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The General (α, β)-metrics with Quadratic Curvatures

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Abstract. In this paper, we consider the Riemannian curvature of one of the most important metric in Finsler geometry called general (α, β)-metrics, where α is a Riemannian metric and β is 1-form. We give a complete classification of the metrics to be R-quadratic and Ricci-quadratic.

Keywords: General (α, β)-metric, R-quadratic, Ricci-quadratic.

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1. Introduction

Finsler spaces are a natural generation of Riemannian spaces. Let $(M, F = \sqrt{g_{ij}(x,y)y^iy^j})$ be an n-dimensional Finsler space. A spray $G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}$ is a vector field induced by F as follows

$$G^i := \frac{g^{im}}{4}\Big([F^2]_{x^jy^m}y^j - [F^2]_{x^m}\Big),$$

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where $g^{im} = (g_{im})^{-1}$ and $G^i = G^i(x, y)$ are called the spray coefficients of F. For example, the spray coefficients of a Riemannian metric are given by

$$G^i = \frac{1}{2}\Gamma^i_{mn}(x)y^m y^n,$$

where the functions $\Gamma^i_{mn}(x)$ are called the Christoffel symbols. Therefore the spray coefficients of a Riemannian metric are quadratic in $y \in T_xM$ at every point $x \in M$. Finsler metrics with this property form a special category in Finsler geometry called Berwald metrics. One can easily see this quadratic property in the Riemann curvature of a Riemannian metric. The Riemann curvature $R_y = R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$ of a Finsler metric is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \tag{1.1}$$

F is R-quadratic if its Riemann curvature [7, 16]

$$R^i{}_k = R_m{}^i{}_{kn}(x)y^m y^n. \tag{1.2}$$

Note that there are R-quadratic non Berwald Finsler metrics. For example the following Finsler metric

$$F = \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - < x, y >^2)} + < x, y >)^2}{(1 - |x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - < x, y >^2)}}$$

is R-flat i.e. $R^i{}_k = 0$. Then F is R-quadratic metric but it is not a Berwald metric.

Since all Finsler metrics generally are not R-quadratic, therefore the study of the property in Finsler spaces has attracted the attention of many Finslerian authors. In [3], Li-Shen obtained equations that characterize Randers spaces with this property and concluded that the S-curvature must be constant under these conditions. S. Bacso and B. Rezaei proved that any Einstein Finsler metric with two special conditions, R-quadratic and non Ricci flat, is a Riemannian metric [1]. Rezaei-Gabrani and Sadeghzadeh studied spherically symmetric Finsler metrics of quadratic curvatures [6, 4]. Our goal in this paper is to investigate this important property for a wide range of Finsler metrics.

In Finsler geometry, several classes of metrics have been introduced, such as: (α, β)-metrics, m-th root Finsler metrics, spherically symmetric Finsler metrics, etc (see [4, 5, 8]). Most of them contain or have a special relationship with the previous one. In [11], Yu-Zhu introduced general (α, β)-metrics. By definition, they are in the following form

$$F = \alpha\, \phi(b^2, \frac{\beta}{\alpha}),$$

where α is a Riemannian metric, β is a 1-form, $b := ||\beta||_\alpha$ and ϕ is a positive smooth function [11]. In [12], Yu and Zhu completely determined classification of general (α, β)-metrics with constant flag curvature under some conditions. Yu gave a classification of locally dually flat general (α, β)-metrics in [13]. Zhu characterized the metrics with isotropic Berwald curvature [14]. M. Zohrehvand and H. Maleki studied the unicorn problem for a class of the metrics [15]. In [10], Q. Xia studied the scalar flag curvature of a general (α, β)-metric, under certain conditions. M. Gabrani and B. Rezaei gave a characterization of a class of general (α, β)-metrics with isotropic E-curvature [2].

We are going to study general (α, β)-metrics that have two conditions below

$$ {}^{\alpha}R^i{}_j = \mu(\alpha^2 \delta^i{}_j - y^i y_j), \quad b_{i|j} = c(x) a_{ij}, \tag{1.3} $$

where $\mu = constant$, $c := c(x)$ with $c^2 = \kappa - \mu b^2$ for some constant κ and $b_{i|j}$ is the covariant derivation of β with respect to α. Throughout this paper, we denote the derivation with respect to b^2 and s by 1 and 2, respectively.

Our main results are given below:

Theorem 1.1. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β)-metric on an n-dimensional manifold M, where α and β satisfy (1.3). Then F is R-quadratic if and only if*

$$ [R_1]_s - s[R_1]_{ss} = 0, \ [R_2]_s = 0, \ R_3 - s[R_3]_s = 0, \ [R_4]_s = 0, $$

where R_1, R_3 and R_4 are defined in (2.6), (2.8) and (2.9), respectively.

Since general (α, β)-metrics include spherically symmetric metrics, thus the following corollary can be easily deduced by Theorem 1.1

Corollary 1.2. [4] *Let $F = |y|\phi(|x|^2, \dfrac{< x, y >}{|y|})$ be a spherically symmetric metric on $B^n(\nu)$. Then F is R-quadratic if and only if*

$$ [R_1]_s - s[R_1]_{ss} = 0, \ [R_2]_s = 0, \ R_4 - s[R_4]_s = 0, \ [R_5]_s = 0, $$

where R_1, R_2, R_4 and R_5 are not the same terms as in Theorem 1.1. They are given in [4].

From the Riemann curvature, we have the Ricci curvature **Ric** as follow

$$ \mathbf{Ric} := R^m{}_m. $$

By definition, it is a positively homogeneous function of degree two in $y \in TM$. But it is not quadratic in y, in general. Finsler metrics with this property are called *Ricci-quadratic metrics*. We shall discuss this property for general (α, β)-metrics.

Theorem 1.3. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β)-metric on an n-dimensional manifold M, where α and β satisfy (1.3) . Then F is Ricci-quadratic if and only if the following holds*

$$ R_s - sR_{ss} = 0, \tag{1.4} $$

where R is defined in (3.15).

2. Preliminaries

A non-negative function F on TM is a Finsler metric on a manifold M if it has the following conditions

(a) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;
(c) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$g_y(u,v) := \frac{1}{2}\Big[F^2(y+su+tv)\Big]\Big|_{s,t=0}, \qquad u,v \in T_xM.$$

Every Finsler metric F induces a spray $G = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ which is defined by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\}y^jy^k,$$

where (g^{ij}) is the inverse of (g_{ij}). Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α,β)-metric. Assume that β satisfies the second equation of (1.3). Then the spray coefficients of F are given by [11]

$$G^i = {}^{\alpha}G^i + Q^i, \tag{2.1}$$

where

$$Q^i := c\,\psi\,\alpha\,y^i + c\,\chi\,\alpha^2\,b^i, \tag{2.2}$$

$$\chi := \frac{1}{2}\frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{\phi - s\phi_2 + (b^2-s^2)\phi_{22}}, \tag{2.3}$$

$$\psi := \frac{1}{2}\frac{\phi_2 + 2s\phi_1}{\phi} - \frac{\chi}{\phi}[s\phi + (b^2-s^2)\phi_2]. \tag{2.4}$$

When α and β satisfy (1.3), the Riemann curvature of F is computed by Q. Xia [9]:

Lemma 2.1. *The Riemann curvature of a general (α,β)-metric, where α and β satisfy (1.3), is given by*

$$R^i{}_j = R_1\alpha^2\delta^i{}_j + R_2y_jy^i + R_3\alpha b_jy^i - sR_4\alpha y_jb^i + R_4\alpha^2b_jb^i, \tag{2.5}$$

where

$$\begin{aligned}
R_1 &:= \mu(1+s\psi)+c^2[\psi^2-2s\psi_1-\psi_2+2\chi(1+s\psi \\
&\quad +u\psi_2)], && (2.6)\\
R_2 &:= -\mu[1-s(\psi-s\psi_2)]+c^2[\psi_2+s\psi_{22}-\psi(\psi-s\psi_2)-2s(\psi_1-s\psi_{12}) \\
&\quad -2\chi(1+s\psi+u\psi_2)+s\chi_2(1+s\psi+u\psi_2)-2s\chi(\psi-s\psi_2+u\psi_{22})], && (2.7)\\
R_3 &:= -\mu(2\psi-s\psi_2)+c^2[2(2\psi_1-s\psi_{12})-\psi\psi_2-\psi_{22}+2\chi(\psi-s\psi_2+u\psi_{22}) \\
&\quad -\chi_2(1+s\psi+u\psi_2)], && (2.8)\\
R_4 &:= -\mu(2\chi-s\chi_2)+c^2[2(2\chi_1-s\chi_{12})-\chi_{22}+2\chi(2\chi-s\chi_2) \\
&\quad +u(2\chi\chi_{22}-\chi_2^2)], && (2.9)
\end{aligned}$$

here $c^2=\kappa-\mu b^2$ *and* $u=b^2-s^2$. *Observe that*

$$R_2=-R_1-sR_3, \tag{2.10}$$

and R_5 *is defined by*

$$R_5 := \frac{1}{3}([R_1]_s+2R_3). \tag{2.11}$$

We can also rewrite (2.5) as

$$R^i{}_j = R_1(\alpha^2\delta^i{}_j-y_jy^i)+R_3(\alpha b_j-sy_j)y^i+\alpha R_4(\alpha b_j-sy_j)b^i. \tag{2.12}$$

3. R-quadratic and Ricci-quadratic

Now we want to find necessary and sufficient conditions for a general (α,β)-metric to be R-quadratic/Ricci-quadratic.

Proof of Theorem 1.1. Let $F=\alpha\phi(b^2,\frac{\beta}{\alpha})$ be a general (α,β)-metric. One can easily get

$$\begin{aligned}
\alpha_{y^i} &= \frac{y_i}{\alpha},\quad s_{y^i}=\frac{\alpha b_i-sy_i}{\alpha^2}, && (3.1)\\
\alpha^2_{y^i} &= 2y_i,\quad \alpha^2_{y^iy^j}=2a_{ij},\quad \alpha^2_{y^iy^jy^k}=0, && (3.2)\\
\alpha_{y^iy^j} &= \frac{1}{\alpha}(a_{ij}-\frac{y_i}{\alpha}\frac{y_j}{\alpha}), && (3.3)\\
\alpha_{y^iy^jy^k} &= \frac{1}{\alpha^5}\big[3y_iy_jy_k-\alpha^2a_{ij}y_k(i\to j\to k\to i)\big], && (3.4)\\
s_{y^iy^j} &= -\frac{1}{\alpha^2}[sa_{ij}+\frac{1}{\alpha}(b_iy_j+b_jy_i)-\frac{3s}{\alpha^2}y_iy_j], && (3.5)\\
s_{y^iy^jy^k} &= \frac{1}{\alpha^5}\{[\alpha(3sy_i-\alpha b_i)a_{jk}+3b_iy_jy_k](i\to j\to k\to i) \\
&\quad -\frac{15s}{\alpha}y_iy_jy_k\}, && (3.6)
\end{aligned}$$

Then, by (2.12), we have

$$\frac{\partial}{\partial y^k}R^i_j = R_1\big([\alpha^2]_{y^k}\delta^i{}_j-a_{jk}y^i-y_j\delta^i{}_k\big)+[R_1]_s s_{y^k}\big(\alpha^2\delta^i{}_j-y_jy^i\big)$$

$$+\Big[R_3\delta^i{}_k + [R_3]_s s_{y^k} y^i + \big(\alpha_{y^k} R_4 + \alpha[R_4]_s s_{y^k}\big) b^i\Big]\big(\alpha b_j - s y_j\big)$$
$$+\big(\alpha R_4 b^i + R_3 y^i\big)\big(\alpha_{y^k} b_j - s_{y^k} y_j - s a_{jk}\big). \qquad (3.7)$$

By (3.7), we obtain

$$\begin{aligned}
\frac{\partial}{\partial y^l}\Big(\frac{\partial}{\partial y^k} R^i_j\Big) \quad = \quad & R_1\big([\alpha^2]_{y^k y^l}\delta^i{}_j - a_{jk}\delta^i{}_l - a_{jl}\delta^i{}_k\big) \\
& +\big([R_1]_s s_{y^k y^l} + [R_1]_{ss} s_{y^k} s_{y^l}\big)\big(\alpha^2\delta^i{}_j - y_j y^i\big) \\
& +[R_1]_s\Big[s_{y^k}\big([\alpha^2]_{y^l}\delta^i{}_j - a_{jl} y^i - y_j\delta^i{}_l\big) + s_{y^l}\big([\alpha^2]_{y^k}\delta^i{}_j \\
& -a_{jk} y^i - y_j\delta^i{}_k\big)\Big] + \Big\{[R_3]_s\big(s_{y^l}\delta^i{}_k + s_{y^k y^l} y^i + s_{y^k}\delta^i{}_l\big) \\
& +[R_3]_{ss} s_{y^k} s_{y^l} y^i + \Big[R_4\alpha_{y^k y^l} + [R_4]_s\big(\alpha_{y^k} s_{y^l} + \alpha_{y^l} s_{y^k} \\
& +\alpha s_{y^k y^l}\big) + \alpha[R_4]_{ss} s_{y^k} s_{y^l}\Big] b^i\Big\}\big(\alpha b_j - s y_j\big) + \Big[R_3\delta^i{}_k \\
& +[R_3]_s s_{y^k} y^i + (\alpha_{y^k} R_4 + \alpha[R_4]_s s_{y^k}) b^i\Big](\alpha_{y^l} b_j - s_{y^l} y_j \\
& -s a_{jl}\big) + \Big[R_3\delta^i{}_l + [R_3]_s s_{y^l} y^i + R_4\alpha_{y^l} b^i \\
& +\alpha[R_4]_s s_{y^l} b^i\Big](\alpha_{y^k} b_j - s_{y^k} y_j - s a_{jk}\big) + \big(\alpha R_4 b^i \\
& +R_3 y^i\big)\big(\alpha_{y^k y^l} b_j - s_{y^k y^l} y_j - s_{y^k} a_{jl} - s_{y^l} a_{jk}\big). \qquad (3.8)
\end{aligned}$$

Then, by (3.8) and using (3.1)-(3.6), we obtain

$$\begin{aligned}
R^i{}_{ljm.k} \quad = \quad & \frac{1}{\alpha^6}\Big\{\alpha^5[R_1]_{sss} b_k b_l b_m \delta^i{}_j + \alpha^5[R_4]_{sss} b_j b_k b_l b_m b^i \\
& +\alpha^4[R_3]_{sss} b_j b_k b_l b_m y^i - \alpha^3\big(s[R_3]_{sss} + [R_1]_{sss} + 3[R_3]_{ss}\big) b_k b_l b_m y_j y^i \\
& +\alpha^2 s\big(3[R_1]_s - 3s[R_1]_{ss} - s^2[R_1]_{sss}\big) y_k y_l y_m \delta^i{}_j + \alpha^2 s\big(3[R_4]_s \\
& -3s[R_4]_{ss} - s^2[R_4]_{sss}\big) b_j y_k y_l y_m b^i + \alpha s^2\big(15[R_4]_s + 9s[R_4]_{ss} \\
& +s^2[R_4]_{sss}\big) y_j y_k y_l y_m b^i + \alpha\big(3R_3 - 3s[R_3]_s - 6s^2[R_3]_{ss} \\
& -s^3[R_3]_{sss}\big) b_j y_k y_l y_m y^i + s\big(15[R_1]_s + 9s[R_1]_{ss} + s^2[R_1]_{sss} \\
& +15R_3 + 33s[R_3]_s + 12s^2[R_3]_{ss} + s^3[R_3]_{sss}\big) y_j y_k y_l y_m y^i \\
& -\alpha^4\big(3[R_4]_{ss} + s[R_4]_{sss}\big) b_k b_l b_m y_j b^i \\
& +\Big[-\alpha^3\big([R_4]_s - s[R_4]_{ss} - s^2[R_4]_{sss}\big) b_j b_k y_l y_m b^i + \alpha^3\big(2[R_4]_s \\
& +4s[R_4]_{ss} + s^2[R_4]_{sss}\big) b_k b_l y_j y_m b^i - \alpha^3\big([R_3]_{ss} \\
& +s[R_3]_{sss}\big) b_j b_k b_l y_m y^i - \alpha^2 s\big(6[R_4]_s + 6s[R_4]_{ss} \\
& +s^2[R_4]_{sss}\big) b_k y_j y_l y_m b^i - \alpha^4 s[R_4]_{sss} b_j b_k b_l y_m b^i \\
& +\alpha^2\big(4[R_3]_s + 5s[R_3]_{ss} + s^2[R_3]_{sss} + 2[R_1]_{ss} + s[R_1]_{sss}\big) b_k b_l y_j y_m y^i \\
& -\alpha\big(3R_3 + 13s[R_3]_s + 8s^2[R_3]_{ss} + s^3[R_3]_{sss} + 3[R_1]_s + 5s[R_1]_{ss} \\
& +s^2[R_1]_{sss}\big) b_k y_j y_l y_m y^i + \alpha^5\big([R_4]_s - s[R_4]_{ss}\big) a_{kl} b_j b_m b^i
\end{aligned}$$

$$
\begin{aligned}
&-\alpha^4 s\big([R_4]_s - s[R_4]_{ss}\big)a_{kl}b_j y_m b^i - \alpha^5\big(2[R_4]_s + s[R_4]_{ss}\big)a_{jk}b_l b_m b^i\\
&+\alpha^4 s\big(2[R_4]_s + s[R_4]_{ss}\big)\big(a_{jl}y_m + a_{jm}y_l + a_{lm}y_j\big)b_k b^i\\
&-\alpha^3 s^2\big(3[R_4]_s + s[R_4]_{ss}\big)\big(a_{lm}y_j + a_{jm}y_l\big)y_k b^i + \alpha^5 s^2[R_4]_s a_{jk}a_{lm}b^i\\
&+\alpha^5\big([R_1]_s - s[R_1]_{ss}\big)a_{kl}b_m\delta^i{}_j - \alpha^5\big(R_3 + s[R_3]_s\\
&+[R_1]_s\big)\big(a_{jl}\delta^i{}_m + a_{jm}\delta^i{}_l\big)b_k + \alpha^5\big(R_3 - s[R_3]_s\big)a_{kl}b_j\delta^i{}_m\\
&+\alpha^5[R_3]_{ss}b_j b_k b_l\delta^i{}_m - \alpha^4 s\big([R_1]_s - s[R_1]_{ss}\big)a_{kl}y_m\delta^i{}_j\\
&+\alpha^4 s\big(R_3 + s[R_3]_s + [R_1]_s\big)\big(a_{jk}a_{lm}y^i + a_{jk}y_l\delta^i{}_m + a_{jk}y_m\delta^i{}_l\\
&+a_{kl}y_j\delta^i{}_m\big) - \alpha^4 s[R_1]_{sss}b_k b_l y_m\delta^i{}_j - \alpha^4\big(2[R_3]_s + s[R_3]_{ss}\\
&+[R_1]_{ss}\big)a_{jk}b_l b_m y^i - \alpha^4 s[R_3]_{ss}\big(a_{lm}y^i + y_l\delta^i{}_m + y_m\delta^i{}_l\big)b_j b_k\\
&-\alpha^4\big(2[R_3]_s + s[R_3]_{ss} + [R_1]_{ss}\big)b_k b_l y_j\delta^i{}_m - \alpha^3\big([R_1]_s - s[R_1]_{ss}\\
&-s^2[R_1]_{sss}\big)b_k y_l y_m\delta^i{}_j + \alpha^3\big(R_3 + 3s[R_3]_s + s^2[R_3]_{ss} + [R_1]_s\\
&+s[R_1]_{ss}\big)\big(a_{jk}b_l y_m y^i + a_{jk}b_m y_l y^i + a_{kl}b_m y_j y^i + b_k y_j y_l\delta^i{}_m\\
&+b_k y_j y_m\delta^i{}_l\big) - \alpha^3\big(R_3 - s[R_3]_s - s^2[R_3]_{ss}\big)\big(a_{kl}y_m y^i\\
&+y_k y_l\delta^i{}_m\big)b_j - \alpha^2 s\big(3R_3 + 5s[R_3]_s + s^2[R_3]_{ss} + 3[R_1]_s\\
&+s[R_1]_{ss}\big)\big(a_{jk}y_l y_m y^i + a_{kl}y_j y_m y^i + y_j y_k y_l\delta^i{}_m\big)\\
&+\alpha^2 s\big(3[R_3]_{ss} + s[R_3]_{sss}\big)b_j b_k y_l y_m y^i\Big](k\to l\to m\to k)\Big\},
\end{aligned}
\tag{3.9}
$$

where

$$
R^i{}_{ljm.k} := \frac{\partial}{\partial y^m}\frac{\partial}{\partial y^l}\Big(\frac{\partial}{\partial y^k}R^i_j\Big).
$$

By definition, F is R-quadratic if and only if $R^i{}_{ljm.k} = 0$. Then if F be R-quadratic, (3.9) yields

$$
[R_1]_s - s[R_1]_{ss} = 0,\ R_3 - s[R_3]_s = 0,\ R_3 + s[R_3]_s + [R_1]_s = 0,\ [R_4]_s = 0. \tag{3.10}
$$

By (2.10), the third equation of (3.10) can be rewritten as follows

$$
R_{2s} = 0.
$$

Thus (3.10) can be rewritten as follows

$$
[R_1]_s - s[R_1]_{ss} = 0,\ R_3 - s[R_3]_s = 0,\ R_{2s} = 0,\ [R_4]_s = 0.
$$

On the other hand by solving (3.10), we have

$$
R_1 = f_2(r)s^2 + f_1(r),\ R_3 = f_3(r)s,\ R_4 = f_4(r), \tag{3.11}
$$

and

$$
f_3(r) + f_2(r) = 0. \tag{3.12}
$$

Plug (3.11) and (3.12) into (2.12), we have

$$\begin{aligned} R^i{}_j &= [f_2(r)s^2+f_1(r)](\alpha^2\delta^i{}_j - y_jy^i) + f_3(r)s(\alpha b_j - sy_j)y^i \\ &\quad +\alpha f_4(r)(\alpha b_j - sy_j)b^i \\ &= f_2(r)\beta^2\delta^i{}_j - f_2(r)s^2y_jy^i + f_1(r)(\alpha^2\delta^i{}_j - y_jy^i) + f_3(r)\beta b_jy^i \\ &\quad -f_3(r)s^2y^iy_j + \alpha^2 f_4(r)b_jb^i - f_4(r)\beta y_jb^i \\ &= f_2(r)\beta^2\delta^i{}_j + f_1(r)(\alpha^2\delta^i{}_j - y_jy^i) + f_3(r)\beta b_jy^i + \alpha^2 f_4(r)b_jb^i \\ &\quad -f_4(r)\beta y_jb^i \\ &= y^my^n\Big\{[f_2(r)b_mb_n + f_1(r)a_{mn}]\delta^i{}_j + [f_3(r)b_mb_j - f_1(r)a_{mj}]\delta^i{}_n \\ &\quad +f_4(r)[a_{mn}b_j - b_ma_{nj}]b^i\Big\}. \end{aligned} \tag{3.13}$$

□

Proof of Theorem 1.3. Let F be a general (α,β)-metric. By (2.5) and (2.10), one can easily obtain a formula for the Ricci curvature $\mathbf{Ric} = R^i{}_i$.

$$\begin{aligned} \mathbf{Ric} &= nR_1\alpha^2 + R_2\alpha^2 + sR_3\alpha^2 + (b^2-s^2)R_4\alpha^2 \\ &= \alpha^2 R, \end{aligned} \tag{3.14}$$

where

$$R := (n-1)R_1 + (b^2-s^2)R_4. \tag{3.15}$$

From (3.14), we have

$$\frac{\partial}{\partial y^j}\mathbf{Ric} = [\alpha^2]_{y^j}R + \alpha^2 R_s s_{y^j}. \tag{3.16}$$

By (3.16), we obtain

$$\begin{aligned} \frac{\partial}{\partial y^k}\Big(\frac{\partial}{\partial y^j}\mathbf{Ric}\Big) &= [\alpha^2]_{y^jy^k}R + [\alpha^2]_{y^j}R_s s_{y^k} + [\alpha^2]_{y^k}R_s s_{y^j} + \alpha^2 R_{ss}s_{y^k}s_{y^j} \\ &\quad +\alpha^2 R_s s_{y^jy^k}. \end{aligned} \tag{3.17}$$

Differentiating (3.17), we have

$$\begin{aligned} \frac{\partial}{\partial y^l}\frac{\partial}{\partial y^k}\Big(\frac{\partial}{\partial y^j}\mathbf{Ric}\Big) &= \frac{1}{\alpha}\Big(R_s - sR_{ss}\Big)a_{jk}b_l(j\to k\to l\to j) \\ &\quad -\frac{s}{\alpha^2}\Big[(R_s - sR_{ss})a_{jk}y_l + R_{sss}b_jb_ky_l\Big](j\to k\to l\to j) \\ &\quad -\frac{1}{\alpha^3}\Big(R_s - sR_{ss} - s^2R_{sss}\Big)b_jy_ky_l(j\to k\to l\to j) \\ &\quad +\frac{s}{\alpha^4}\Big(3R_s - 3sR_{ss} - s^2R_{sss}\Big)y_jy_ky_l + \frac{1}{\alpha}R_{sss}b_jb_kb_l. \end{aligned}$$

By definition, if a Finsler metric F is Ricci-quadratic metric then

$$\frac{\partial}{\partial y^l}\frac{\partial}{\partial y^k}\Big(\frac{\partial}{\partial y^j}\mathbf{Ric}\Big) = 0.$$

From above formula, a general (α, β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a Ricci-quadratic metric if and only if R satisfy

$$R_s - sR_{ss} = 0. \tag{3.18}$$

Then one gets that

$$R = g_2(r)s^2 + g_1(r). \tag{3.19}$$

Plug (3.19) into (3.14), we have

$$\begin{aligned} \mathbf{Ric} &= \alpha^2[g_2(r)s^2 + g_1(r)] \\ &= g_2(r)\beta^2 + g_1(r)\alpha^2 \\ &= y^m y^n \Big\{ g_2(r)b_m b_n + g_1(r)a_{mn} \Big\}. \end{aligned} \tag{3.20}$$

This completes the proof. □

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