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On the Diophantine Equation $x^6 + ky^3 = z^6 + kw^3$

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Abstract. Given the positive integers m, n, solving the well known symmetric Diophantine equation $x^m + ky^n = z^m + kw^n$, where k is a rational number, is a challenge. By computer calculations, we show that for all integers $1 \leq k \leq 500$ the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$ has infinitely many nontrivial ($y \neq w$ and $x \neq z$) rational solutions. Clearly, the same result holds for positive integers k whose cube-free part is not greater than 500. We exhibit a collection of (probably infinitely many) rational numbers k for which this Diophantine equation is satisfied. Finally, appealing these observations we conjecture that the above result is true for all rational numbers k.

Keywords: Diophantine equation, Elliptic curve.

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1. Introduction

A symmetric Diophantine equation is an equation of the form

$$f(x, y) = f(z, w),$$

where f is a 2-variable polynomial with integer coefficients. Choudhry [1] used certain properties of rational binary forms to solve several symmetric

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Diophantine equation. Inose [2] studied the quartic symmetric Diophantine equations using Kummer surfaces (see also [4]). Of symmetric Diophantine equations, the equation

$$x^m + ky^n = z^m + kw^n, \tag{1.1}$$

where k is a rational number, is of more importance. In [5] the equation (1.1) is considered via analytic number theory.

A technique to claim the existence for a Diophantine equation is to find an elliptic fibration and use the specialization process to construct an elliptic curve with positive rank. Using this technique, the authors of [3] proved that the equation $x^6 + 6y^3 = z^6 \pm 6w^3$ has infinitely many nontrivial integral solutions. The exhibited fibration works for some other k's, but not for all. Here, by a trivial solution we mean a solution with $y = w$ and $x = z$ as well as any relation leading to these qualities. In these circumstances, k and kd^3, for all integers d, doesn't really make a difference for the type of results we are after.

In general, it is complicated to find a fibration works for all or, infinitely many k's. In this article we find a fibration for the Diophantine equation

$$x^6 + ky^3 = z^6 + kw^3. \tag{1.2}$$

that works for the positive integers k up to 500 and establish the following main result.

Theorem 1.1. *For each integer k with $1 \leq k \leq 500$, the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$ has infinitely many nontrivial rational solutions.*

If (x, y, z, w) is a solution to $x^6 + ky^3 = z^6 + kw^3$ for a specific k in the range 1 to 500, then (mx, my, mz, mw) is also a solution for km^3, for all integers m. The following corollary is obvious.

Corollary 1.2. *For all integers k whose cube-free part is not greater than 500, the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$ has infinitely many nontrivial rational solutions.*

The Diophantine equation (1.2) is also true for infinitely many rational numbers. For example, from Theorem 1.1 it is concluded that (1.2) is true for all rational numbers of the form $\frac{k}{e^6}$, where $1 \leq k \leq 500$ and $e \in \mathbb{Z}^+$. The following results exhibit other collection of (probably infinitely many) rational numbers k for which the Diophantine equation (1.2) is satisfied.

Theorem 1.3. *Let a be a cube-free non-zero integer such that the elliptic curve*

$$E_A : Y^2 = X^3 + AX + A + 1, \;\; A = 3a^8 + a^4 + 4,$$

has positive rank. Then the Diophantine equation

$$x^6 + \frac{1}{a}y^3 = z^3 + \frac{1}{a}w^3$$

has infinitely many nontrivial rational solutions.

In the case $a = 1$, the elliptic curve E_8 has positive rank. One sees that the Diophantine equation of Theorem 1.3 for $a = 1$ is the same as that of Theorem 1.3 for $k = 1$. A quick search shows that for $a = 2, 3, 4$, the corresponding elliptic curves E_A have also positive ranks. This approach gives an isotrivial family of positive rank elliptic curves, increasing the number of nontrivial rational solutions of (1.2).

With the above observations we now claim that,

Conjecture 1.4. *For each rational number k, the Diophantine equation $x^6 + ky^3 = z^6 + kw^3$ has infinitely many nontrivial rational solutions.*

2. Preliminaries

Let K be a field and C be the algebraic curve defined over K with quartic affine model

$$v^2 = au^4 + bu^3 + cu^2 + du + e, \ a \neq 0, \tag{2.1}$$

Suppose C has a K-rational affine point $(u, v) = (p, q)$. We may assume $p = 0$ by changing u to $u + p$, if necessary. Then $e = q^2$ and the equation (2.1) turns to

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2, \ a \neq 0. \tag{2.2}$$

Suppose $q = 0$. If $d = 0$, then the curve (2.2) will have a singularity at $(u, v) = (0, 0)$. Therefore, assume $d \neq 0$. Dividing both side of (2.2) by u^4 we get

$$(\frac{v}{u^2})^2 = d(\frac{1}{u})^3 + c(\frac{1}{u})^2 + b(\frac{1}{u}) + a,$$

and putting $X = 1/u$ and $Y = 1/u^2$ we obtain the elliptic curve $Y^2 = dX^3 + cX^2 + bX + a$ which is clearly turned to the Weierstrass form. The harder case is when $q \neq 0$. In this case we have the following result [6].

Theorem 2.1. *Let K be a field of characteristic not 2 and C be the algebraic curve defined over K by*

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2, \ q \neq 0. \tag{2.3}$$

Suppose C has a K-rational point (p, q). Let

$$X = \frac{2q(v + q) + du}{u^2}, \ Y = \frac{4q^2(v + q) + 2q(du + cu^2) - (d^2u^2/2q)}{u^3}.$$

Define

$$a_1 = d/q, \ a_2 = c - (d^2/4q^2), \ a_3 = 2qb, \ a_4 = -4q^2a, \ a_6 = a_2a_4.$$

Then the curve C is in one to one corresponding with the elliptic curve

$$E : Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

The inverse transformation is

$$u = \frac{2q(X + c) - (d^2/2q)}{Y}, \ v = -q + \frac{u(uX - d)}{2q}. \tag{2.4}$$

The point $(u,v)=(0,q)$ *on* C *corresponds to the point* $(X,Y)=\infty$ *on* E *and* $(u,v)=(0,-q)$ *on* C *corresponds to* $(X,Y)=(-a_2,a_1a_2-a_3)$ *on* E*.*

3. Proof of the main theorem

Proof of Theorem 1.1. Consider the intersection of the 3-fold $x^6+ky^3=z^6+kw^3$ with the hyperplane

$$x=u+\frac{4}{k}s^2,\ y=v-\frac{u}{2},\ z=u-\frac{4}{k}s^2,\ w=u+y=v+\frac{u}{2}. \tag{3.1}$$

Put $\frac{4}{k}s^2=t$. With some straightforward calculations, we get

$$12u^5t+40u^3t^3+12ut^5=k(u^3+3u^2y+3uy^2).$$

Taking $u\neq 0$ we have

$$12u^4t+40u^2t^3+12t^5=k(u^2+3uy+3y^2),$$

or,

$$v^2=\frac{4}{k}u^4t+(\frac{40t^3}{3k}-\frac{1}{12})u^2+\frac{4}{k}t^5,$$

where $v=y+u/2$. Putting back $t=\frac{4}{k}s^2$, we have

$$v^2=\big(\frac{4}{k}s\big)^2u^4+\big(\frac{10}{3}\big(\frac{4}{k}\big)^4s^6-\frac{1}{12}\big)u^2+\big(\frac{4}{k}\big)^6s^{10}. \tag{3.2}$$

Now we use Theorem 2.1 for

$$a=\big(\frac{4}{k}s\big)^2,\ b=0,\ c=\frac{10}{3}\big(\frac{4}{k}\big)^4s^6-\frac{1}{12},\ d=0,\ q=\big(\frac{4}{k}\big)^3s^5. \tag{3.3}$$

Therefor we get

$$a_1=0,\ a_2=\frac{10}{3}\big(\frac{4}{k}\big)^4s^6-\frac{1}{12},\ a_3=0,\ a_4=-4\big(\frac{4}{k}\big)^8s^{12}\ \ a_6=a_2a_4$$

and the curve in (3.2) transfers to the elliptic curve

$$E_{k,s}:Y^2=X^3+\big(\frac{10}{3}\big(\frac{4}{k}\big)^4s^6-\frac{1}{12}\big)X^2-4\big(\frac{4}{k}\big)^8s^{12}X-\frac{40}{3}\big(\frac{4}{k}\big)^{12}s^{18}+\frac{1}{3}\big(\frac{4}{k}\big)^8s^{12},$$

over $\mathbb{Q}(k,s)$. This elliptic curve can trivially be put into short Weierstrass form, denoted again by $E_{k,s}$.

$$E_{k,s}:Y^2=X^3+AX+(A+1),\ \text{where } A=\frac{3s^{12}+s^6k^4+4k^8}{k^8} \tag{3.4}$$

We now use the software MWRANK. We take a fixed k and find $s=s_k$ so that the elliptic curve has positive rank. The result of implementation is recorded in Table 1.

Now let (X,Y) be a point on the elliptic curve E_{k,s_k}. Substituting the equations of (3.3) in (2.4) we find u,v and putting these in (3.1) we obtain a rational solution (x,y,z,w) for the Diophantine equation $x^6+ky^3=z^6+kw^3$ which, in turn, gives a rational solution. Since the rank of E_{k,s_k} is positive, there are infinitely many rational solutions. This completes the proof. □

k=1,3,7,9,10,11,13,14,16,29,30,32,38,39,44,47,52,58,61,68,81,82,86,100, 103,114,116,126,127,136,139,152,184,191,199,213,220,227,238,241,242, 251,256,271,292,293,323,324,332,346,364,367,378,412,422,436,473,476	s_k=1/2,r=1
k=2,6,15,18,19,25,27,28,37,46,51,53,54,56,59,62,63,64,66,71,72,74,75,83,88,104, 123,125,154,174,187,192,216,217,226,248,298,308,328,338,433,456,472,488,498	s_k=1, r=1
k=4,8,20,22,24,31,43,60,80,85,92,111,112,115,120,121,129,132,140,141, 143,145,147,148,150,156,166,171,194,212,218,219,221,228,232,240, 247,249,255,304,312,319,352,360,373,376,396,406,416,424,428,499	s_k=2, r=1
k=5,23,35,41,48,49,50,65,73,79,89,93,113,124,155,157,159,161,163,164,168,169,170, 173,177,178,180,186,189,193,195,206,209,210,223,224,225,230,233,235,239,258,262, 265,269,272,275,302,310,345,349,355,356,357,362,363,365,370,371,374,375,388,390, 393,399,401,413,414,415,417,419,421,427,434,435,440,443,453,455,461,474,493,495	s_k=k/4, r=1
k=12,87,144,198,204,211,252,276,297,301,320,409,464,485,496	s_k=2, r=2
k=17,33,40,67,69,76,84,102,106,137,142,158,176,208,383,438,446,497	s_k=1/2, r=2
k=21,34,55,57,97,101,122,133,138,146,179,190,197, 202,203,281,295,347,359,361,389,395,405,478,491	s_k=k/2, r=1
k=26,36,42,45,70,118,128,165,167,172,264,321,348	s_k=1, r=2
k=90,109,131,183,196,229,250,279,280,285,286,289,290,306,309, 311,315,316,317,325,326,329,330,335,336,337,341,344,353,379, 382,384,387,400,402,403,425,441,442,448,450,458,465,467,470	s_k=k/8, r=1
k=91,95,96,98,99,153,181,205,207,215,231,234,236,245,253,254,257,259,260,261, 266,267,291,303,313,331,350,351,369,397,398,407,411,418,420,429,432,452,457,471	s_k=k/4, r=2

k=77,175,243,246, 263,380,386,404,469	s_k=2, $1 \le r \le 2$	k=117,149,151,268, 287,342,426,500	s_k=1, $1 \le r \le 2$	k=162,343,381	s_k=k/2, r=3
k=78,110,182,237, 277,307,314,368	s_k=1/2, 1,$\le r \le 2$	k=119,322,394, 437,460,481	s_k=k, r=1	k=222,278,296,299,300, 305,451,466,477,479,483	s_k=k/8, r=2
k=94,107,135,185, 188,200,372,377,463	s_k=k/2, r=2	k=130,333,482	s_k=13, r=1	k=273,294,339, 392,445,492	s_k=1, $1 \le r \le 3$
k=105,201,244,284, 288,340,408,410,444	s_k=1/2, $1 \le r \le 3$	k=134,354	s_k=2, r=3	k=366,431,486	s_k=k/8, r=3
k=108,391	s_k=1, r=3	k=141,318,327, 334,358,439	s_k=k/2, $1 \le r \le 3$	k=449,459,475	s_k=k, $1 \le r \le 2$

k=160	s_k=k/4, r=3	k=423	s_k=k/8, $1 \le r \le 2$	k=480	s_k=6, r=1
k=214	s_k=8, r=2	k=430	s_k=19, $1 \le r \le 2$	k=484	s_k=4, $1 \le r \le 3$
k=270	s_k=5, $1 \le r \le 2$	k=447	s_k=k/2, r=4	k=487	s_k=14, $1 \le r \le 2$
k=274	s_k=k, $1 \le r \le 3$	k=454	s_k=7, $1 \le r \le 2$	k=489	s_k=k, r=3
k=283	s_k=3k/4, r=2	k=462	s_k=9, r=1	k=490	s_k=8, $1 \le r \le 3$
k=385	s_k=12, $1 \le r \le 2$	k=468	s_k=14, r=1	k=494	s_k=19, r=1

TABLE 1. The results of implementation

Proof of Theorem 1.3. We first prove that the Diophantine equation

$$A = \frac{3s^{12} + s^6k^4 + 4k^8}{k^8} \tag{3.5}$$

has rational solutions. Write (3.5) as

$$3s^4\left(\frac{s}{k}\right)^8 + s^2\left(\frac{s}{k}\right)^4 + 4 - A = 0.$$

Take $s = h$ and $g = \frac{h}{k}$. Then the above equation changes to a quadratic polynomial in h^2g^4, positive root of which is

$$h^2g^4 = \frac{-1 + \sqrt{1 - 12(4 - A)}}{6} = a^4.$$

Putting $h = t^2$ and $g = \frac{a}{t}$ we obtain the solutions $s = t^2, k = \frac{1}{a}t^3$. With this solutions, the elliptic curve E_A coincides with the elliptic curve $E_{k,s}$ in (3.4) in the proof of Theorem 1.1. The result now comes from the last paragraph of the proof of Theorem 1.1. □

4. Closing comments

Finding the curves of arithmetic genus zero over the 3-fold (1.1) may be of importance. The following example shows the existence of these curves.

Example 4.1. The identity

$$a^6 + \frac{1}{2}(b^2 + ab - a^2)^3 = b^6 + \frac{1}{2}(a^2 + ab - b^2)^3$$

establish a parametric solution for the 3-fold $x^6 + \frac{1}{2}y^3 = z^6 + \frac{1}{2}w^3$.

The family of Diophantine equations $x^6 + ky^3 = z^6 + kw^3$ may be studied other ways. To see this we need the next result.

Proposition 4.2. *The cubic curve $C_k : x^3 + y^3 = k$ is birationally equivalent to the elliptic curve $E_k : Y^2 = X^3 - 432k^2$.*

Proof. The rational map

$$X = \frac{12k}{y + x},\ Y = \frac{36k(y - x)}{y + x}$$

is a birational map from C_k to E_k with the inverse

$$x = \frac{Y + 36k}{6X},\ y = \frac{Y - 36k}{6X}.$$

□

For example consider the equation

$$x^6 + 7y^3 = z^6 + 7w^3. \tag{4.1}$$

We may write

$$(x^3 - z^3)(x^3 + z^3) = 7(w^3 - y^3).$$

Putting $x = 2, z = -1$ we have

$$w^3 + (-y)^3 = 9. \tag{4.2}$$

By the substitutions in the proof of Theorem 4.2 we obtain the elliptic curve $Y^2 = X^3 - 34992$. Now the software MWRANK to compute the ranks of elliptic curves. This shows that the Diophantine equation (4.1) has infinitely many nontrivial rational solutions

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