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On Hybrid Ideals and Hybrid Bi-ideals in Semigroups

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Abstract. In this paper, we introduce the notion of hybrid bi-ideals in semigroups and investigate some of their important properties. We also give various equivalent conditions for a semigroup to be regular and hybrid structures to be hybrid bi-ideals of S.

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1. Introduction

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as $AB = \{ab\ : a \in A \text{ and } b \in B\}$. Let A be a non-empty subset of S. A is called a subsemigroup of S if $S^2 \subseteq S$. A subsemigroup X of S is called a left (resp., right) ideal of S if $SX \subseteq X(resp., XS \subseteq X)$. If X is both a left and right ideal of S, then X is called a two-sided ideal or ideal of S. It can easily be verified that for any $a \in S,\ L(a) = \{a, Sa\}$ (resp., $R(a) = \{a, aS\}$) is a left (resp., right) ideal of S generated by a in S. A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA \subseteq A$. It is clear that for any

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$a \in S$, $B(a) = \{a, a^2, aSa\}$ is a bi-ideal of S generated by an element $a \in S$. A semigroup S is called left (resp., right) duo if every left (resp., right) ideal of S is an ideal of S. A semigroup S is called duo if it is both left and right duo. A semigroup S is called regular if for each $a \in S$, there exists an element x in S such that $a = axa$.

In his classic paper [10], Zadeh introduced the notion of a fuzzy subset A of a set X as a mapping from X into $[0, 1]$. Rosenfeld [9] and Kuroki [6] applied this concept in group theory and semigroup theory. Since then, several authors have been pursued the study of fuzzy algebraic structure in many directions such as groups, rings, modules, vector spaces and so on (See[5, 6, 7]). As a new mathematical tool for dealing with uncertainties, Molodtsov [8] introduced the soft set theory. In the past few years, the fundamentals of soft set theory have been studied by various researchers (See [1, 3, 4]). In 2017, Anis et al.[2] introduced the notions of hybrid sub-semigroups and hybrid left(resp., right) ideals in semigroups and obtained several properties.

In this paper, we introduce and discuss some properties of hybrid bi-ideal of a semigroup, which is an extension of the concept of a hybrid sub-semigroup of S, and characterize regular semigroups, and both intra-regular and left quasiregular semigroups in terms of hybrid bi-ideals.

2. Hybrid structures in semigroups

In this section, we present some elementary definitions of hybrid structures in semigroup that we will use later in this paper.

Definition 2.1. [2] Let I be the unit interval and $\mathscr{P}(U)$ denote the power set of an initial universal set U. A hybrid structure in S over U is defined to be a mapping $\tilde{f}_\lambda := (\tilde{f}, \lambda) : S \to \mathscr{P}(U) \times I,\ x \mapsto (\tilde{f}(x), \lambda(x))$, where $\tilde{f} : S \to \mathscr{P}(U)$ and $\lambda : S \to I$ are mappings.

Let us denote by $H(S)$ the set of all hybrid structures in S over U. We define an order $\ll$ in $H(S)$ as follows : For all $\tilde{f}_\lambda, \tilde{g}_\gamma \in H(S),\ \tilde{f}_\lambda \ll \tilde{g}_\gamma$ if and only if $\tilde{f} \mathrel{\tilde{\subseteq}} \tilde{g}, \lambda \succeq \gamma$, where $\tilde{f} \mathrel{\tilde{\subseteq}} \tilde{g}$ means that $\tilde{f}(x) \subseteq \tilde{g}(x)$ and $\lambda \succeq \gamma$ means that $\lambda(x) \geq \gamma(x)$ for all $x \in S$. Note that $(H(S), \ll)$ is a poset.

Definition 2.2. [2] Let S be a semigroup. A hybrid structure $\tilde{f}_\lambda$ in S is called a hybrid subsemigroup of S over U if $\tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y)$ and $\lambda(xy) \leq \bigvee\{\lambda(x), \lambda(y)\}$ for all $x, y \in S$.

Definition 2.3. [2] Let S be a semigroup. A hybrid structure $\tilde{f}_\lambda$ in S over U is called a hybrid left (resp., right) ideal of S over U if

(i) $\tilde{f}(xy) \supseteq \tilde{f}(y)$ (resp., $\tilde{f}(xy) \supseteq \tilde{f}(x)$),

(ii) $\lambda(xy) \leq \lambda(y)(res., \lambda(xy) \leq \lambda(x))$ for all $x, y \in S$.

$\tilde{f}_\lambda$ is called a hybrid ideal of S if it is both a hybrid left and a hybrid right ideal of S over U.

Definition 2.4. A semigroup S is called a hybrid left (resp., right) duo over U if every hybrid left (resp., right) ideal of S over U is a hybrid ideal of S over U. A semigroup S is called a hybrid duo over U if it is both a hybrid left and a hybrid right duo over U.

Definition 2.5. Let S be a semigroup. A hybrid subsemigroup $\tilde{f}_\lambda$ in S over U is called a hybrid bi-ideal of S over U if
(i) $\tilde{f}(xyz) \supseteq \tilde{f}(x) \cap \tilde{f}(z)$,
(ii) $\lambda(xyz) \leq \bigvee\{\lambda(x), \lambda(z)\}$ for all $x, y, z \in S$.

Clearly, every hybrid left and hybrid right ideals of S over U are hybrid bi-deals of S over U, however, hybrid bi-deals of S over U need not be either hybrid left or hybrid right ideals of S over U as can be seen by the following example.

Example 2.6. Let $S = \{0, a, b, c\}$ be a semigroup with the following Cayley table:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	b
c	b	b	b	c

Table 1

Let $\tilde{f}_\lambda$ be a hybrid structure in S over $U = [0, 1]$ defined by $f(0) = [0, 0.6]$; $f(a) = [0, 0.5]; f(b) = [0, 0.4]; f(c) = [0, 0.2]$ and λ be any constant mapping form S to I. Then $\tilde{f}_\lambda$ is a hybrid bi-ideal of S over U, which is neither a hybrid left nor a hybrid right ideal of S over U as $f(ca) \not\supseteq f(a)$ and $f(ac) \not\supseteq f(a)$. □

Definition 2.7. [2] Let A be a non-empty subset of S. Then the characteristic hybrid structure in S over U is denoted by $\chi_A(\tilde{f}_\lambda) = (\chi_A(\tilde{f}), \chi_A(\lambda))$, where

$$\chi_A(\tilde{f}) : S \to \mathscr{P}(U), x \mapsto \begin{cases} U & if\ x \in A \\ \phi & otherwise \end{cases}$$

and

$$\chi_A(\lambda) : S \to I, x \mapsto \begin{cases} 0 & if x \in A \\ 1 & otherwise. \end{cases}$$

3. Main results

Theorem 3.1. *Let A be a non-empty subset of S. Then A is a subsemigroup of S if and only if the characteristic hybrid structure $\chi_A(\tilde{f}_\lambda)$ is a hybrid subsemigroup of S.*

Proof. Assume that A is a subsemigroup of S and $x, y \in S$.

If $xy \in A$, then $\chi_A(\tilde{f})(xy) = U \supseteq \chi_A(\tilde{f})(x) \cap \chi_A(\tilde{f})(y)$ and $\chi_A(\lambda)(xy) = 0 \leq \bigvee\{\chi_A(\lambda)(x), \chi_A(\lambda)(y)\}$. If $xy \notin A$, then either $x \notin A$ or $y \notin A$, so $\chi_A(\tilde{f})(xy) = \phi = \chi_A(\tilde{f})(x) \cap \chi_A(\tilde{f})(y)$ and $\chi_A(\lambda)(xy) = 1 = \bigvee\{\chi_A(\lambda)(x), \chi_A(\lambda)(y)\}$. Therefore $\chi_A(\tilde{f}_\lambda)$ is a hybrid subsemigroup of S.

Conversely, assume that $\chi_A(\tilde{f}_\lambda)$ is a hybrid subsemigroup of S. Let $x, y \in A$. Then $\chi_A(\tilde{f})(xy) \supseteq \chi_A(\tilde{f})(x) = U$ implies $xy \in A$. So A is a subsemigroup of S. □

Theorem 3.2. *Let A be a non-empty subset of S. Then A is a bi-ideal of S if and only if the characteristic hybrid structure $\chi_A(\tilde{f}_\lambda)$ is a hybrid bi-ideal of S.*

Proof. Assume that A is a bi-ideal of S. Let $x, y, z \in S$. If $x \in A$ and $z \in A$, then $xyz \in A$, so $\chi_A(\tilde{f})(xyz) = U = \chi_A(\tilde{f})(x) \cap \chi_A(\tilde{f})(z)$ and $\chi_A(\lambda)(xyz) = 0 = \bigvee\{\chi_A(\lambda)(x), \chi_A(\lambda)(z)\}$. If $x \notin A$ or $z \notin A$, then $\chi_A(\tilde{f})(x) \cap \chi_A(\tilde{f})(z) = \phi \subseteq \chi_A(\tilde{f})(xyz)$ and $\bigvee\{\chi_A(\lambda)(x), \chi_A(\lambda)(z)\} = 1 \geq \chi_A(\lambda)(xyz)$. By Theorem 3.1, $\chi_A(\tilde{f}_\lambda)$ is a hybrid bi-ideal of S.

Conversely, assume that $\chi_A(\tilde{f}_\lambda)$ is a hybrid bi-ideal of S. Let $x, z \in A$ and $y \in S$. Then $\chi_A(\tilde{f})(xyz) \supseteq \chi_A(\tilde{f})(x) \cap \chi_A(\tilde{f})(z) = \phi$ which implies $xyz \in A$. By Theorem 3.1, A is a bi-ideal of S. □

Theorem 3.3. *Let S be a regular semigroup. Then the following conditions are equivalent:*

(i) S is left (resp., right) duo,

(ii) S is hybrid left (resp., right) duo.

Proof. Assume that S is left duo. Then for any $a, b \in S$, we have $ab \in (aSa)b \subseteq (Sa)S \subseteq Sa$ as Sa is a left ideal of S and S is regular, so there exists $x \in S$ such that $ab = xa$. Let $\tilde{f}_\lambda$ be any hybrid left ideal of S. Then $\tilde{f}(ab) = \tilde{f}(xa) \supseteq \tilde{f}(a)$ and $\lambda(ab) = \lambda(xa) \leq \lambda(a)$. Thus $\tilde{f}_\lambda$ is a hybrid right ideal of S and hence S is hybrid left duo.

Conversely, assume that S is hybrid left duo and let A be any left ideal of S. Then by Theorem 3.5 of [2], the characteristic hybrid structure $\chi_A(\tilde{f}_\lambda)$ is a hybrid left ideal of S. Since S is hybrid left duo, we have $\chi_A(\tilde{f}_\lambda)$ is a hybrid ideal of S. By Corollary 3.6 of [2], we have A is an ideal of S. Therefore S is left duo. □

As a consequence of above theorem, we have the following corollary.

Corollary 3.4. *Let S be a regular semigroup. Then the following conditions are equivalent:*

(i) S is duo,

(ii) S is hybrid duo.

Theorem 3.5. *Let S be a regular left duo (resp., right duo, duo) semigroup. Then the following conditions are equivalent:*

(i) $\tilde{f}_\lambda$ is a hybrid bi-ideal of S,

(ii) $\tilde{f}_\lambda$ is a hybrid right ideal (resp., left, ideal) of S.

Proof. $(i) \Rightarrow (ii)$ Assume that $\tilde{f}_\lambda$ is a hybrid bi-ideal of S. Let $x, y \in S$. Since S is regular, we have $x = xtx \in xS \cap Sx$ for some $t \in S$ which implies $xy \in (xS \cap Sx)S \subseteq xS \cap Sx$ as S is left duo. So $xy = sx$ and $xy = xs'$ for some $s, s' \in S$. By the regularity of S, there exists $r \in S$ with $xy = xyrxy = xs'rsx = x(s'rs)x$. Since $\tilde{f}_\lambda$ is a hybrid bi-ideal of S, we have $\tilde{f}(xy) = \tilde{f}(x(s'rs)x) \supseteq \tilde{f}(x)$ and $\lambda(xy) = \lambda(x(s'rs)x) \leq \lambda(x)$. Therefore $\tilde{f}_\lambda$ is a hybrid right ideal of S. $(ii) \Rightarrow (i)$ It is trivial. □

Theorem 3.6. *Let S be a regular semigroup. Then the following conditions are equivalent:*

(i) Every bi-ideal of S is a right ideal (resp., left ideal, ideal) of S.

(ii) Every hybrid bi-ideal of S is a hybrid right ideal (resp., left ideal, ideal) of S.

Proof. $(i) \Rightarrow (ii)$ Assume that (i) holds. Let $\tilde{f}_\lambda$ be a hybrid bi-ideal of S and $a, b \in S$. Then aSa is a right ideal of S and $a \in aSa$ as S is regular. so $ab \in aSa$ implies $ab = asa$ for some $s \in S$. So $\tilde{f}(ab) = \tilde{f}(asa) \supseteq \tilde{f}(a)$ and $\lambda(ab) = \lambda(asa) \leq \lambda(a)$. Therefore $\tilde{f}_\lambda$ is a hybrid right ideal of S.

$(ii) \Rightarrow (i)$ Assume that (ii) holds and let A be any bi-ideal of S. Then by Theorem 3.2, $\chi_A(\tilde{f}_\lambda)$ is a hybrid bi-ideal of S, by assumption it becomes a hybrid right ideal of S. By Theorem 3.5 of [2], A is a right ideal of S. □

Definition 3.7. [2] For any hybrid structures $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ in S over U, the hybrid product of $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ in S is defined to be a hybrid structure $\tilde{f}_\lambda \odot \tilde{g}_\gamma = (\tilde{f} \tilde{\circ} \tilde{g}, \lambda \tilde{\circ} \gamma)$ in S over U, where

$$(\tilde{f} \tilde{\circ} \tilde{g})(x) = \begin{cases} \bigcup_{x=yz} \{\tilde{f}(y) \cap \tilde{g}(z)\} & if\ \exists\ y, z \in S\ such\ that \text{ x=yz} \\ \phi & otherwise \end{cases}$$

and

$$(\lambda \tilde{\circ} \gamma)(x) = \begin{cases} \bigwedge_{x=yz} \bigvee \{\lambda(y), \lambda(z)\} & if\ \exists\ y, z \in S\ such\ that\ x = yz \\ 1 & otherwise \end{cases}$$

for all $x \in S$.

Theorem 3.8. *Let $\tilde{f}_\lambda$ be a hybrid structure of S. Then the following conditions are equivalent:*

(i) $\tilde{f}_\lambda$ is a hybrid bi-ideal of S,

(ii) $\tilde{f}_\lambda \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$ and $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$ for any hybrid structure $\tilde{g}_\mu$ of S.

Proof. $(i) \Rightarrow (ii)$ Assume that $\tilde{f}_\lambda$ is a hybrid bi-ideal of S and $x \in S$. If $x \neq yz$ for all $y, z \in S$, then clearly $\tilde{f}_\lambda \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$ and $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$. Assume that $x = yz$ for some $y, z \in S$. Then $(\tilde{f} \tilde{\circ} \tilde{f})(x) = \bigcup\limits_{x=yz} \{\tilde{f}(y) \cap \tilde{f}(z)\} \subseteq \bigcup\limits_{x=yz} \tilde{f}(yz) = \tilde{f}(x)$ and $(\lambda \tilde{\circ} \lambda)(x) = \bigwedge\limits_{x=yz} \bigvee\{\lambda(y), \lambda(z)\} \geq \bigwedge\limits_{x=yz} \lambda(yz) = \lambda(x)$. Therefore $\tilde{f}_\lambda \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$.

We now prove that $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$.

Let $a \in S$. Suppose that there exist $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Then

$$\begin{aligned}
(\tilde{f} \tilde{\circ} \chi_S(\tilde{g}) \tilde{\circ} \tilde{f})(a) &= \bigcup_{a=xy} \{(\tilde{f} \tilde{\circ} \chi_S(\tilde{g}))(x) \cap \tilde{f}(y)\} \\
&= \bigcup_{a=xy} \{ \bigcup_{x=pq} \{\tilde{f}(p) \cap \chi_S(\tilde{g})(q)\} \cap \tilde{f}(y)\} \\
&= \bigcup_{a=xy} \{ \bigcup_{x=pq} \{\tilde{f}(p) \cap U\} \cap \tilde{f}(y)\} \\
&= \bigcup_{a=pqy} \{\tilde{f}(p) \cap \tilde{f}(y)\} \\
&\subseteq \bigcup_{a=pqy} \tilde{f}(pqy) = \tilde{f}(a),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda \tilde{\circ} \chi_S(\mu) \tilde{\circ} \lambda)(a) &= \bigwedge_{a=xy} \bigvee\{(\lambda \tilde{\circ} \chi_S(\mu))(x), \lambda(y)\} \\
&= \bigwedge_{a=xy} \bigvee\{ \bigwedge_{x=pq} \bigvee\{(\lambda(p), \chi_S(\mu)(q)\}, \lambda(y)\} \\
&= \bigwedge_{a=xy} \bigvee\{ \bigwedge_{x=pq} \bigvee\{(\lambda(p), 0\}, \lambda(y)\} \\
&= \bigwedge_{a=pqy} \bigvee\{(\lambda(p), \lambda(y)\} \\
&\geq \bigwedge_{a=pqy} \lambda(pqy) = \lambda(a).
\end{aligned}$$

Otherwise $a \neq xy$ or $x \neq pq$ for all $x, y, p, q \in S$. Then $(\tilde{f} \tilde{\circ} \chi_S(\tilde{g}) \tilde{\circ} \tilde{f})(a) = \phi \subseteq \lambda(a)$ and $(\lambda \tilde{\circ} \chi_S(\mu) \tilde{\circ} \lambda)(a) = 1 \geq \lambda(a)$.

Therefore $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds and $x, y \in S$. Then $\tilde{f}(xy) \supseteq (\tilde{f} \tilde{\circ} \tilde{f})(xy) \supseteq \tilde{f}(x) \cup \tilde{f}(y)$ and $\lambda(xy) \leq (\lambda \tilde{\circ} \lambda)(xy) \leq \bigvee\{\lambda(x), \lambda(y)\}$. So $\tilde{f}_\lambda$ is a hybrid sub-semigoup of S.

Now, let $x, y, z \in S$. Then $\tilde{f}(xyz) \supseteq (\tilde{f} \tilde{\circ} \chi_S(\mu) \tilde{\circ} \tilde{f})(xyz)$

$$\begin{aligned}&\supseteq (\tilde{f}\tilde{\circ}\chi_S(\mu))(xy)\cap\tilde{f}(z)\\&\supseteq (\tilde{f}(x)\cap\chi_S(\mu)(y))\cap\tilde{f}(z)\\&=(\tilde{f}(x)\cap U)\cap\tilde{f}(z)\\&=\tilde{f}(x)\cap\tilde{f}(z),\end{aligned}$$

and

$$\begin{aligned}\lambda(xyz)&\leq(\lambda\tilde{\circ}\chi_S(\mu)\tilde{\circ}\lambda)(xyz)\\&\leq\bigvee\{\lambda\tilde{\circ}\chi_S(\mu))(xy),\lambda(z)\}\\&\leq\bigvee\{\bigvee\{\lambda(x),\chi_S(\mu)(y)\},\lambda(z)\}\\&=\bigvee\{\bigvee\{\lambda(x),0\},\lambda(z)\}\\&=\bigvee\{\lambda(x),\lambda(z)\}.\end{aligned}$$

Therefore $\tilde{f}_\lambda$ is a hybrid bi-ideal of S. □

Definition 3.9. [2] Let $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ be hybrid structures in S over U. Then the hybrid intersection of $\tilde{f}_\lambda$ and $\tilde{g}_\gamma$ is denoted by $\tilde{f}_\lambda \Cap \tilde{g}_\gamma$ and is defined to be a hybrid structure $\tilde{f}_\lambda \Cap \tilde{g}_\gamma : S \to \mathscr{P}(U)\times I, x\mapsto ((\tilde{f}\tilde{\cap}\tilde{g})(x),(\lambda\vee\gamma)(x))$, where $\tilde{f}\tilde{\cap}\tilde{g} : S \longrightarrow \mathscr{P}(U), x\mapsto \tilde{f}(x)\cap\tilde{g}(x)$ and $\lambda\vee\gamma : S\to I, x\mapsto\bigvee\{\lambda(x),\gamma(x)\}$.

Theorem 3.10. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda = \tilde{f}_\mu\odot\tilde{g}_\lambda\odot\tilde{f}_\mu$ for every hybrid bi-ideal $\tilde{f}_\mu$ and every hybrid ideal $\tilde{g}_\lambda$ of S.

Proof. $(i)\Rightarrow(ii)$ Assume that S is regular, and let $\tilde{f}_\mu$ be a hybrid bi-ideal and $\tilde{g}_\lambda$ a hybrid ideal of S. Then by Theorem 3.8, we have $\tilde{f}_\mu\odot\chi_S(\tilde{g}_\lambda)\odot\tilde{f}_\mu \ll \tilde{f}_\lambda$. So $\tilde{f}_\mu\odot\tilde{g}_\lambda\odot\tilde{f}_\mu \ll \tilde{f}_\mu\odot\chi_S(\tilde{g}_\lambda)\odot\tilde{f}_\mu \ll \tilde{f}_\lambda$. By Theorem 3.13 and Theorem 3.14 of [2], we have $\tilde{f}_\mu\odot\tilde{g}_\lambda\odot\tilde{f}_\mu \ll \chi_S(\tilde{f}_\mu)\odot\tilde{g}_\lambda\odot\chi_S(\tilde{f}_\mu) \ll \tilde{g}_\lambda$. So $\tilde{f}_\mu\odot\tilde{g}_\lambda\odot\tilde{f}_\mu \ll \tilde{f}_\mu\Cap\tilde{g}_\lambda$.

Let $a\in S$. Then by the reqularity of S, there exists $x\in S$ such that $a = axa = axaxa$.

Now $(\tilde{f}\tilde{\circ}\tilde{g}\tilde{\circ}\tilde{f})(a) = \bigcup_{a=uv}\{\tilde{f}(u)\cap(\tilde{g}\tilde{\circ}\tilde{f})(v)\}$

$$\begin{aligned}&\supseteq\tilde{f}(a)\cap(\tilde{g}\tilde{\circ}\tilde{f})(xaxa)\\&\supseteq\tilde{f}(a)\cap\tilde{g}(xax)\cap\tilde{f}(a)\\&\supseteq\tilde{f}(a)\cap\tilde{g}(a)=(\tilde{f}\cap\tilde{g})(a),\end{aligned}$$

and $(\mu\tilde{\circ}\lambda\tilde{\circ}\mu)(a) = \bigwedge_{a=uv}\bigvee\{\mu(u),(\lambda\tilde{\circ}\mu)(v)\}$

$$\begin{aligned}&\leq\bigvee\{\mu(a),(\lambda\tilde{\circ}\mu)(xaxa)\}\\&\leq\bigvee\{\mu(a),\bigvee\{\lambda(xax),\mu(a)\}\}\\&\leq\bigvee\{\mu(a),\lambda(a)\}=(\lambda\cap\mu)(a).\end{aligned}$$

Thus $\tilde{f}_\mu\Cap\tilde{g}_\lambda \ll \tilde{f}_\mu\odot\tilde{g}_\lambda\odot\tilde{f}_\mu$ and hence $\tilde{f}_\mu\Cap\tilde{g}_\lambda = \tilde{f}_\mu\odot\tilde{g}_\lambda\odot\tilde{f}_\mu$.

$(ii)\Rightarrow(i)$ Assume that (ii) holds. Then by Corollary 3.6 of [2] and assumption, we have $\tilde{f}_\mu\Cap\chi_S(\tilde{f}_\mu) = \tilde{f}_\mu\odot\chi_S(\tilde{f}_\mu)\odot\tilde{f}_\mu$. But $\tilde{f}_\mu\Cap\chi_S(\tilde{f}_\mu)=\tilde{f}_\mu$, so $\tilde{f}_\mu = \tilde{f}_\mu\odot\chi_S(\tilde{f}_\mu)\odot\tilde{f}_\mu$ for every hybrid bi-ideal $\tilde{f}_\mu$ of S. Let $a\in S$. Then by Theorem 3.2, $\chi_{B(a)}(\tilde{f}_\mu)$ is a hybrid bi-ideal of S and $\chi_{B(a)}(\tilde{f}_\mu) = \chi_{B(a)}(\tilde{f}_\lambda)\odot$

$\chi_S(\tilde{f}_\mu) \odot \chi_{B(a)}(\tilde{f}_\mu) = \chi_{B(a)SB(a)}(\tilde{f}_\lambda)$ by Lemma 3.11 of [2]. Since $a \in B(a)$, we have $\chi_{B(a)SB(a)}(\tilde{f})(a) = \chi_{B(a)}(\tilde{f})(a) = U$ which implies $a \in B(a)SB(a)$. Therefore S is regular. □

Theorem 3.11. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{f}_\mu \odot \tilde{g}_\lambda$ for every hybrid bi-ideal $\tilde{f}_\mu$ and every hybrid left ideal $\tilde{g}_\lambda$ of S.

Proof. $(i) \Rightarrow (ii)$ Assume that S is regular and let $\tilde{f}_\mu$ be a hybrid bi-ideal and $\tilde{g}_\lambda$ be a hybrid left ideal of S. Let $a \in S$. Then there exists $x \in S$ such that $a = axa$. Then

$$\begin{aligned}(\tilde{f}\tilde{\circ}\tilde{g})(a) &= \bigcup_{a=uv}\{\tilde{f}(u) \cap \tilde{g}(v)\}\\ &\supseteq \tilde{f}(a) \cap \tilde{g}(xa) \supseteq \tilde{f}(a) \cap \tilde{g}(a) = (\tilde{f} \cap \tilde{g})(a),\end{aligned}$$

and

$$\begin{aligned}(\mu\tilde{\circ}\lambda)(a) &= \bigwedge_{a=uv}\bigvee\{\mu(u), \lambda(v)\}\\ &\leq \bigvee\{\mu(a), \lambda(xa)\} \leq \bigvee\{\mu(a), \lambda(a)\} = (\mu \cap \lambda)(a).\end{aligned}$$

Therefore $\tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{f}_\mu \odot \tilde{g}_\lambda$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds, and let $\tilde{f}_\mu$ be a hybrid right ideal and $\tilde{g}_\lambda$ be a hybrid left ideal of S. Since every hybrid right ideal of S is a hybrid bi-ideal of S, so $\tilde{f}_\mu$ is a bi-ideal of S. Then by assumption, we have $\tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{f}_\mu \odot \tilde{g}_\lambda$. By Theorem 3.13 and Theorem 3.14 of [2], we can get $\tilde{f}_\mu \odot \tilde{g}_\lambda \ll \tilde{f}_\mu \Cap \tilde{g}_\lambda$. So for any hybrid right ideal $\tilde{f}_\mu$ and hybrid left ideal $\tilde{g}_\lambda$ of S, we have $\tilde{f}_\mu \Cap \tilde{g}_\lambda = \tilde{f}_\mu \odot \tilde{g}_\lambda$.

Let R be a right ideal and L a left ideal of S. Then for any $a \in R \cap L$, we have $\chi_R(\tilde{f}_\mu)(a) \Cap \chi_L(\tilde{f}_\mu)(a) = \chi_R(\tilde{f}_\mu)(a) \odot \chi_L(\tilde{f}_\mu)(a)$. By Lemma 3.11 of [2], we have $\chi_{R\cap L}(\tilde{f}_\mu) = \chi_{RL}(\tilde{f}_\mu)$. Since $a \in R \cap L$ and $\chi_{R\cap L}(\tilde{f})(a) = U$, we have $\chi_{RL}(\tilde{f})(a) = U$ which implies $a \in RL$. Thus $R \cap L \subseteq RL \subseteq R \cap L$ and hence $R \cap L = RL$. Therefore S is regular. □

Theorem 3.12. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{g}_\lambda \odot \tilde{f}_\mu$ for every hybrid bi-ideal $\tilde{f}_\mu$ and every hybrid right ideal $\tilde{g}_\lambda$ of S.

Proof. The proof is similar to that of Theorem 3.11. □

Theorem 3.13. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda \Cap \tilde{h}_\nu \ll \tilde{f}_\mu \odot \tilde{g}_\lambda \odot \tilde{h}_\nu$ for every hybrid right ideal $\tilde{f}_\mu$, hybrid bi-ideal $\tilde{g}_\lambda$ and every hybrid left ideal of $\tilde{h}_\nu$ of S.

Proof. $(i) \Rightarrow (ii)$ Let S be a regular semigroup, $\tilde{f}_\mu$ a hybrid right ideal, $\tilde{g}_\lambda$ a hybrid bi-ideal and $\tilde{h}_\nu$ a hybrid left ideal of S. Let $a \in S$. Then there exists $x \in S$ such that $a = axa$.

$$\begin{aligned}\text{Now } (\tilde{f}\tilde{\circ}\tilde{g}\tilde{\circ}\tilde{h})(a) &= \bigcup_{a=uv} \{\tilde{f}(u) \cap (\tilde{g}\tilde{\circ}\tilde{h})(v)\} \\ &\supseteq \tilde{f}(ax) \cap \{\bigcup_{a=pq} \{\tilde{g}(p) \cap \tilde{h}(q)\}\} \\ &\supseteq \tilde{f}(ax) \cap \tilde{g}(a) \cap \tilde{h}(xa) \\ &\supseteq \tilde{f}(a) \cap \tilde{g}(a) \cap \tilde{h}(a) = (\tilde{f} \cap \tilde{g} \cap \tilde{h})(a),\end{aligned}$$

$$\begin{aligned}\text{and } (\mu\tilde{\circ}\lambda\tilde{\circ}\nu)(a) &= \bigwedge_{a=uv} \bigvee\{\mu(u), (\lambda\tilde{\circ}\nu)(v)\} \\ &\leq \bigvee\{\mu(ax), \bigwedge_{a=pq} \bigvee\{(\lambda(p), \nu(q)\}\} \\ &\leq \bigvee\{\mu(ax), \lambda(a), \nu(xa)\} \\ &\leq \bigvee\{\mu(a), \lambda(a), \nu(a)\} = (\mu \cap \lambda \cap \nu)(a).\end{aligned}$$

Therefore $\tilde{f}_\mu \Cap \tilde{g}_\lambda \Cap \tilde{h}_\nu \ll \tilde{f}_\mu \odot \tilde{g}_\lambda \odot \tilde{h}_\nu$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds, and let $\tilde{f}_\mu$ be a hybrid right ideal and $\tilde{h}_\nu$ a hybrid left ideal of S. Then $\tilde{f}_\mu \Cap \tilde{h}_\nu = \tilde{f}_\mu \Cap \chi_S(\tilde{g}_\lambda) \Cap \tilde{h}_\nu \ll \tilde{f}_\mu \odot \chi_S(\tilde{g}_\lambda) \odot \tilde{h}_\nu \ll \tilde{f}_\mu \odot \tilde{h}_\nu$.

By Theorem 3.13 and Theorem 3.14 of [2], we can get $\tilde{f}_\mu \odot \tilde{h}_\nu \ll \tilde{f}_\mu \Cap \tilde{h}_\nu$. So for any hybrid right ideal $\tilde{f}_\mu$ and hybrid left ideal $\tilde{h}_\nu$ of S, we have $\tilde{f}_\mu \Cap \tilde{h}_\nu = \tilde{f}_\mu \odot \tilde{h}_\nu$.

Let R be a right ideal and L be a left ideal of S and $a \in R \cap L$. Then $\chi_R(\tilde{f}_\mu)(a) \Cap \chi_L(\tilde{f}_\mu)(a) = \chi_R(\tilde{f}_\mu)(a) \odot \chi_L(\tilde{f}_\mu)(a)$. By Lemma 3.11 of [2], we have $\chi_{R\cap L}(\tilde{f}_\mu) = \chi_{RL}(\tilde{f}_\mu)$. Since $a \in R \cap L$ and $\chi_{R\cap L}(\tilde{f})(a) = U$, we have $\chi_{RL}(\tilde{f})(a) = U$ which implies $a \in RL$. Thus $R \cap L \subseteq RL \subseteq R \cap L$ and hence $R \cap L = RL$. Therefore S is regular. □

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