigma)}(\{x\}) \cap c_{(\zeta,\varsigma)}(\{y\}) = \emptyset$, by Corollary 4.3, (X,μ) is a ζ_ς-R_0 space. $\square$

Proposition 4.5. *For a strong generalized topological space (X,μ), the following properties are equivalent:*

(1) *(X,μ) is a ζ_ς-R_0 space;*
(2) *for any nonempty set A and $G \in (\zeta,\varsigma)O(X,\mu)$ such that $A \cap G \neq \emptyset$, there exists $F \in (\zeta,\varsigma)C(X,\mu)$ such that $A \cap F \neq \emptyset$ and $F \subseteq G$;*
(3) *for any $G \in (\zeta,\varsigma)O(X)$, $G = \cup\{F \in (\zeta,\varsigma)C(X,\mu) \mid F \subseteq G\}$;*
(4) *for any $F \in (\zeta,\varsigma)C(X,\mu)$, $F = \cap\{G \in (\zeta,\varsigma)O(X,\mu) \mid F \subseteq G\}$;*
(5) *for any $x \in X$, $c_{(\zeta,\varsigma)}(\{x\}) \subseteq \zeta_{(\zeta,\varsigma)}(\{x\})$.*

Proof. (1) $\Rightarrow$ (2): Let A be a nonempty subset of X and let $G \in (\zeta,\varsigma)O(X,\mu)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \in (\zeta,\varsigma)O(X,\mu)$, $c_{(\zeta,\varsigma)}(\{x\}) \subseteq G$. Put $F = c_{(\zeta,\varsigma)}(\{x\})$, then $F \in (\zeta_\varsigma)C(X,\mu)$, $F \subseteq G$ and $A \cap F \neq \emptyset$.

(2) $\Rightarrow$ (3): Let $G \in (\zeta,\varsigma)O(X,\mu)$, then $G \supseteq \cup\{F \in (\zeta,\varsigma)C(X,\mu) \mid F \subseteq G\}$. Let x be any point of G. There exists $F \in (\zeta,\varsigma)C(X,\mu)$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in (\zeta,\varsigma)C(X,\mu) \mid F \subseteq G\}$ and hence $G = \cup\{F \in (\zeta,\varsigma)C(X,\mu) \mid F \subseteq G\}$.

(3) $\Rightarrow$ (4): The proof is obvious.

(4) $\Rightarrow$ (5): Let x be any point of X and let $y \notin \zeta_{(\zeta,\varsigma)}(\{x\})$. Then, there exists $V \in (\zeta,\varsigma)O(X,\mu)$ such that $x \in V$ and $y \notin V$; hence $c_{(\zeta,\varsigma)}(\{y\}) \cap V = \emptyset$. By (4), we have $(\cap\{G \in (\zeta,\varsigma)O(X,\mu) \mid c_{(\zeta,\varsigma)}(\{y\}) \subseteq G\}) \cap V = \emptyset$ and there exists $G \in (\zeta,\varsigma)O(X,\mu)$ such that $x \notin G$ and $c_{(\zeta,\varsigma)}(\{y\}) \subseteq G$. Therefore, $c_{(\zeta,\varsigma)}(\{x\}) \cap G = \emptyset$ and $y \notin c_{(\zeta,\varsigma)}(\{x\})$. Consequently, we obtain

$$c_{(\zeta,\varsigma)}(\{x\}) \subseteq \zeta_{(\zeta,\varsigma)}(\{x\}).$$

(5) $\Rightarrow$ (1): Let $G \in (\zeta,\varsigma)O(X,\mu)$ and let $x \in G$. Let $y \in \zeta_{(\zeta,\varsigma)}(\{x\})$, then $x \in c_{(\zeta,\varsigma)}(\{y\})$ and $y \in G$. This implies that $\zeta_{(\zeta,\varsigma)}(\{x\}) \subseteq G$. Therefore, we obtain $x \in c_{(\zeta,\varsigma)}(\{x\}) \subseteq \zeta_{(\zeta,\varsigma)}(\{x\}) \subseteq G$. This shows that (X,μ) is a ζ_ς-R_0 space. □

Corollary 4.6. *For a strong generalized topological space (X,μ), the following properties are equivalent:*

(1) *(X,μ) is a ζ_ς-R_0 space;*
(2) *$c_{(\zeta,\varsigma)}(\{x\}) = \zeta_{(\zeta,\varsigma)}(\{x\})$ for all $x \in X$.*

Proof. (1) $\Rightarrow$ (2): Suppose that (X,μ) is a ζ_ς-R_0 space. By Proposition 4.5, $c_{(\zeta,\varsigma)}(\{x\}) \subseteq \zeta_{(\zeta,\varsigma)}(\{x\})$ for each $x \in X$. Let $y \in \zeta_{(\zeta,\varsigma)}(\{x\})$, then $c_{(\zeta,\varsigma)}(\{x\}) \cap c_{(\zeta,\varsigma)}(\{y\}) \neq \emptyset$. By Corollary 4.3, $c_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{y\})$. Thus, $y \in c_{(\zeta,\varsigma)}(\{x\})$ and hence $\zeta_{(\zeta,\varsigma)}(\{x\}) \subseteq c_{(\zeta,\varsigma)}(\{x\})$. This shows that

$$\zeta_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{x\}).$$

(2) $\Rightarrow$ (1): By Proposition 4.5. □

Corollary 4.7. *Let (X,μ) be a strong ζ_ς-R_0 generalized topological space and let $x \in X$. If $\langle x\rangle_\varsigma = \{x\}$, then $c_{(\zeta,\varsigma)}(\{x\}) = \{x\}$.*

Proof. This is a consequence of Corollary 4.6. □

Proposition 4.8. *For a strong generalized topological space (X,μ), the following properties are equivalent:*

(1) *(X,μ) is a ζ_ς-R_0 space;*
(2) *$x \in c_{(\zeta,\varsigma)}(\{y\})$ if and only if $y \in c_{(\zeta,\varsigma)}(\{x\})$.*

Proof. (1) $\Rightarrow$ (2): Suppose that (X,μ) is a ζ_ς-R_0 space. Let $x \in c_{(\zeta,\varsigma)}(\{y\})$ and let U be any (ζ,ς)-open set such that $y \in U$. Hence, $\zeta_{(\zeta,\varsigma)}(\{y\}) \subseteq U$. Since $x \in c_{(\zeta,\varsigma)}(\{y\})$ and (X,μ) is ζ_ς-R_0, by Corollary 4.3, $x \in \zeta_{(\zeta,\varsigma)}(\{y\}) \subseteq U$. Therefore, every (ζ,ς)-open set containing y contains x. Thus, $y \in c_{(\zeta,\varsigma)}(\{x\})$.

(2) $\Rightarrow$ (1): Let U be a (ζ,ς)-open set and let $x \in U$. If $y \notin U$, then $x \notin c_{(\zeta,\varsigma)}(\{y\})$ and hence $y \notin c_{(\zeta,\varsigma)}(\{x\})$. This implies that $c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$. Hence, (X,μ) is a ζ_ς-R_0 space. □

Proposition 4.9. *For a strong generalized topological space (X,μ), the following properties are equivalent:*

(1) *(X,μ) is a ζ_ς-R_0 space;*
(2) *$\langle x\rangle_\varsigma = c_{(\zeta,\varsigma)}(\{x\})$ for each $x \in X$;*
(3) *$\langle x\rangle_\varsigma$ is (ζ,ς)-closed for each $x \in X$.*

Proof. (1) $\Rightarrow$ (2): By Corollary 4.7, $c_{(\zeta,\varsigma)}(\{x\}) = \zeta_{(\zeta,\varsigma)}(\{x\})$ for each $x \in X$. Thus, $c_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{x\}) \cap \zeta_{(\zeta,\varsigma)}(\{x\}) = \langle x\rangle_\varsigma$.

(2) $\Rightarrow$ (1): Since $\langle x\rangle_\varsigma = c_{(\zeta,\varsigma)}(\{x\})$ for each $x \in X$, $c_{(\zeta,\varsigma)}(\{x\}) \subseteq \zeta_{(\zeta,\varsigma)}(\{x\})$, by Proposition 4.5, (X,μ) is ζ_ς-R_0.

(2) $\Leftrightarrow$ (3): This is a consequence of Proposition 3.14. □

5. ζ_ς-R_1 STRONG GENERALIZED TOPOLOGICAL SPACES

In this section, we introduce the concept of ζ_ς-R_1 strong generalized topological spaces. Moreover, some characterizations of ζ_ς-R_1 strong generalized topological spaces are discussed.

Definition 5.1. A strong generalized topological space (X,μ) is said to be a ζ_ς-R_1 *space* if, for any points x, y in X with $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$, there exist disjoint (ζ,ς)-open sets U and V such that $c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$ and $c_{(\zeta,\varsigma)}(\{y\}) \subseteq V$.

Proposition 5.2. *If a strong generalized topological space (X,μ) is ζ_ς-R_1, then (X,μ) is ζ_ς-R_0.*

Proof. Let U be a (ζ,μ)-open set and let $x \in U$. If $y \notin U$, then since $x \notin c_{(\zeta,\varsigma)}(\{y\})$, $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$ and there exists a (ζ,ς)-open set V_y such that $c_{(\zeta,\varsigma)}(\{y\}) \subseteq V_y$ and $x \notin V_y$, which implies $y \notin c_{(\zeta,\varsigma)}(\{x\})$. Thus, $c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$. Hence, (X,μ) is ζ_ς-R_0. □

Theorem 5.3. *For a strong generalized topological space (X,μ), the following properties are equivalent:*

(1) *(X,μ) is a ζ_ς-R_1 space;*
(2) *for each $x, y \in X$ one of the following hold:*
   (i) *for any (ζ,ς)-open set U, $x \in U$ if and only if $y \in U$,*
   (ii) *there exist disjoint (ζ,ς)-open sets U and V such that $x \in U$ and $y \in V$;*

(3) *for each* $x, y \in X$ *such that* $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$, *there exist* (ζ,ς)-*closed sets* F_x *and* F_y *such that* $x \in F_x, y \notin F_x, y \in F_y, x \notin F_y$ *and* $X = F_x \cup F_y$.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$. Then, (i) $c_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{y\})$ or (ii) $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$. If $c_{(\zeta,\varsigma)}(\{x\}) = c_{(\zeta,\varsigma)}(\{y\})$ and U is any (ζ,ς)-open set, by Proposition 5.2, $x \in U$ implies $y \in c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$ and also $y \in U$ implies $x \in c_{(\zeta,\varsigma)}(\{y\}) \subseteq U$. If $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$, then by (1) there exist disjoint (ζ,ς)-open sets U and V such that $x \in c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$ and $y \in c_{(\zeta,\varsigma)}(\{y\}) \subseteq V$.

(2) $\Rightarrow$ (3): Let $x, y \in X$ such that $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$. Then, we have $x \notin c_{(\zeta,\varsigma)}(\{y\})$ or $y \in c_{(\zeta,\varsigma)}(\{x\})$, say $x \notin c_{(\zeta,\varsigma)}(\{y\})$. Then, there exists a (ζ,ς)-open set G such that $x \in G$ and $y \notin G$. This shows that (ii) holds. There exist disjoint (ζ,ς)-open sets U and V such that $x \in U$ and $y \in V$. Put $F_x = X - V$ and $F_y = X - U$. Then, F_x and F_y are (ζ,ς)-closed sets such that $x \in F_x, y \notin F_x, y \in F_y, x \notin F_y$ and $X = F_x \cup F_y$.

(3) $\Rightarrow$ (1): First, we shall show that (X,μ) is a ζ_ς-R_1 space. Let U be a (ζ,ς)-open set and let $x \in U$. Suppose that $y \notin U$. Then, $c_{(\zeta,\varsigma)}(\{y\}) \cap U = \emptyset$ and hence $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$ since $x \in U$. By (3), there exist (ζ,ς)-closed sets F_x and F_y such that $x \in F_x, y \notin F_x, y \in F_y, x \notin F_y$ and $X = F_x \cup F_y$. Then, $y \in X - F_x, x \notin X - F_x$ and $X - F_x$ is (ζ,ς)-open. Therefore, we have $y \notin c_{(\zeta,\varsigma)}(\{x\})$ and hence $c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$. This shows that (X,μ) is a ζ_ς-R_0 space. Next, we show that (X,μ) is a ζ_ς-R_1 space. Let $x, y \in X$ such that $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$. There exist (ζ,ς)-closed sets F_x and F_y such that $x \in F_x, y \notin F_x, y \in F_y, x \notin F_y$ and $X = F_x \cup F_y$. Now, put $U = X - F_y$ and $V = X - F_x$. Then, $x \in U, y \in V$ and U, V are disjoint (ζ,ς)-open sets. Moreover, we have $c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$ and $c_{(\zeta,\varsigma)}(\{y\}) \subseteq V$ since (X,μ) is a ζ_ς-R_0 space. Thus, (X,μ) is a ζ_ς-R_1 space. □

Definition 5.4. A strong generalized topological space (X,μ) is said to be:

(i) ζ_ς-T_0 if for any distinct pair of points x and y in X, there exists a (ζ,ς)-open set containing one of the points but not the other;
(ii) ζ_ς-T_1 if for any distinct pair of points x and y in X, there exist (ζ,ς)-open sets U and V of X such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$;
(iii) ζ_ς-T_2 if for any distinct pair of points x and y in X, there exist (ζ,ς)-open sets U and V of X such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Proposition 5.5. *Let* (X,μ) *be a* ζ_ς-T_1 *strong generalized topological space. Then, the following properties are equivalent:*

(1) (X,μ) *is a* ζ_ς-T_1 *space;*
(2) *for any point* $x \in X$, $\{x\}$ *is* (ζ,ς)-*closed;*
(3) (X,μ) *is a* ζ_ς-R_0 *and* ζ_ς-T_0 *space.*

Proof. (1) $\Rightarrow$ (2): Let x be any point of X. Let $y \in X$ such that $y \neq x$. There exists a (ζ, ς)-open set U such that $y \in U$ and $x \notin U$. This shows that $y \notin c_{(\zeta,\varsigma)}(\{x\})$. Therefore, we have $c_{(\zeta,\varsigma)}(\{x\}) = \{x\}$ and hence $\{x\}$ is (ζ, ς)-closed.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (1): Let x and y be any distinct points of X. Since (X, μ) is a ζ_ς-T_0 space, there exists a (ζ, ς)-open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. In case $x \in U$ and $y \notin U$, we have $x \in c_{(\zeta,\varsigma)}(\{x\}) \subseteq U$ and hence $y \in X - U \subseteq X - c_{(\zeta,\varsigma)}(\{x\})$. Since the proof of the other is quite similar, (X, μ) is a ζ_ς-T_1 space. □

Theorem 5.6. *For a strong generalized topological space (X, μ), the following properties are equivalent:*

(1) *(X, μ) is a ζ_ς-T_2 space;*
(2) *(X, μ) is a ζ_ς-R_1 and ζ_ς-T_1 space;*
(3) *(X, μ) is a ζ_ς-R_1 and ζ_ς-T_0 space.*

Proof. (1) $\Rightarrow$ (2): Since (X, μ) is a ζ_ς-T_2 space, (X, μ) is ζ_ς-T_1. Let x and y be points of X such that $c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\})$. Then, by Proposition 5.5, $\{x\} = c_{(\zeta,\varsigma)}(\{x\}) \neq c_{(\zeta,\varsigma)}(\{y\}) = \{y\}$ and there exist disjoint (ζ, ς)-open sets U and V such that $c_{(\zeta,\varsigma)}(\{x\}) = \{x\} \subseteq U$ and $c_{(\zeta,\varsigma)}(\{y\}) = \{y\} \subseteq V$. This shows that (X, μ) is a ζ_ς-R_1 space.

(2) $\Rightarrow$ (3): The proof is obvious.

(3) $\Rightarrow$ (1): Let (X, μ) be a ζ_ς-R_1 and ζ_ς-T_0 space. By Proposition 5.2 and 5.5, (X, μ) is a ζ_ς-T_1 space and every singleton is (ζ, ς)-closed. Let x and y be distinct points of X. Then, $c_{(\zeta,\varsigma)}(\{x\}) = \{x\} \neq \{y\} = c_{(\zeta,\varsigma)}(\{y\})$ and there exist disjoint (ζ, ς)-open sets U and V such that $x \in U$ and $y \in V$. This shows that (X, μ) is a ζ_ς-T_2 space. □

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