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On A Class of Soc-Injective Modules

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Abstract. Let R be a ring. The class of SA-injective right R-modules (SAI_R) is introduced as a class of soc-injective right R-modules. Let N be a right R-module. A right R-module M is said to be SA-N-injective if every R-homomorphism from a semi-artinian submodule of N into M extends to N. A module M is called SA-injective, if M is SA-R-injective. We characterize rings over which every right module is SA-injective. Conditions under which the class SAI_R is closed under quotient (resp. direct sums, pure homomorphic images) are given. The definability of the class SAI_R is studied. Finally, relations between SA-injectivity and certain generalizations of injectivity are given.

Keywords: Semi-artinian submodule, Definable class, Injective module, Noetherian module, Flat module.

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1. Introduction

Throughout R is an associative ring with identity and all modules are unitary R-modules. If not otherwise specified, by a module (resp. homomorphism) we will mean a right R-module (resp. right R-homomorphism). We use R-Mod (resp. Mod-R) to denote the class of left (resp. right) R-modules. We will use M^* to denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a right module

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M. Let $\mathcal{G}$ (resp. $\mathcal{F}$) be a class of right (resp. left) R-modules. A pair $(\mathcal{F}, \mathcal{G})$ is called almost dual pair if $\mathcal{G}$ is closed under summands and direct products, and for any left R-module M, $M \in \mathcal{F}$ if and only if $M^* \in \mathcal{G}$ [12, p. 66]. An exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ of right R-modules is said to be pure if the sequence $0 \to \mathrm{Hom}_R(N, A) \to \mathrm{Hom}_R(N, B) \to \mathrm{Hom}_R(N, C) \to 0$ is exact, for every finitely presented right R-module N and we called that $\alpha(A)$ is a pure submodule of B [21]. A right R-module M is called FP-injective if every monomorphism $\alpha : M \to N$ is pure. A right R-module M is called pure injective if M is injective with respect to all pure short exact sequences [21]. If a subclass $\mathcal{G}$ of Mod-R is closed under pure submodules, direct limits and direct products, then it is called a definable class[16]. We denote by $\mathrm{Soc}(M)$ to the socle of a module M. A right R-module M is called semi-artinian if for any proper submodule N of M we have $\mathrm{Soc}(M/N) \neq 0$ [9, p. 238]. We will denote to the sum of all semi-artinian submodules of a right R-module M by $\mathrm{Sa}(M)$. If N is a submodule of a right R-module M, the notation $N \subseteq^{sa} M$ means that N is a semi-artinian submodule of M.
We refer the reader to [2], [9], [16], [18] and [21], for general background materials.

Injective modules have been studied extensively, and several generalizations for these modules are given, for example, soc-injective modules [1], $\mathcal{L}$-injective Modules [13], and n-FP-injective modules [5]. If $\mathrm{Ext}^1(R/K, M) = 0$, for any semisimple right ideal K of R, then a right R-module M is called soc-injective [1], where $\mathrm{Ext}^1(A, B)$ is defined as the first right derived functor of $\mathrm{Hom}_R(A, B)$, for any two right R-modules A, B (see [4, Ch. VI] for more details).

In section 2 of this paper, we introduce the class of SA-injective modules. This class of modules lies between injective modules and soc-injective modules. We first give examples to show that the notion of SA-injectivity is distinct from that of injectivity and soc-injectivity. We characterize rings over which every module is SA-injective. We prove the equivalence of the following statements: (1) Every right R-module is SA-injective; (2) Every semi-artinian module is SA-injective; (3) Every semi-artinian right ideal of R is SA-injective; (4) Every semi-artinian right ideal of R is a direct summand of R. Conditions under which the class of SA-injective right R-modules (SAI_R) is closed under quotient are given. For instance, we prove that the equivalence of the following: (1) The class SAI_R is closed under quotient; (2) Sums of any two SA-injective submodules of any right R-module is SA-injective; (3) All semi-artinian right ideals of R are projective. Finally, we give conditions such that the class SAI_R is closed under direct sums. For instance, we prove that the following are equivalent. (1) $Sa(R_R)$ is noetherian; (2) Any direct sum of SA-injective right R-modules is SA-injective; (3) The class SAI_R is closed under pure submodules; (4) All FP-injective modules are SA-injective.

In section 3, we study the definability of the class SAI_R. It is shown that the following assertions are equivalent: (1) SAI_R is definable; (2) The class SAI_R is closed under pure submodules and pure homomorphic images; (3) Every semi-artinian right ideal in R is finitely presented; (4) A module $M \in SAI_R$ iff $M^* \in (SAI_R)^{\ominus}$; (5) A module $M \in SAI_R$ iff $M^{**} \in SAI_R$. Finally, we prove that if the class SAI_R is a definable, then the class of flat left R-modules and the class $(SAI_R)^{\ominus}$ are coincide iff all modules in SAI_R are FP-injective iff all pure-injective modules in SAI_R are injective.

In section 4, we give relations between SA-injectivity and certain generalizations of injectivity (in particular, quasi-injectivity and F-injectivity). Firstly, we prove that a ring R is a right semi-artinian ring iff every SA-injective right R-module is quasi-injective iff every cyclic SA-injective right R-module is quasi-injective. Then, we prove that a commutative ring R is semisimple if and only if R is a semi-artinian ring and every quasi-injective R-module is SA-injective. Also, we prove that $\mathrm{Sa}(R_R)$ is a noetherian right R-module if and only if every F-injective right R-module is SA-injective. Finally, we prove that a ring R is a (von Neumann) regular and every P-injective right R-module is SA-injective if and only if every SA-injective right R-module is P-injective and every semi-artinian right ideal of R is a direct summand of R_R.

2. SA-Injective Modules

Definition 2.1. Let N be a module. A module M is called SA-N-injective, if for any semi-artinian submodule K of N, any homomorphism $f : K \to M$ extends to N. M is called SA-injective if M is SA-R-injective. A ring R is called SA-injective if the module R_R is SA-injective.

We will use SAI_R to denote the class of SA-injective right R-modules.

Examples 2.2. **(1)** All injective modules are SA-injective. Since 0 is the only semi-artinian right ideal in $\mathbb{Z}$, we have the right $\mathbb{Z}$-module $\mathbb{Z}$ is a SA-injective but it is not injective. Hence SA-injectivity is a proper generalization of injectivity.

(2) Since every semisimple module is semi-artinian, we have every SA-injective module is soc-injective. The converse is not true in general, for example: let $R = \mathbb{Z}_2[x_1, x_2, ...]$ where $x_i^3 = 0$ for all i, $x_i^2 = x_j^2 \neq 0$ for all i and j and $x_i x_j = 0$ for all $i \neq j$. By [1, Example 5.7], R is a semiprimary commutative and soc-injective ring but it is not self injective. By [18, Example 1, p. 184], R is a right semi-artinian ring, so that Proposition 2.5 in [18, p. 183] implies that $I \subseteq^{sa} R_R$ for any right ideal I in R and hence R is not SA-injective ring.

(3) Clearly, if $\mathrm{Soc}(N_R) = 0$, then 0 is the only semi-artinian submodule of N and hence every module is SA-N-injective. Particularly, all $\mathbb{Z}$-modules are SA-injective.

(4) All modules with zero socles are SA-injective, this follows from the fact that $\mathrm{Soc}(M) = 0$ if and only if $\mathrm{Sa}(M) = 0$, for any module M.

Proposition 2.3. *Let N be a module. Then following statements hold:*
(1) *The class of SA-N-injective modules is closed under isomorphic copies, direct products, direct summands and finite direct sums.*
(2) *For any submodule K of N, if M is SA-N-injective module, then M is SA-K-injective.*
(3) *If M is SA-N-injective module, then M is SA-K-injective, for any module K isomorphic to N.*

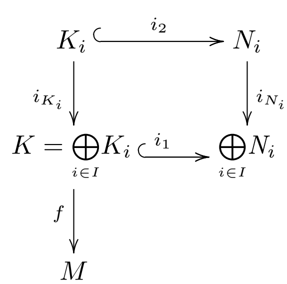
Proof. Clear. □

Corollary 2.4. *The class of SA-injective right R-modules (SAI_R) is closed under isomorphic copies, direct products, direct summands and finite direct sums.*

Proposition 2.5. *Let M be a module and $\{N_i : i \in I\}$ be a family of modules. If $\bigoplus_{i\in I} N_i$ is a multiplication module, then M is SA-$\bigoplus_{i\in I} N_i$-injective iff M is SA-N_i-injective, for all $i \in I$.*

Proof. ($\Rightarrow$) By Proposition 2.3((2),(3)).

($\Leftarrow$) Let $K \subseteq^{s.a} \bigoplus_{i\in I} N_i$. Since $\bigoplus_{i\in I} N_i$ is a multiplication module (by hypothesis), we have from [20, Theorem 2.2, p. 3844] that $K = \bigoplus_{i\in I} K_i$ with K_i is a submodule of N_i, for all $i \in I$. By [9, p. 238], $K_i \subseteq^{s.a} N_i$. For $i \in I$, consider the following diagram:



where i_{K_i}, i_{N_i} are injection maps and i_1, i_2 are inclusion maps. The hypothesis implies that there exists homomorphism $h_i : N_i \longrightarrow M$ such that $h_i \circ i_2 = f \circ i_{K_i}$. By [9, Theorem 4.1.6(2)], there exists exactly one homomorphism $h : \bigoplus_{i\in I} N_i \longrightarrow M$ satisfying $h_i = h \circ i_{N_i}$. Thus $f \circ i_{K_i} = h_i \circ i_2 = h \circ i_{N_i} \circ i_2 = h \circ i_1 \circ i_{K_i}$ for all $i \in I$. Let $(a_i)_{i\in I} \in \bigoplus_{i\in I} K_i$, thus $a_i \in K_i$, for all $i \in I$ and $f((a_i)_{i\in I}) = f(\sum_{i\in I} i_{K_i}((a_i)_{i\in I})) = (h \circ i_1)((a_i)_{i\in I})$. Thus $f = h \circ i_1$ and the proof is complete. □

Recall that a ring R is called a right invariant if each of its right ideals is an ideal of R [20, p. 3839].

Corollary 2.6. (1) *Let M be a module over a right invariant ring R and $1 = \lambda_1 + \lambda_2 + ... + \lambda_m$ in R such that λ_j are orthogonal idempotent. Then M is SA-injective iff M is SA-$\lambda_j R$-injective for every $j = 1, 2, ..., m$.*
(2) *If M is SA-aR-injective module and $aR \cong bR$, where a and b are idempotents of R, then M is SA-bR-injective.*

Proof. (1) By [2, Corollary 7.3], $R = \bigoplus_{j=1}^{m} \lambda_j R$. Since R is a right invariant ring, we get from [20, Proposition 3.1, p. 3855] that R is a multiplication module and hence Proposition 2.5 implies that M is SA-injective iff M is SA-$\lambda_j R$-injective for all $1 \leq j \leq m$.

(2) By Proposition 2.3(3). □

Proposition 2.7. *The following statements are equivalent for a module M.*
(1) *All modules are SA-M-injective.*
(2) *All semi-artinian modules are SA-M-injective.*
(3) *All semi-artinian submodules of M are SA-M-injective.*
(4) *All semi-artinian submodules of M are direct summands of M.*

Proof. Straightforward. □

Proposition 2.7 implies the next result.

Corollary 2.8. *For a ring R, the following conditions are equivalent.*
(1) *Mod-$R = SAI_R$.*
(2) *All semi-artinian modules are SA-injective.*
(3) *All semi-artinian right ideals of R are SA-injective.*
(4) *If $I \subseteq^{sa} R_R$, then I is a direct summand of R_R.*

Corollary 2.9. *A module M is semisimple if and only if M is semi-artinian and all modules are SA-M-injective.*

Proof. ($\Rightarrow$) It is obvious.

($\Leftarrow$) If K is a submodule of M, then K is semi-artinian by [9, p. 238] and hence Proposition 2.7 implies that K is a direct summand of M. Thus M is a semisimple module. □

As a special case of Corollary 2.9, we have the following corollary.

Corollary 2.10. *A ring R is a right semisimple ring if and only if it is a right semi-artinian ring and Mod-$R = SAI_R$.*

In general, not every semi-artinian submodule of a projective module is projective, for example, if $M = \mathbb{Z}_4$ as $\mathbb{Z}_4$-module and $K = \overline{2}\mathbb{Z}_4$, then $K \subseteq^{sa} M$ but K is not a projective $\mathbb{Z}_4$-module.

Theorem 2.11. *The following conditions are equivalent for a projective module M.*
(1) *The class of SA-M-injective modules is closed under quotient.*
(2) *Every quotient of an injective module is SA-M-injective.*
(3) *If K_1 and K_2 are two SA-M-injective submodules of a module N, then $K_1 + K_2$ is SA-M-injective.*
(4) *If K_1 and K_2 are two injective submodules of a module N, then $K_1 + K_2$ is SA-M-injective.*
(5) *If $K \subseteq^{sa} M$, then K is projective.*

Proof. (1) ⇒ (2) and (3) ⇒ (4) are obvious.

(2) ⇒ (5) Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \stackrel{i}{\hookrightarrow} & M \\ & & \downarrow f & & \\ E & \stackrel{h}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

where N and E are modules, K is a semi-artinian submodule of M, h is an epimorphism and f is a homomorphism. We can assume that E is injective (see, e.g. [3, Proposition 5.2.10]). By SA-M-injectivity of N, f can be extended to a homomorphism $g : M \longrightarrow N$. By projectivity of M, there is a homomorphism $\tilde{g} : M \longrightarrow E$ such that $h \circ \tilde{g} = g$. Let $\tilde{f} : K \longrightarrow E$ be the restriction of $\tilde{g}$ over K. It is clear that $h \circ \tilde{f} = f$. Then K is projective.

(5) ⇒ (1) Let L and N be modules such that N is SA-M-injective and $h : N \longrightarrow L$ is an epimorphism. If $K \subseteq^{sa} M$ and $f : K \longrightarrow L$ is any homomorphism, then the hypothesis implies that K is projective and hence there is a homomorphism $g : K \longrightarrow N$ with $h \circ g = f$. By SA-M-injectivity of N, there is a homomorphism $\tilde{g} : M \longrightarrow N$ with $\tilde{g} \circ i = g$. Let $\beta = h \circ \tilde{g} : M \longrightarrow L$. Then $\beta \circ i = h \circ \tilde{g} \circ i = h \circ g = f$. and hence L is an SA-M-injective module.

(1) ⇒ (3) Let K_1 and K_2 be two SA-M-injective submodules of a module K. Thus $K_1 + K_2$ is a homomorphic image of the direct sum $K_1 \oplus K_2$. SA-M-injectivity of $K_1 \oplus K_2$ and the hypothesis imply that $K_1 + K_2$ is SA-M-injective.

(4) ⇒ (2) Let F be an injective module with submodule D. Let $B = F \oplus F$, $L = \{(x, x) \mid x \in D\}$, $\bar{B} = B/L$, $K_1 = \{b + L \in \bar{B} \mid b \in F \oplus 0\}$, $K_2 = \{b + L \in \bar{B} \mid b \in 0 \oplus F\}$. Then $\bar{B} = K_1 + K_2$. Since $(F \oplus 0) \cap L = 0$ and $(0 \oplus F) \cap L = 0$, $F \cong K_i$, $i = 1, 2$. Since $K_1 \cap K_2 = \{b + L \in \bar{B} \mid b \in D \oplus 0\}= \{b + L \in \bar{B} \mid b \in 0 \oplus D\}$, $K_1 \cap K_2 \cong D$ under $b \mapsto b + L$ for all $b \in D \oplus 0$. By hypothesis, $\bar{B}$ is SA-M-injective. Injectivity of K_1 implies that $\bar{B} = K_1 \oplus A$ for some submodule A of $\bar{B}$, so $A \cong (K_1 + K_2)/K_1 \cong K_2/K_1 \cap K_2 \cong F/D$. By Proposition 2.3(5), F/D is SA-M-injective. □

Theorem 2.11 implies the following result.

Corollary 2.12. *The following statements are equivalent.*
(1) *The class SAI_R is closed under quotient.*
(2) *Every quotient of an injective module is SA-injective.*
(3) *For any module N, if N_1 and N_2 are submodules of N with N_1, $N_2 \in SAI_R$, then $N_1 + N_2 \in SAI_R$.*
(4) *For any module N, if N_1 and N_2 are injective submodules of N, then $N_1 + N_2 \in SAI_R$.*
(5) *If $I \subseteq^{sa} R_R$, then I is projective.*

Theorem 2.13. *If M is a finitely generated module, then the following statements are equivalent.*
(1) $\mathrm{Sa}(M)$ *is noetherian.*
(2) *The class of SA-M-injective modules is closed under direct sums.*
(3) *Direct sums of injective modules are SA-M-injective.*
(4) *If K is injective module, then $K^{(S)}$ is SA-M-injective for any index set S,*
.
(5) *If K is injective module, then $K^{(\mathbb{N})}$ is SA-M-injective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) Clear.

(1) $\Rightarrow$ (2) Let $E = \bigoplus_{i \in I} M_i$, where M_i are SA-M-injective modules and $f : K \to E$ be a homomorphism with $K \subseteq^{sa} M$. Since $\mathrm{Sa}(M)$ is a noetherian module, we have K is finitely generated and hence $f(K) \subseteq \bigoplus_{j \in I_1} M_j$, for some finite subset I_1 of I and hence $\bigoplus_{j \in I_1} M_j$ is SA-injective. Then f can be extended to a homomorphism $g : M \to E$ and so E is SA-injective.

(5) $\Rightarrow$ (1) Let $K_1 \subseteq K_2 \subseteq ...$ be a chain of submodules of $\mathrm{Sa}(M)$. For each $i \geq 1$, let $F_i = E(M/K_i)$, $F = \bigoplus_{i=1}^{\infty} F_i$ and $M_i = \prod_{j=1}^{\infty} F_j = F_i \oplus (\prod_{\substack{j=1 \\ j \neq i}}^{\infty} F_j)$, then M_i is injective. By hypothesis, $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} F_i) \oplus (\bigoplus_{i=1}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{\infty} F_j)$ is SA-M-injective and hence Proposition 2.3(1) implies that F it self is SA-M-injective.
Define $f : H = \bigcup_{i=1}^{\infty} K_i \longrightarrow F$ by $f(a) = (a + K_i)_i$. Clearly, f is a well defined homomorphism. Since $\mathrm{Sa}(M) \subseteq^{sa} M$ (by [9, p. 238]), we have $\bigcup_{i=1}^{\infty} K_i \subseteq^{sa} M$ and hence f can be extended to a homomorphism $g : M \longrightarrow F$. Since M is finitely generated, we have $g(M) \subseteq \bigoplus_{i=1}^{n} E(M/K_i)$ for some n and hence $f(\bigcup_{i=1}^{\infty} K_i) \subseteq \bigoplus_{i=1}^{n} E(M/K_i)$. Since $\pi_i f(x) = \pi_i (x + K_j)_{j \geq 1} = x + K_i$, for all $x \in H$ and $i \geq 1$, where $\pi_i : \bigoplus_{j \geq 1} E(M/K_j) \longrightarrow E(M/K_i)$ is the projection map, $\pi_i f(H) = H/K_i$ for all $i \geq 1$. Since $f(H) \subseteq \bigoplus_{i=1}^{n} E(M/K_i)$, $H/K_i = \pi_i f(H) = 0$,

for all $i \geq n+1$, so $H = K_i$ for all $i \geq n+1$ and hence the chain $K_1 \subseteq K_2 \subseteq \ldots$ terminates at K_{n+1}. Thus $\mathrm{Sa}(M)$ is a noetherian module. □

Proposition 2.14. *The following statements are equivalent.*
(1) $\mathrm{Sa}(R_R)$ *is noetherian.*
(2) *The class* SAI_R *is closed under direct sums.*
(3) *Any direct sum of injective modules is* SA*-injective.*
(4) *If* K *is injective module, then* $K^{(S)}$ *is* SA*-injective for any index set* S.
(5) *If* K *is injective module, then* $K^{(\mathbb{N})}$ *is* SA*-injective.*
(6) *The class* SAI_R *is closed under pure submodules.*
(7) *All* FP*-injective modules are* SA*-injective.*

Proof. By applying Theorem 2.13, we have the equivalent of $(1), (2), (3), (4)$ and (5).

(1) ⇒ (6). Let $N \in SAI_R$ and K a pure submodule of N. Let $C \subseteq^{sa} R_R$, thus the hypothesis implies that C is finitely generated and so R/C is a finitely presented. Hence the sequence $\mathrm{Hom}_R(R/C, N) \to \mathrm{Hom}_R(R/C, N/K) \to 0$ is exact. By [8, Theorem XII.4.4 (4), p. 491], the sequence $\mathrm{Hom}_R(R/C, N) \to \mathrm{Hom}_R(R/C, N/K) \to \mathrm{Ext}^1(R/C, K) \to \mathrm{Ext}^1(R/C, N)$ is exact. Thus $\mathrm{Ext}^1(R/C, K) = 0$ and hence $K \in SAI_R$. Therefore, the class SAI_R is closed under pure submodules.

(6) ⇒ (7). If M is any FP-injective module, then M is a pure submodule of a SA-injective module. By hypothesis, $M \in SAI_R$.

(7) ⇒ (1). Let I be a submodule of $\mathrm{Sa}(R_R)$, thus $I \subseteq^{sa} R_R$. Let $\alpha : I \to M$ be a homomorphism, where M is a FP-injective module. By hypothesis, M is SA-injective and hence α extends to R_R. By [6], I is finitely generated and hence $\mathrm{Sa}(R_R)$ is a noetherian module. □

3. Definability of the class SAI_R

If $\mathcal{X} \subseteq$ Mod-R, then we write $\mathcal{X}^{\ominus} = \{M \in R\text{-Mod} \mid M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathcal{X}\}$ and $\mathcal{X}^{+} = \{M \in \text{Mod-}R \mid M \text{ is a pure submodule of a module in } \mathcal{X}\}$.

Lemma 3.1. *The pair* $((SAI_R)^{\ominus}, SAI_R)$ *is an almost dual pair over a ring* R.

Proof. By Corollary 2.4 and [12, Proposition 4.2.11, p. 72]. □

Corollary 3.2. *Consider the following conditions for the class* SAI_R *over a ring* R.
(1) *The class* SAI_R *is definable.*
(2) $(SAI_R, (SAI_R)^{\ominus})$ *is an almost dual pair over a ring* R.
(3) $(SAI_R)^* \subseteq (SAI_R)^{\ominus}$.
(4) $(SAI_R)^{**} \subseteq SAI_R$.
(5) *The class* SAI_R *is closed under pure homomorphic images.*

Then $(1) \Leftrightarrow (2), (1) \Rightarrow (3), (1) \Rightarrow (5)$ *and* $(3) \Leftrightarrow (4)$. *Moreover, if* $Sa(R_R)$ *is noetherian, then all five conditions are equivalent.*

Proof. (1) $\Leftrightarrow$ (2). By Lemma 3.1 and [12, Proposition 4.3.8, p. 89].

(1) $\Rightarrow$ (3). Since SAI_R is a definable class, it is closed under pure submodules and hence $(SAI_R)^+ = SAI_R$. Since $((SAI_R)^\ominus, SAI_R)$ is an almost dual (by Lemma 3.1), it follows from [12, Theorem 4.3.2, p. 85], that $(SAI_R)^* \subseteq (SAI_R)^\ominus$.

(1) $\Rightarrow$ (5). By [16, 3.4.8, p. 109].

(3) $\Rightarrow$ (4). By Lemma 3.1 and [12, Theorem 4.3.2, p. 85].

(4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1). Suppose that $\mathrm{Sa}(R_R)$ is a noetherian module. By Proposition 2.14, the class SAI_R is closed under pure submodules and hence $(SAI_R)^+ = SAI_R$. Thus the results follow from [12, Theorem 4.3.2, p. 85]. □

Corollary 3.3. *If every SA-injective modules is pure-injective, then the following statements are equivalent for a class SAI_R over a ring R.*
(1) *SAI_R is definable.*
(2) *The class SAI_R is closed under direct sums.*
(3) $(SAI_R)^+ = SAI_R$
(4) $\mathrm{Sa}(R_R)$ *is a noetherian module.*

Proof. By Proposition 2.14, Lemma 3.1 and [12, Theorem 4.5.1, p. 103]. □

If A is a a right R-module and B is a left R-module, then $\mathrm{Tor}_1(A, B)$ is defined as the first left derived functor of the tensor product $A \otimes_R B$ (see [4, Ch. VI] for more details).

Lemma 3.4. *A left R-module $M \in (SAI_R)^\ominus$ iff $Tor_1(R/I, M) = 0$, for any semi-artinian right ideal I of a ring R.*

Proof. Let M be a left R-module and $I \subseteq^{sa} R_R$. By [7, Theorem 3.2.1, p. 75], $\mathrm{Ext}^1(R/I, M^*) \cong (\mathrm{Tor}_1(R/I, M))^*$, so that $\mathrm{Tor}_1(R/I, M) = 0$ if and only if $M^* \in SAI_R$. Hence $({}_RSAF, SAI_R)$ is an almost dual, where ${}_RSAF = \{M \in R\text{-}\mathrm{Mod} \mid \mathrm{Tor}_1(R/I, M) = 0$, for any semi-artinian right ideal I of a ring $R\}$. By [12, Proposition 4.2.11, p. 72], $(SAI_R)^\ominus =_R SAF$. □

A module M is called n-presented if there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$, with each F_i is a finitely generated free modules [5].

Theorem 3.5. *The following statements are equivalent for a class SAI_R over a ring R.*
(1) *SAI_R is definable.*
(2) *The class SAI_R is closed under pure submodules and pure homomorphic images.*
(3) *Every semi-artinian right ideal in R is finitely presented.*
(4) *A module $M \in SAI_R$ iff $M^* \in (SAI_R)^\ominus$.*
(5) *A module $M \in SAI_R$ iff $M^{**} \in SAI_R$.*

Proof. (1) $\Rightarrow$ (2). By [16, 3.4.8, p. 109].

(2) $\Rightarrow$ (3). Let N be any FP-injective module, thus there is an injective module H with pure exact sequence $0 \to N \stackrel{i}{\to} H \stackrel{\pi}{\to} H/N \to 0$. By hypothesis, $H/N \in SAI_R$. Let $K \subseteq^{sa} R_R$, thus $\mathrm{Ext}^1(R/K, H/N)$
$= 0$. By [8, Theorem 4.4 (4), p. 491], the sequence $0 = \mathrm{Ext}^1(R/K, H/N) \to \mathrm{Ext}^2(R/K, N) \to$
$\mathrm{Ext}^2(R/K, H) = 0$ is exact and hence $\mathrm{Ext}^2(R/K, N) = 0$. By [8, Theorem 4.4 (3), p. 491], the sequence $0 = \mathrm{Ext}^1(R, N) \to \mathrm{Ext}^1(K, N) \to \mathrm{Ext}^2(R/K, N) = 0$ is exact, so that $\mathrm{Ext}^1(K, N) = 0$. By hypothesis, SAI_R is closed under pure submodules, so that K is finitely generated by Proposition 2.14 and hence [6, Proposition, p. 361] implies that K is finitely presented.

(3) $\Rightarrow$ (1). Let $M \in SAI_R$. Let $K \subseteq^{sa} R_R$, thus K is finitely presented (by hypothesis) and hence there is an exact sequence $F_2 \stackrel{\alpha_2}{\to} F_1 \stackrel{\alpha_1}{\to} K \to 0$, where F_1, F_2 are finitely generated free modules. Let $\beta = i\alpha_1$, where $i : K \to R$ is the inclusion mapping, thus the sequence $F_2 \stackrel{\alpha_2}{\to} F_1 \stackrel{\beta}{\to} R \stackrel{\pi}{\to} R/K \to 0$ is exact, where $\pi : R \to R/K$ is the natural epimorphism. Hence R/K is a 2-presented module, so that from [5, Lemma 2.7 (2)] we have $\mathrm{Tor}_1(R/K, M^*) \cong (\mathrm{Ext}^1(R/K, M))^* = 0$. By Lemma 3.4, $M^* \in (SAI_R)^\ominus$ and hence $(SAI_R)^* \subseteq (SAI_R)^\ominus$. By hypothesis, every semi-artinian right ideal in R is finitely generated, so that $\mathrm{Sa}(R_R)$ is noetherian. By Corollary 3.2, SAI_R is a definable class.

(1) $\Rightarrow$ (4). By Corollary 3.2, $(SAI_R, (SAI_R)^\ominus)$ is an almost dual pair and hence a module $M \in SAI_R$ iff $M^* \in (SAI_R)^\ominus$.

(4) $\Rightarrow$ (5). By hypothesis, $(SAI_R)^* \subseteq (SAI_R)^\ominus$. By Corollary 3.2, $(SAI_R)^{**} \subseteq SAI_R$. Hence for any module M, if $M \in SAI_R$, then $M^{**} \in SAI_R$.
Conversely, if $M^{**} \in SAI_R$, then $M^* \in (SAI_R)^\ominus$. By hypothesis, $M \in SAI_R$.

(5) $\Rightarrow$ (1). Let N be a FP-injective module, thus there is a pure exact sequence $0 \to N \to E \to E/N \to 0$, where E is an injective module. By [21, 34.5, p. 286], the sequence $0 \to N^{**} \to E^{**} \to (E/N)^{**} \to 0$ is split. By hypothesis, $E^{**} \in SAI_R$ and hence $N^{**} \in SAI_R$. By hypothesis, $N \in SAI_R$ so that $\mathrm{Sa}(R_R)$ is noetherian by Proposition 2.14. Thus SAI_R is definable class by Corollary 3.2. □

Note that if the class SAI_R is closed under pure submodules, then $(SAI_R)^+ = SAI_R$. Thus we have the following corollary.

Corollary 3.6. *The class SAI_R is a definable if and only if it is closed under pure submodules and the class $(SAI_R)^+$ is a definable.*

Corollary 3.7. *If the class SAI_R is a definable, then the following are equivalent.*
(1) *The class of flat left R-modules and the class $(SAI_R)^\ominus$ are coincide.*
(2) *Every module in SAI_R is FP-injective.*
(3) *Every pure-injective module in SAI_R is injective.*

Proof. (1) ⇒ (2). Let $M \in SAI_R$, thus $M^* \in (SAI_R)^{\ominus}$ by Corollary 3.2. By hypothesis, M^* is a flat left R-module and hence [10, Theorem, p. 239] implies that M^{**} is injective. Since M is a pure submodule in M^{**}, we have M is FP-injective by [21, 35.8, p. 301].

(2) ⇒ (3). Let M be any pure-injective module in SAI_R. Let $\mathcal{E} : 0 \to M \to N \to K \to 0$ be an exact sequence. By hypothesis, M is FP-injective. By [17, Proposition 2.6], the sequence $\mathcal{E}$ is pure and hence pure-injectivity of M implies that the sequence $\mathcal{E}$ is split by [21, 33.7, p. 279]. Therefore, M is injective.

(3) ⇒ (1). Let M be a flat left R-module, thus $\mathrm{Tor}_1(N, M) = 0$, for any right R-module N. By Lemma 3.4, $M \in (SAI_R)^{\ominus}$. Conversely, if $M \in (SAI_R)^{\ominus}$, then $M^* \in SAI_R$. By [16, Proposition 4.3.29, p. 149], M^* is a pure injective module. By hypothesis, M^* is injective and hence M is flat by [10, Theorem, p. 239]. □

4. Relations between SA-injectivity and certain generalizations of injectivity

A right R-module M is called quasi-injective if, for every submodule N of M, every right R-homomorphism from N to M can be extended to a right R-endomorphism of M [3, p. 169].

In general, if M is SA-injective right R-module, then M need not be quasi-injective, for example $\mathbb{Z}$ as $\mathbb{Z}$-module is SA-injective (by Example 2.2(1)) but it is not quasi-injective. Also, the converse is not true in general, for example in the ring $\mathbb{Z}_4$, the ideal $I =< \bar{2} >$ is a quasi-injective $\mathbb{Z}_4$-module but it is not SA-injective $\mathbb{Z}_4$-module.

The following theorem gives a relation between SA-injective modules and quasi-injective modules.

Theorem 4.1. *The following statements are equivalent for a ring R.*
(1) *R is a right semi-artinian ring.*
(2) *Every SA-injective right R-module is injective.*
(3) *Every SA-injective right R-module is quasi-injective.*
(4) *Every cyclic SA-injective right R-module is quasi-injective.*

Proof. (1) ⇒ (2) Let M be any SA-injective right R-module. Let I be any right ideal of a ring R and $f : I \to M$ be any right R-homomorphism. Since R is a right semi-artinian ring (by hypothesis), it follows from [9, Exercise 7(8), p. 238] that I is a semi-artinian right ideal of R. Since M is an SA-injective right R-module (by hypothesis), f extends to R and hence M is an injective right R-module.

(2) ⇒ (3) and (3) ⇒ (4) are clear.

(4) $\Rightarrow$ (1) Let M be any nonzero cyclic right R-module. We will prove that $\mathrm{Soc}(M) \neq 0$. Assume that $\mathrm{Soc}(M) = 0$. Let N be a nonzero submodule of M. Thus $\mathrm{Soc}(N) = 0$ and hence from Example 2.2(4) that M and N are SA-injective right R-modules. By Corollary 2.4, $N \oplus M$ is an SA-injective right R-module. By hypothesis, $N \oplus M$ is a quasi-injective right R-module. By [15, Proposition 1.17, p. 8], N is an M-injective right R-module and hence N is a direct summand of M. Thus M is semisimple and hence $M = \mathrm{Soc}(M) = 0$ and this is a contradiction. Thus $\mathrm{Soc}(M) \neq 0$ for any nonzero cyclic right R-module M and hence from [18, p. 183] we have that R is a right semi-artinian ring. □

Since every left perfect ring is right semi-artinian [9, Theorem 11.6.3, p. 294], we have the following corollary immediately from Theorem 4.1.

Corollary 4.2. *If R is a left perfect ring, then every SA-injective right R-module is injective (quasi-injective).*

In the following proposition, we give another connection between SA-injective modules and quasi-injective modules.

Proposition 4.3. *A commutative ring R is semisimple if and only if R is a semi-artinian ring and every quasi-injective R-module is SA-injective.*

Proof. ($\Rightarrow$) By Corollary 2.10.
($\Leftarrow$) Let M be any quasi-injective R-module. By hypothesis, M is SA-injective. Since R is a semi-artinian ring (by hypothesis), it follows from Theorem 4.1 that M is injective and hence from [19, Corollary 2.2] we get that R is a semisimple ring. □

The following corollary is immediately from Theorem 4.1 and Proposition 4.3.

Corollary 4.4. *The following statements are equivalent for a commutative ring R.*
(1) *R is semisimple.*
(2) *For each R-module M, M is SA-injective if and only if it is quasi-injective .*

A right R-module M is called P-injective (resp. F-injective) if, for every principally (resp. finitely generated) right ideal I of R, every right R-homomorphism from I to M can be extended to a right R-homomorphism from R into M (see, for example [11] and [22]).

If M is SA-injective right R-module, then M need not be P-injective (resp. F-injective) in general, for example $\mathbb{Z}$ as $\mathbb{Z}$-module is SA-injective (by Example 2.2(1)) but it is not P-injective (resp. F-injective). Also, the converse is not true in general, for example: let $F = \mathbb{Z}_2$ be the field of two elements, $F_n = F$ for $n = 1, 2, ...$, $Q = \prod_{i=1}^{\infty} F_i$, $S = \oplus_{i=1}^{\infty} F_i$. If R is the subring of Q generated by 1 and S, then R is a F-injective right R-module (by [1, Example 4.5]) and hence R_R is a P-injective module. Thus Example 4.5 in [1] implies that R is not a soc-injective right R-module and so R is not a SA-injective module. Thus R is F-injective (P-injective) right R-module but it is not SA-injective.

The following proposition gives a condition under which every F-injective right R-module is SA-injective.

Proposition 4.5. *Let R be a ring. Then $Sa(R_R)$ is a noetherian right R-module if and only if every F-injective right R-module is SA-injective.*

Proof. (⇒) Let M be any F-injective right R-module. Let I be a semi-artinian right ideal of R and let $f : I \to M$ be any right R-homomorphism. Since $\mathrm{Sa}(R_R)$ is noetherian and $I \subseteq \mathrm{Sa}(R_R)$, it follows that I is a finitely generated right ideal. By F-injectivity of M, f extends to a right R-homomorphism from R into M and hence M is SA-injective.
(⇐) Let $\{M_i\}_{i\in I}$ be a family of injective right R-modules. Thus M_i are F-injective modules. By [22, Proposition 2.1(c)], $\bigoplus_{i\in I} M_i$ is an F-injective module. By hypothesis, $\bigoplus_{i\in I} M_i$ is a SA-injective module and hence from Proposition 2.14 we get that $\mathrm{Sa}(R_R)$ is a noetherian right R-module. □

Directly from Proposition 4.5 and Proposition 2.14, we have the following corollary.

Corollary 4.6. *Let R be a ring. Then every F-injective right R-module is SA-injective if and only if every FP-injective right R-module is SA-injective.*

A ring R is called (von Neumann) regular if for any $a \in R$, there is $b \in R$ such that $a = aba$ [9, p. 38].

Proposition 4.7. *The following statements are equivalent.*
(1) *R is a (von Neumann) regular ring and every P-injective right R-module is SA-injective.*
(2) *R is a (von Neumann) regular ring and $Sa(R_R)$ is a noetherian right R-module.*
(3) *Every SA-injective right R-module is P-injective and every semi-artinian right ideal of R is a direct summand of R_R.*

Proof. (1) ⇒ (2) Since every F-injective right R-module is P-injective, we have from hypothesis that every F-injective right R-module is SA-injective. By Proposition 4.5, $\mathrm{Sa}(R_R)$ is a noetherian right R-module.
(2) ⇒ (3) Since R is a (von Neumann) regular ring, it follows from [14, Lemma 2] that every SA-injective right R-module is P-injective. Let I be any semi-artinian right ideal of R. Thus $I \subseteq \mathrm{Sa}(R_R)$. Since $\mathrm{Sa}(R_R)$ is a noetherian right R-module (by hypothesis), we have that I is a finitely generated right ideal. By [9, Exercise 13, p. 38], I is a direct summand of R_R.

(3) ⇒ (1) Since every semi-artinian right ideal of R is a direct summand of R_R (by hypothesis), it follows that from Corollary 2.8 that every right R-module is SA-injective and hence every P-injective right R-module is SA-injective. Since every SA-injective right R-module is P-injective (by hypothesis), we have that every right R-module is P-injective. By [14, Lemma 2], R is a (von Neumann) regular ring. □

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