On the Diophantine Equation $x^6 + ky^3 = z^6 + kw^3$

H. Shabani-Solt., N. Yusefnejad, A. S. Janfada*

Department of Mathematics, Urmia University,
Urmia 57561-51818, Iran.
E-mail: h.shabani.solt@gmail.com
E-mail: yusefnejadnazanin@yahoo.com
E-mails: a.sjanfada@urmia.ac.ir; asjanfada@gmail.com

Abstract. Given the positive integers $m, n$, solving the well known symmetric Diophantine equation $x^m + ky^n = z^m + kw^n$, where $k$ is a rational number, is a challenge. By computer calculations, we show that for all integers $1 \leq k \leq 500$ the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$ has infinitely many nontrivial ($y \neq w$ and $x \neq z$) rational solutions. Clearly, the same result holds for positive integers $k$ whose cube-free part is not greater than 500. We exhibit a collection of (probably infinitely many) rational numbers $k$ for which this Diophantine equation is satisfied. Finally, appealing these observations we conjecture that the above result is true for all rational numbers $k$.

Keywords: Diophantine equation, Elliptic curve.

2010 Mathematics Subject Classification: 11Y50, 11G05, 11D41.

1. Introduction

A symmetric Diophantine equation is an equation of the form

$$f(x, y) = f(z, w),$$

where $f$ is a 2-variable polynomial with integer coefficients. Choudhry [1] used certain properties of rational binary forms to solve several symmetric

*Corresponding Author

Received 07 January 2017; Accepted 17 March 2018
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Diophantine equation. Inose [2] studied the quartic symmetric Diophantine equations using Kummer surfaces (see also [4]). Of symmetric Diophantine equations, the equation

\[ x^m + ky^n = z^m + kw^n, \quad (1.1) \]

where \( k \) is a rational number, is of more importance. In [5] the equation (1.1) is considered via analytic number theory.

A technique to claim the existence for a Diophantine equation is to find an elliptic fibration and use the specialization process to construct an elliptic curve with positive rank. Using this technique, the authors of [3] proved that the equation \( x^6 + 6y^3 = z^6 + 6w^3 \) has infinitely many nontrivial integral solutions. The exhibited fibration works for some other \( k \)'s, but not for all. Here, by a trivial solution we mean a solution with \( y = w \) and \( x = z \) as well as any relation leading to these qualities. In these circumstances, \( k \) and \( kd^3 \), for all integers \( d \), doesn’t really make a difference for the type of results we are after.

In general, it is complicated to find a fibration works for all or, infinitely many \( k \)’s. In this article we find a fibration for the Diophantine equation

\[ x^6 + ky^3 = z^6 + kw^3. \quad (1.2) \]

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that works for the positive integers \( k \) up to 500 and establish the following main result.

**Theorem 1.1.** For each integer \( k \) with \( 1 \leq k \leq 500 \), the Diophantine equation \( x^6 + ky^3 = z^6 + kw^3 \) has infinitely many nontrivial rational solutions.

If \( (x, y, z, w) \) is a solution to \( x^6 + ky^3 = z^6 + kw^3 \) for a specific \( k \) in the range 1 to 500, then \((mx, my, mz, mw)\) is also a solution for \( km^3 \), for all integers \( m \). The following corollary is obvious.

**Corollary 1.2.** For all integers \( k \) whose cube-free part is not greater than 500, the Diophantine equation \( x^6 + ky^3 = z^6 + kw^3 \) has infinitely many nontrivial rational solutions.

The Diophantine equation (1.2) is also true for infinitely many rational numbers. For example, from Theorem 1.1 it is concluded that (1.2) is true for all rational numbers of the form \( \frac{k}{e^6} \), where \( 1 \leq k \leq 500 \) and \( e \in \mathbb{Z}^+ \). The following results exhibit other collection of (probably infinitely many) rational numbers \( k \) for which the Diophantine equation (1.2) is satisfied.

**Theorem 1.3.** Let \( a \) be a cube-free non-zero integer such that the elliptic curve

\[ E_A : Y^2 = X^3 + AX + A + 1, \quad A = 3a^8 + a^4 + 4, \]

has positive rank. Then the Diophantine equation

\[ x^6 + \frac{1}{a}y^3 = z^3 + \frac{1}{a}w^3 \]

has infinitely many nontrivial rational solutions.
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In the case $a = 1$, the elliptic curve $E_8$ has positive rank. One sees that the Diophantine equation of Theorem 1.3 for $a = 1$ is the same as that of Theorem 1.3 for $k = 1$. A quick search shows that for $a = 2, 3, 4$, the corresponding elliptic curves $E_A$ have also positive ranks. This approach gives an isotri-vial family of positive rank elliptic curves, increasing the number of nontrivial rational solutions of (1.2).

With the above observations we now claim that,

**Conjecture 1.4.** For each rational number $k$, the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$ has infinitely many nontrivial rational solutions.

### 2. Preliminaries

Let $K$ be a field and $C$ be the algebraic curve defined over $K$ with quartic affine model

$$v^2 = au^4 + bu^3 + cu^2 + du + e, \quad a \neq 0, \quad (2.1)$$

Suppose $C$ has a $K$-rational affine point $(u, v) = (p, q)$. We may assume $p = 0$ by changing $u$ to $u + p$, if necessary. Then $e = q^2$ and the equation (2.1) turns to

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2, \quad a \neq 0. \quad (2.2)$$

Suppose $q = 0$. If $d = 0$, then the curve (2.2) will have a singularity at $(u, v) = (0, 0)$. Therefore, assume $d \neq 0$. Dividing both side of (2.2) by $u^4$ we get

$$\left(\frac{v}{u^2}\right)^2 = d\left(\frac{1}{u}\right)^3 + c\left(\frac{1}{u}\right)^2 + b\left(\frac{1}{u}\right) + a,$$

and putting $X = 1/u$ and $Y = 1/u^2$ we obtain the elliptic curve $Y^2 = dX^3 + cX^2 + bX + a$ which is clearly turned to the Weierstrass form. The harder case is when $q \neq 0$. In this case we have the following result [6].

**Theorem 2.1.** Let $K$ be a field of characteristic not 2 and $C$ be the algebraic curve defined over $K$ by

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2, \quad q \neq 0. \quad (2.3)$$

Suppose $C$ has a $K$-rational point $(p, q)$. Let

$$X = \frac{2q(v + q) + du}{u^2}, \quad Y = \frac{4q^2(v + q) + 2q(du + cu^2) - (d^2u^2/2q)}{u^3}.$$

Define

$$a_1 = d/q, \quad a_2 = c - (d^2/4q^2), \quad a_3 = 2qb, \quad a_4 = -4q^2a, \quad a_6 = a_2a_4.$$

Then the curve $C$ is in one to one corresponding with the elliptic curve

$$E : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

The inverse transformation is

$$u = \frac{2q(X + c) - (d^2/2q)}{Y}, \quad v = -q + \frac{u(uX - d)}{2q} \quad (2.4)$$
The point \((u, v) = (0, q)\) on \(C\) corresponds to the point \((X, Y) = \infty\) on \(E\) and \((u, v) = (0, -q)\) on \(C\) corresponds to \((X, Y) = (-a_2, a_1a_2 - a_3)\) on \(E\).

3. PROOF OF THE MAIN THEOREM

Proof of Theorem 1.1. Consider the intersection of the 3-fold \(x^6 + ky^3 = z^6 + kw^3\) with the hyperplane

\[
x = u + \frac{4}{k}s^2, \quad y = v - \frac{u}{2}, \quad z = u - \frac{4}{k}s^2, \quad w = u + v + \frac{u}{2},
\]

(3.1)

Put \(\frac{4}{k}s^2 = t\). With some straightforward calculations, we get

\[
12u^5t + 40u^3t^3 + 12ut^5 = k(u^3 + 3u^2y + 3uy^2).
\]

Taking \(u \neq 0\) we have

\[
12u^4t + 40u^2t^3 + 12t^5 = k(u^2 + 3uy + y^2),
\]
or,

\[
v^2 = \frac{4}{k}u^4t + \left(\frac{40t^3}{3k} - \frac{1}{12}\right)u^2 + \frac{4}{k}t^5,
\]

where \(v = y + u/2\). Putting back \(t = \frac{1}{k}s^2\), we have

\[
v^2 = \left(\frac{4}{k}s\right)^2u^4 + \left(\frac{10}{3}\left(\frac{4}{k}\right)^4s^6 - \frac{1}{12}\right)u^2 + \left(\frac{4}{k}\right)^6s^{10}.
\]

(3.2)

Now we use Theorem 2.1 for

\[
a = \left(\frac{4}{k}s\right)^2, \quad b = 0, \quad c = \frac{10}{3}\left(\frac{4}{k}\right)^4s^6 - \frac{1}{12}, \quad d = 0, \quad q = \left(\frac{4}{k}\right)^3s^5.
\]

(3.3)

Therefor we get

\[
a_1 = 0, \quad a_2 = \frac{10}{3}\left(\frac{4}{k}\right)^4s^6 - \frac{1}{12}, \quad a_3 = 0, \quad a_4 = -4\left(\frac{4}{k}\right)^8s^{12}, \quad a_6 = a_2a_4
\]

and the curve in (3.2) transfers to the elliptic curve

\[
E_{k,s} : Y^2 = X^3 + \left(\frac{4}{k}\right)^4s^6 - \frac{1}{12})X^2 - 4\left(\frac{4}{k}\right)^8s^{12}X - \frac{40}{3}\left(\frac{4}{k}\right)^{12}s^{18} + \frac{1}{3}\left(\frac{4}{k}\right)^6s^{12},
\]

over \(Q(k, s)\). This elliptic curve can trivially be put into short Weierstrass form, denoted again by \(E_{k,s}\).

\[
E_{k,s} : Y^2 = X^3 + AX + (A + 1), \quad \text{where} \ A = \frac{3s^{12} + s^6k^4 + 4k^8}{k^6}
\]

(3.4)

We now use the software MWRANK. We take a fixed \(k\) and find \(s = s_k\) so that the elliptic curve has positive rank. The result of implementation is recorded in Table 1.

Now let \((X, Y)\) be a point on the elliptic curve \(E_{k,s_k}\). Substituting the equations of (3.3) in (2.4) we find \(u, v\) and putting these in (3.1) we obtain a rational solution \((x, y, z, w)\) for the Diophantine equation \(x^6 + ky^3 = z^6 + kw^3\) which, in turn, gives a rational solution. Since the rank of \(E_{k,s_k}\) is positive, there are infinitely many rational solutions. This completes the proof. \(\square\)
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<table>
<thead>
<tr>
<th>$k$</th>
<th>$s_k/2$</th>
<th>$r=1$</th>
<th>$s_k/k$, $r=1$</th>
<th>$s_k/k/2$, $r=1$</th>
<th>$s_k/k/8$, $r=1$</th>
</tr>
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</table>

**Table 1.** The results of implementation
Proof of Theorem 1.3. We first prove that the Diophantine equation

$$A = \frac{3s^{12} + s^{6}k^4 + 4k^8}{k^8}$$

has rational solutions. Write (3.5) as

$$3s^4\left(\frac{s}{k}\right)^8 + s^2\left(\frac{s}{k}\right)^4 + 4 - A = 0.$$

Take $s = h$ and $g = \frac{h}{k}$. Then the above equation changes to a quadratic polynomial in $h^2g^4$, positive root of which is

$$h^2g^4 = -1 + \frac{\sqrt{1 - 12(4 - A)}}{6} = a^4.$$

Putting $h = t^2$ and $g = 2$ we obtain the solutions $s = t^2, k = \frac{1}{a}t^3$. With this solutions, the elliptic curve $E_A$ coincides with the elliptic curve $E_{k,s}$ in (3.4) in the proof of Theorem 1.1. The result now comes from the last paragraph of the proof of Theorem 1.1. □

4. Closing Comments

Finding the curves of arithmetic genus zero over the 3-fold (1.1) may be of importance. The following example shows the existence of these curves.

Example 4.1. The identity

$$a^6 + \frac{1}{2}(b^2 + ab - a^2)^3 = b^6 + \frac{1}{2}(a^2 + ab - b^2)^3$$

establish a parametric solution for the 3-fold $x^6 + \frac{1}{2}y^3 = z^6 + \frac{1}{2}w^3$.

The family of Diophantine equations $x^6 + ky^3 = z^6 + kw^3$ may be studied other ways. To see this we need the next result.

Proposition 4.2. The cubic curve $C_k : x^3 + y^3 = k$ is birationally equivalent to the elliptic curve $E_k : Y^2 = X^3 - 432k^2$.

Proof. The rational map

$$X = \frac{12k}{y + x}, \quad Y = \frac{36k(y - x)}{y + x}$$

is a birational map from $C_k$ to $E_k$ with the inverse

$$x = \frac{Y + 36k}{6X}, \quad y = \frac{Y - 36k}{6X}.$$

□

For example consider the equation

$$x^6 + 7y^3 = z^6 + 7w^3.$$  \hspace{1cm} (4.1)

We may write

$$(x^3 - z^3)(x^3 + z^3) = 7(w^3 - y^3).$$
Putting $x = 2, z = -1$ we have

$$w^3 + (-y)^3 = 9.$$  \hfill (4.2)

By the substitutions in the proof of Theorem 4.2 we obtain the elliptic curve $Y^2 = X^3 - 34992$. Now the software MWRANK to compute the ranks of elliptic curves. This shows that the Diophantine equation (4.1) has infinitely many nontrivial rational solutions.

Acknowledgments

The authors would like to thank the helpful comments and suggestions of the referee(s) which motivate the authors to state and prove Theorem 1.3 and also improve the paper.

References