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The Number of Subgroups of a Given Type in Certain Finite Groups

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Abstract. The number of subgroups, normal subgroups and characteristic subgroups of a finite group G are denoted by $Sub(G)$, $NSub(G)$ and $CSub(G)$, respectively. The main goal of this paper is to present a matrix model for computing these positive integers for dicyclic groups, semi-dihedral groups, and three sequences U_{6n}, V_{8n} and $H(n)$ of groups that can be presented as follows:

$$\begin{aligned} U_{6n} &= \langle a, b \mid a^{2n} = b^3 = e,\ bab = a\rangle, \\ V_{8n} &= \langle a, b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b\rangle, \\ H(n) &= \langle a, b, c \mid a^{2^{n-2}} = b^2 = c^2 = e, [a, b] = [b, c] = e, a^c = ab\rangle. \end{aligned}$$

For each group, a matrix model containing all information is given.

Keywords: Subgroup, Normal subgroup, Characteristic subgroup.

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1. Introduction

Throughout this paper all groups are assumed to be finite. The number of subgroups, normal subgroups and characteristic subgroups in a finite group G are denoted by $Sub(G)$, $NSub(G)$ and $CSub(G)$, respectively. The dihedral

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group D_{2n} can be presented as $D_{2n} = \langle a, b \mid a^n = b^2 = e,\ b^{-1}ab = a^{-1}\rangle$. Suppose $\tau(n)$ and $\sigma(n)$ denote the number and the sum of positive divisors of n, respectively. Then it is well-known that $Sub(D_{2n}) = \tau(n) + \sigma(n)$. For a proof of this known result we refer to [2, Example 1]. The aim of this paper is to extend this result to some other classes of finite groups. Our motivation is to ask whether, if we cannot list all subgroups of a given group, we can at least count how many subgroups, normal subgroups and characteristic subgroups there are. We solve this problem for the semi-dihedral group SD_{2^n}, dicyclic group T_{4n} and the groups U_{6n}, V_{8n} and $H(n)$. These groups have the following presentations:

$$\begin{aligned}
T_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle,\\
U_{6n} &= \langle a, b \mid a^{2n} = b^3 = e,\ bab = a\rangle,\\
V_{8n} &= \langle a, b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b\rangle,\\
SD_{2^n} &= \langle a, b \mid a^{2^{n-1}} = b^2 = e, b^{-1}ab = a^{-1+2^{n-2}}\rangle,\\
H(n) &= \langle a, b, c \mid a^{2^{n-2}} = b^2 = c^2 = e, [a, b] = [b, c] = e, a^c = ab\rangle.
\end{aligned}$$

The number of subgroups of a finite group G is the order of the subgroup lattice $\mathcal{L}(G)$ and by [4, Theorem 2] the number of normal subgroup of a finite group G is the same as the number of all topologies on G which makes G a topological group. So, the number of subgroups and normal subgroups of a given finite group G are important in some other branches of mathematics. Calhoun [2], generalized the formula of the number of subgroups in a dihedral group to the class of groups which can be formed as cyclic extensions of cyclic groups. The dihedral groups and the abelian groups of the form $Z_m \times Z_n$ are examples of this class of finite groups. Tărnăuceanu [9], presented an arithmetic method for determining the number of some types of subgroups of finite abelian groups. Tărnăuceanu and Tóth [10] studied the number of subgroups of a given exponent in a finite abelian group and obtained formulas for the number of subgroups of rank two and three.

Throughout this paper our notations are standard and our main references are the books of James and Liebeck [6] and Tărnăuceanu [11]. We refer the interested readers to [6], for the main properties of the groups T_{4n}, U_{6n} and V_{8m}, where n is arbitrary positive integer m is an odd positive integer. For the groups V_{8m}, m is even, the group $H(n)$ and the semi-dihedral group SD_{2^n}, we refer to the papers [3], [1] and [5], respectively. Our calculations are done with the aid of GAP [12]. This work is a continuation of our last work in [7, 8].

2. Main Results

Suppose G is one of the groups introduced in Section 1. The aim of this section is to find exact formulas for $Sub(G)$, $NSub(G)$ and $CSub(G)$, where $G \in$

$\{T_{4n}, U_{6n}, V_{8n}, SD_{2^n}, H(n)\}$. To do this, we first define a table $OT(G) = [a_{ij}]$, the **order table** of G, which is crucial in our calculations. The columns of this table are labeled by the powers of 2 and its rows by the odd divisors of $|G|$. This model is suitable for groups of even orders. Suppose $|G| = 2^r \cdot m$, where m is odd and $c_0 = 1 < c_1 < c_2 < \cdots < c_t$ are all the odd divisors of $|G|$. Then

$$a_{ij} = \begin{cases} 2^{j-1} & i = 1 \\ 2^{j-1}c_{i-1} & i \neq 1 \end{cases}.$$

The order table of G is given in Table 1.

It is easy to see that this table has exactly $\tau(\frac{n}{2^r})$ rows and $r+1$ columns and so this matrix has $(r+1)\tau(\frac{n}{2^r})$ entries. Suppose $x = p_1^{a_1}p_2^{a_2}...p_s^{a_s}$ is the prime factorization of positive integer x. Then it is well-known that

$$\tau(x) = (a_1+1)(a_2+1)...(a_s+1) \text{ and } \sigma(x) = \frac{p_1^{a_1+1}-1}{p_1-1}\cdots\frac{p_s^{a_s+1}-1}{p_s-1}.$$

Table 1. Orders of subgroups of G, when $|G| = n = 2^r p_1^{a_1}p_2^{a_2}\dots p_s^{a_s}$.

	j	1	2	3	…	r+1
i		1	2	4	…	2^r
1	1	1	2	4	…	2^r
2	c_1	c_1	$2c_1$	$4c_1$	…	$2^r c_1$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$i+1$	c_i	c_i	$2c_i$	$4c_i$	…	$2^r c_i$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$t+1$	c_t	c_t	$2c_t$	$4c_t$	…	$2^r c_t$

2.1. **Dihedral Group D_{2n}.** In this subsection we will compute the number of normal and characteristic subgroups of dihedral group D_{2n}. It is worth mentioning here that Tărnăuceanu [9] computed these numbers, but we reprove his results to explain our method for dihedral groups. Define $D_{2n} = \langle a, b | a^n = b^2 = e, bab = a^{-1}\rangle$. Note that an arbitrary subgroup of D_{2n} has one of the following forms:

(1) A subgroup of $\langle a^i \rangle$ $i|n$;
(2) A subgroup of $\langle a^i, a^j b\rangle$, where $i|n$ and $1 \leq j \leq i$.

Since $ab = ba^{-1}$, $(a^i b)$ has order two. It is possible to conclude from this conclusion that the number order subgroups of the dihedral group D_{2n} can be computed by the following formula:

$$b_{ij}(D_{2n}) = \begin{cases} 1 & j = 1 \\ \frac{n}{2^{j-2}c_{i-1}} + 1 & 2 \leq j \leq r+1 \\ \frac{n}{2^{j-2}c_{i-1}} & j = r+2 \end{cases}.$$

Table 2. The number of subgroups of each order for D_{2n}.

	j	1	2	$\cdots$	$r+1$	$r+2$
i		1	2	$\cdots$	2^r	2^{r+1}
1	1	1	$n+1$	$\cdots$	$\frac{n}{2^{r-1}}+1$	$\frac{n}{2^r}$
2	c_1	1	$\frac{n}{c_1}+1$	$\cdots$	$\frac{n}{2^{r-1}c_1}+1$	$\frac{n}{2^r c_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	c_{t-1}	1	$\frac{n}{c_{t-1}}+1$	$\cdots$	$\frac{n}{2^{r-1}c_{t-1}}+1$	$\frac{n}{2^r c_{t-1}}$
$t+1$	c_t	1	2^r+1	$\cdots$	$2+1$	1

Now we compute the number of normal subgroups in D_{2n}.

$$n_{ij}(D_{2n}) = \begin{cases} 0 & 1 \leq j \leq r+2, 1 \leq i \leq t \\ 1 & 1 \leq j \leq r, i = t+1 \ or \ j = r+2, i = t+1 \\ 3 & j = r+1, i = t+1 \end{cases} .$$

Table 3. The number of normal subgroups of each order for D_{2n}.

	j	1	2	$\cdots$	$r+1$	$r+2$
i		1	2	$\cdots$	2^r	2^{r+1}
1	1	1	1	$\cdots$	1	0
2	c_1	1	1	$\cdots$	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	c_{t-1}	1	1	$\cdots$	1	0
$t+1$	c_t	1	1	$\cdots$	3	1

The number of characterstic subgroups of the group D_{2n} can be computed as follows:

$$c_{ij}(D_{2n}) = \begin{cases} 0 & j = r+2, 1 \leq i \leq t \\ 1 & 1 \leq j \leq r+1, 1 \leq i \leq t+1 \ or \ j = r+2, i = t+1 \end{cases} .$$

Table 4. The number of characterstic subgroups of D_{2n}.

	j	1	2	$\cdots$	$r+1$	$r+2$
i		1	2	$\cdots$	2^r	2^{r+1}
1	1	1	1	$\cdots$	1	0
2	c_1	1	1	$\cdots$	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	c_{t-1}	1	1	$\cdots$	1	0
$t+1$	c_t	1	1	$\cdots$	1	1

We record our calculations in the following theorem:

Theorem 2.1. *Suppose n is a given positive integer. Then the number of all subgroups, normal subgroups and characteristic subgroups of D_{2n} can be computed by the following formulas:*

- $Sub(D_{2n}) = \tau(n) + \sigma(n)$.
- $NSub(D_{2n}) = \begin{cases} \tau(n)+3 & 2 \mid n \\ \tau(n)+1 & 2 \nmid n \end{cases}$.
- $CSub(D_{2n}) = \tau(n) + 1$.

2.2. **Dicyclic groups T_{4n}.** The aim of this subsection is to compute the number of subgroups, normal subgroups and characteristic subgroups of the dicyclic group T_{4n}. By our assumption, $|G| = 2^{r+2}p_1^{a_1}.p_2^{a_2}\dots p_s^{a_s}$. Note that an arbitrary subgroup of T_{4n} has one of the following forms:

(1) A subgroup of $\langle a^i\rangle$, $i|2n$;
(2) A subgroup of $\langle a^i, a^j b\rangle$, where $i|n$ and $1 \le j \le i$.

Since a cyclic group of order m has exactly $\tau(m)$ subgroups, there are $\tau(2n)$ subgroups of the first type. On the other hand, by the structure of our second type subgroups, T_{4n} has exactly $\sum_{i|n} i = \sigma(n)$ subgroups of the second type. Hence $Sub(T_{4n}) = \tau(2n) + \sigma(n)$.

We are now ready to present a matrix form for our calculations on subgroup lattice of dicyclic groups. To do this, we first define:

$$b_{ij}(T_{4n}) = \begin{cases} 1 & j = 1,2 \\ \frac{n}{2^{j-3}c_{i-1}} + 1 & 3 \le j \le r+2 \\ \frac{n}{2^{j-3}c_{i-1}} & j = r+3 \end{cases}.$$

If $B = [b_{ij}]$ then the matrix B gives Table 5, for the number of subgroups of each order in the dicyclic group T_{4n}.

Table 5. The number of subgroups of each order in T_{4n}.

	j	1	2	3	$\cdots$	$r+2$	$r+3$
i		1	2	4	$\cdots$	2^{r+1}	2^{r+2}
1	1	1	1	$n+1$	$\cdots$	$\frac{n}{2^{r-1}}+1$	$\frac{n}{2^r}$
2	c_1	1	1	$\frac{n}{c_1}+1$	$\cdots$	$\frac{n}{2^{r-1}c_1}+1$	$\frac{n}{2^r c_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	c_{t-1}	1	1	$\frac{n}{c_{t-1}}+1$	$\cdots$	$\frac{n}{2^{r-1}c_{t-1}}+1$	$\frac{n}{2^r c_{t-1}}$
$t+1$	c_t	1	1	2^r+1	$\cdots$	$2+1$	1

We are now ready to compute the number of normal subgroups of T_{4n}. By above discussions, all subgroups of the first type are normal in T_{4n} and among all subgroups of the second type, only T_{4n}, $\langle a^2, ab\rangle$ and $\langle a^2, a^2b\rangle$ are normal in T_{4n}. Therefore, $NSub(T_{4n}) = \tau(2n) + 3$.

To present a matrix form for the number of normal subgroups in the group T_{4n}, we define:

$$n_{ij}(T_{4n}) = \begin{cases} 0 & 1 \le i \le t, 1 \le j \le r+3 \\ 1 & i = t+1, 1 \le j \le r+1 \text{ } or \text{ } i = t+1, j = r+3 \\ 3 & i = t, j = r+2 \end{cases}.$$

If $C = [n_{ij}]$ then the entries of the matrix C can be computed by Table 6 and the number of normal subgroups of T_{4n} is as follows:

Table 6. The number of normal subgroups of each order in T_{4n}.

	j	1	2	3	$\cdots$	$r+2$	$r+3$
i		1	2	4	$\cdots$	2^{r+1}	2^{r+2}
1	1	1	1	1	$\cdots$	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
t	c_{t-1}	1	1	1	$\cdots$	1	0
$t+1$	c_t	1	1	1	$\cdots$	3	1

Our calculations given above give the following theorem:

Theorem 2.2. *Suppose n is a given positive integer. Then the number of all subgroups, normal subgroups and characteristic subgroups of T_{4n} can be computed by the following formulas:*

- $Sub(T_{4n}) = \tau(2n) + \sigma(n)$.
- $NSub(T_{4n}) = \begin{cases} \tau(2n)+3 & 2 \mid n \\ \tau(2n)+1 & 2 \nmid n \end{cases}$.
- $CSub(T_{4n}) = \tau(2n) + 1$.

2.3. **Group U_{6n}.** In this subsection, we compute the number of subgroups, normal subgroups and characteristic subgroups of the group U_{6n} of order $6n$, where $n = 2^r 3^k p_1^{\alpha_1} \ldots p_s^{\alpha_s}$. The presentation of this group were given in Section 1. This group has four types of subgroups as follows:

- A subgroup of $G_1 = \langle a^i \rangle$, where $i|2n$;
- A subgroup of $G_2 = \langle a^i, b \rangle$, where $i|2n$;
- A subgroup of $G_3 = \langle a^i b \rangle$, where $i|2n$ and $2 \cdot 3^k \nmid i$.
- A subgroup of $G_4 = \langle a^i b^2 \rangle$, where $i|2n$ and $2 \cdot 3^k \nmid i$.

Suppose $n = 2^r 3^k p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ and define:

$$b_{ij}(U_{6n}) = \begin{cases} 1 & 3 \nmid c_{(i-1)} \text{ and } 1 \leq j \leq r+1, \ \textit{or} \ 3^{k+1} | c_{(i-1)} \\ 3 & 3 \nmid c_{(i-1)} \text{ and } j = r+2 \\ 4 & 3 | c_{(i-1)} \text{ and } 3^{k+1} \nmid c_{(i-1)} \end{cases}.$$

There are b_{ij} subgroups of order a_{ij} and Table 7 gives the number of subgroups of each order in the group U_{6n}. We are now counting the number of normal subgroups in U_{6n}. In Table 8, we record the number of normal subgroups of each order. By this table, one can see that the number of all normal subgroups of order a_{ij} can be computed by the following:

$$n_{ij}(U_{6n}) = \begin{cases} 0 & 3 \nmid c_{(i-1)} \ \textit{and} \ (j = r+2 \ \textit{or} \ 1 \leq j \leq r+1), \\ 1 & 3^{k+1} | c_{(i-1)}, \ 1 \leq j \leq r+2, \ \textit{or} \ 3 \mid c_{(i-1)}, \ j = r+2 \\ 2 & 3 | c_{(i-1)}, \ 1 \leq j \leq r+1, \ 3^{k+1} \nmid c_{(i-1)} \end{cases}.$$

Table 7. The number of subgroups of each order in the group U_{6n}.

	j	1	2	3	$\cdots$	$r+1$	$r+2$
i		1	2	4	$\cdots$	2^r	2^{r+1}
1	1	1	1	1	$\cdots$	1	3
2	c_1	4	4	4	$\cdots$	4	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$i+1$	c_i	1	1	1	$\cdots$	1	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$t+1$	c_t	1	1	1	$\cdots$	1	1

Table 8. The number of normal subgroups of each order in the group U_{6n}.

	j	1	2	$\cdots$	$r+1$	$r+2$
i		1	2	$\cdots$	2^r	2^{r+1}
1	1	1	1	$\cdots$	1	0
2	c_1	2	2	$\cdots$	2	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$i+1$	c_i	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$t+1$	c_t	1	1	$\cdots$	1	1

On the other hand, each normal subgroup of U_{6n} is characteristics and we have proved the following result:

Theorem 2.3. *Suppose $n = 2^r 3^k m$, $6 \nmid m$, is a given positive integer. Then the number of all subgroups, normal subgroups and characteristic subgroups of U_{6n} can be computed by the following formulas:*

- $Sub(U_{6n}) = 4\tau(2n) - 2\tau(\frac{n}{3^k}) = 2[\tau(2n) + \tau(\frac{n}{3}) + \tau(\frac{n}{2^r})]$. *Here, if* $3 \nmid n$ *then we define* $\tau(\frac{n}{3}) = 0$.
- $NSub(U_{6n}) = CSub(U_{6n}) = \tau(\frac{n}{2^r})(2r+3) = 2\tau(n) + \tau(\frac{n}{2^r})$.

Proof. To prove $Sub(U_{6n}) = 4\tau(2n) - 2\tau(\frac{n}{3^k}) = 2(\tau(2n) + \tau(\frac{n}{3}) + \tau(\frac{n}{2^r}))$, we first note that $\langle a\rangle$ has exactly $\tau(2n)$ subgroups. On the other hand, for each divisor i of $2n$ we have $\tau(2n)$ subgroups $\langle a^i, b\rangle$. To complete the proof, we have to count the number of subgroups of the forms $\langle a^i b\rangle$ and $\langle a^i b^2\rangle$ which are different from each other and the subgroups presented above.

Suppose $n = 2^r 3^k p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ and $i = 2^f 3^j p_1^{\alpha_1} \ldots p_s^{\alpha_s}$, where $1 \le f \le r+1$ and $1 \le j \le k$. Then one can easily seen that $\langle a^i, b\rangle = \langle a^i b^t\rangle$, where $t = 1, 2$. Note that in such a case, i is even and that $3 \nmid \frac{2n}{i}$ which means that $3^k \mid i$. Hence, there are $\tau(2n)$ subgroups of the first type and $\tau(2n)$ subgroups of the second type. Moreover, the number of subgroups of the forms $\langle a^i b\rangle$ and $\langle a^i b^2\rangle$, $6|i$, are $2\tau(\frac{n}{3^k})$ and so we have $2(\tau(2n) - \tau(\frac{n}{3^k}))$ subgroups of the third and fourth type together. Therefore, $Sub(U_{6n}) = 4\tau(2n) - 2\tau(\frac{n}{3^k}) = 2[\tau(2n) + \tau(\frac{n}{3}) + \tau(\frac{n}{2^r})]$.

Furthermore, a subgroup of the form $\langle a^i\rangle$, $i|2n$, is normal in U_{6n} if and only if i is even. All subgroups of the form $\langle a^i, b\rangle$ are normal in U_{6n} and all subgroups of the third and fourth types are not normal. Therefore, the number of all normal subgroups are $\tau(2n) + \tau(n)$. Since all normal subgroups are characteristic, $NSub(G) = CSub(G) = \tau(2n) + \tau(n)$. This completes our argument. □

2.4. **Group V_{8n}.** In this subsection, we compute the number of subgroups, normal subgroups and characteristic subgroups of the group V_{8n}. Suppose $n = 2^r p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$. Then clearly $|V_{8n}| = 2^{r+3} p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$. There are nine different types for the subgroups of V_{8n}. In what follows, these types together with the number of subgroups in each type are given.

(1) The cyclic subgroups $G_1(i) = \langle a^i\rangle$, where $i \mid 2n$. For each divisor i of $2n$, there exists exactly one subgroup of this type and so we obtain $\tau(2n)$ cyclic subgroups contained in $\langle a\rangle$.
(2) The cyclic subgroups $G_2(i) = \langle a^i b^2\rangle$, $i|n$. All subgroups in this form are different from those are given in part (1). On the other hand, for each divisor i of n we will have a unique cyclic subgroup of this form. Thus, we find exactly $\tau(n)$ subgroups in the form of $G_2(i)$.
(3) The subgroups $G_3(i) = \langle a^i, b^2\rangle \simeq C_{\frac{2n}{i}} \times C_2$, where $i|2n$. A similar argument as Part (1) shows that there are $\tau(2n)$ subgroups in this form and all such subgroups are different from those are given in Parts (1) and

(2). Thus there are $2\tau(2n)+\tau(n)$ subgroups of the form $G_1(i), G_2(i)$ and $G_3(i)$.

(4) The subgroups $G_4(i,j)=\langle a^i, a^j b\rangle$, $i|2n$, $1\le j\le i$, i is even and j is odd. In this case, it is easy to see that $\langle a^i, a^j b\rangle = \langle a^u, a^v b\rangle$ if and only if $i=u$ and $j=v$. Since i is even, all divisors of $2n$ are $2n$, $\frac{2n}{2}$, $\ldots$, 2 and since j is odd there are $\sum_{d|n} d=\sigma(n)$ subgroups in this form.

(5) The subgroups $G_5(i,j)=\langle a^i, a^j b^3\rangle$, $i|2n$, $1\le j\le i$, i is even and j is odd. It is easy to see that $\langle a^i, a^j b^3\rangle = \langle a^u, a^v b^3\rangle$ if and only if $i=u$ and $j=v$. Since i is even, all such divisors of $2n$ are $2n$, $\frac{2n}{2}$, $\ldots$, 2 and since i is odd there are $\sum_{d|n} d=\sigma(n)$ subgroups in this form.

(6) The subgroups $G_6(i,j)=\langle a^i b^2, a^j b\rangle$, $i|n, 1\le j\le i$ and i is even. It is easy to see that $n+\frac{n}{2}+\cdots+2=2[\frac{n}{2}+\frac{n}{4}+\cdots+1]=2\sigma(\frac{n}{2})$. So, there are $2\sigma(\frac{n}{2})$ subgroups in the form of $G_6(i,j)$.

(7) The subgroups $G_7(i,j)=\langle a^i, b^2, a^j b\rangle$, $i|2n$, $1\le j\le i$ and i, j are even. The number of these subgroups are the same as the number of subgroups in Part 4.

(8) The subgroups $G_8(i,j)=\langle a^i, b^2, a^j b\rangle$, $i|2n$, $1\le j\le i$, i is even and j is odd. Then there are the same number of subgroups as in the Part 7, i.e. there are $\sigma(n)$ subgroups in the form of $G_8(i,j)$.

(9) The subgroups $G_9(i,j)=\langle a^i, b^2, a^j b\rangle$, $i|2n, 1\le j\le i$ and i is odd. In this case, $2n=2^{r+1}p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ and so $\frac{n}{2^r}$ is odd. So, there are $\sum_{d|(\frac{n}{2^r})} d=\sigma(\frac{n}{2^r})$ subgroups in this form.

The summation of these numbers are as follows:

$$\begin{aligned}
Sub(V_{8n}) &= \tau(n)+2\tau(2n)+4\sigma(n)+2\sigma(\frac{n}{2})+\sigma(\frac{n}{2^r})\\
&= \tau(n)+2\tau(2n)+4\sigma(n)+2\sigma(2^{r-1})\sigma(\frac{n}{2^r})+\sigma(\frac{n}{2^r})\\
&= \tau(n)+2\tau(2n)+4\sigma(n)+\sigma(\frac{n}{2^r})[2\sigma(2^{r-1})+1]\\
&= \tau(n)+2\tau(2n)+4\sigma(n)+\sigma(\frac{n}{2^r})[2^r+2^{r-1}+2^{r-2}+\cdots+2^2+2+1]\\
&= \tau(n)+2\tau(2n)+4\sigma(n)+\sigma(\frac{n}{2^r})\sigma(2^r)\\
&= \tau(n)+2\tau(2n)+5\sigma(n).
\end{aligned}$$

In Table 9, the number of subgroups of each order for the group V_{8n} are given.

Table 9. The number subgroups of each order in the group V_{8n}.

	j	1	2	3	4	$\cdots$	$r+2$	$r+3$	$r+4$
i		1	2	2^2	2^3	$\cdots$	2^{r+1}	2^{r+2}	2^{r+3}
1	1	1	$\frac{2^3n}{2^2}+3$	$\frac{2^3n}{2^1}+3$	$\frac{2^3n}{2^2}+3$	$\cdots$	$\frac{2^3n}{2^r}+3$	$\frac{2^3n}{2^{r+2}}+1$	$\frac{2^3n}{2^{r+3}}$
2	c_1	1	$\frac{2^3n}{2^2c_1}+3$	$\frac{2^3n}{2^1c_1}+3$	$\frac{2^3n}{2^2c_1}+3$	$\cdots$	$\frac{2^3n}{2^rc_1}+3$	$\frac{2^3n}{2^{r+2}c_1}+1$	$\frac{2^3n}{2^{r+3}c_1}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$t+1$	c_t	1	$\frac{2^3n}{2^2c_t}+3$	$\frac{2^3n}{2^1c_t}+3$	$\frac{2^3n}{2^2c_t}+3$	$\cdots$	$\frac{2^3n}{2^rc_t}+3$	$\frac{2^3n}{2^{r+2}c_t}+1$	$\frac{2^3n}{2^{r+3}c_t}$

Similar to other groups, we assume that b_{ij} denotes the number of subgroups of order a_{ij} in V_{8n}. From Table 9, one can see that the 2−dimensional sequence b_{ij} can be computed by the following formula:

$$b_{ij}(V_{8n}) = \begin{cases} 1 & j=1, 1 \le i \le t+1 \\ \frac{2^3n}{2^2c_{i-1}}+3 & j=2, 1 \le i \le t+1 \\ \frac{2^3n}{2^kc_{i-1}}+3 & 3 \le j \le r+2, 1 \le i \le t+1, k=j-2 \\ \frac{2^3n}{2^{r+2}c_{i-1}}+1 & j=r+3, 1 \le i \le t+1 \\ \frac{2^3n}{2^{r+3}c_{i-1}}+3 & j=r+4, 1 \le i \le t+1 \end{cases} .$$

It is easy to see that all normal subgroups of V_{8n} have types (1−3), i is even, and types (4,5), $i = 1, 2$ and $j = 1$. In Table 10, the number of normal subgroups of a given order in V_{8n} are computed. In what follows the number of normal subgroups of V_{8n} in all cases are computed.

(1) The number of normal subgroups of types (1), (2) and (3), i is even, are $\tau(2n)-\tau(\frac{n}{2^r}) = \tau(n)$, $\tau(n)-\tau(\frac{n}{2^r}) = r\tau(x)$ and $\tau(2n)$, respectively.
(2) There are two normal subgroups of type (6), one normal subgroup of type (7) and one normal subgroup of type (8).
(3) The trivial normal subgroups $\{e\}$ and V_{8n}.

The summation of these numbers give the formula $NSup(V_{8n}) = 3\tau(n) + 5$ for the number of normal subgroups.

Table 10. The number normal subgroups of each order in the group V_{8n}.

	j	1	2	3	$\cdots$	$r+1$	$r+2$	$r+3$	$r+4$
i	*	1	2	4	$\cdots$	2^r	2^{r+1}	2^{r+2}	2^{r+3}
1	1	1	3	3	$\cdots$	3	1	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	c_{t-1}	1	3	3	$\cdots$	3	1	1	0
$t+1$	c_t	1	3	3	$\cdots$	3	3	3	1

The number n_{ij} of normal subgroups of order a_{ij} in V_{8n} can be computed as follows:

$$n_{ij}(V_{8n}) = \begin{cases} 0 & 1 \leq i \leq t, j = r+4 \text{ } or \text{ } 1 \leq i \leq t+1,\ j = 1, \\ 1 & 1 \leq i \leq t,\ j = r+2,\ r+3 \text{ } or \text{ } i = t+1,\ j = r+4 \\ 3 & 1 \leq i \leq t+1,\ 2 \leq j \leq r+1 \text{ } or \text{ } i = t+1,\ j = r+2, r+3. \end{cases}$$

In Table 11, the number of characteristic subgroups of each order in the group V_{8n} are given.

Table 11. The number characteristic subgroups of each order in the group V_{8n}.

	j	1	2	3	$\cdots$	$r+1$	$r+2$	$r+3$	$r+4$
i	$*$	1	2	4	$\cdots$	2^r	2^{r+1}	2^{r+2}	2^{r+3}
1	1	1	3	3	$\cdots$	3	1	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	c_{t-1}	1	3	3	$\cdots$	3	1	1	0
$t+1$	c_t	1	3	3	$\cdots$	3	1	3	1

Based on our calculations in Table 11, we compute the function c_{ij} for the number of characteristic subgroups of order a_{ij} as follows:

$$c_{ij}(V_{8n}) = \begin{cases} 0 & 1 \leq i \leq t,\ j = r+4 \text{ } or \text{ } 1 \leq i \leq t+1,\ j = 1,\ r+2, \\ 1 & 1 \leq i \leq t,\ j = r+3 \text{ } or \text{ } i = t+1,\ j = r+4, \\ 3 & 1 \leq i \leq t+1,\ 2 \leq j \leq r+1 \text{ } or \text{ } i = t+1,\ j = r+3. \end{cases}$$

Hence, there are $3\tau(2n) + 3$ characteristic subgroup in V_{8n}. Therefore, we proved the following theorem:

Theorem 2.4. *Suppose n is a given positive integer. Then the number of subgroups, normal subgroup and characteristic subgroups of the group V_{8n} is computed as follows:*

- $Sub\,(V_{8n}) = \tau(n) + 2\tau(2n) + 5\sigma(n)$.
- $NSup\,(V_{8n}) = 3\tau(n) + 5$.
- $CSup\,(V_{8n}) = 3\tau(n) + 3$.

2.5. **The Semi-dihedral group SD_{2^n}.** The semi-dihedral group SD_{8n} can be presented as $SD_{2^n} = \langle a, b | a^{2^{n-1}} = b^2 = e, b^{-1}ab = a^{-1+2^{n-2}} \rangle$, where $n \geq 4$. The number of subgroups and normal subgroups of this group was given by Tărnăuceanu [11]. In this section, we reprove the results of Tărnăuceanu. Furthermore, details of all types of the subgroups of this group are given. At first, we note that this group has two types of subgroups as follows:

(1) The subgroups $G_1(i) = \langle a^i \rangle$, $i|2^{n-1}$. In this case, for each i there exists exactly one subgroup of the given form and so we obtain $\tau(2^{n-1}) = n$ subgroups.
(2) The subgroups $G_2(i,j) = \langle a^i, a^j b \rangle$, where $i|2^{n-1}$ and $1 \leq j \leq i$. In this case, there are $\sigma(n)$ subgroups of the form G_2.

In Table 12, we used above information to compute the number of subgroups of each order in SD_{2^n}.

Table 12. The number subgroups of each order in the group SD_{2^n}.

j	1	2	3	4	$\cdots$	n	$n+1$
$a_{1j}(SD_{2^n})$	1	2	2^2	2^3	$\cdots$	2^{n-1}	2^n
$b_{1j}(SD_{2^n})$	1	$2^{n-2}+1$	$2^{n-2}+1$	$2^{n-3}+1$	$\cdots$	$2^{n-n+1}+1$	2^{n-n}

Note that there are 2^{n-2} subgroups of order 2, 2^{n-3} subgroups of order 4, 2^{n-4} subgroups of order 8 and so on. Therefore, $Sub(SD_{2^n}) = 2 + 2(2^{n-2} + 1) + \sum_{i=3}^{n-1}(2^{n-i}+1) = n + 3.2^{n-2} - 1$. In Table 13, we record the number of normal and characteristic subgroups of a given order in this group.

Table 13. The number of normal and characteristic subgroups in the group SD_{2^n}.

j	1	2	3	$\cdots$	$n-1$	n	$n+1$
$a_{1j}(SD_{2^n})$	1	2	2^2	$\cdots$	2^{n-2}	2^{n-1}	2^n
$c_{1j} = n_{1j}(SD_{2^n})$	1	1	1	$\cdots$	1	3	1

Therefore, we have proved the following theorem:

Theorem 2.5. *Suppose n is a given positive integer. The number of subgroups, normal subgroup and characteristic subgroups of the group SD_{2^n} are computed as follows:*

- $Sub(SD_{2^n}) = 3.2^{n-2} + n - 1$.
- $CSup = NSup(SD_{2^n}) = n + 3$.

2.6. **The Group H_n.** We recall that this group can be presented as $G = \langle a, b, c | a^{2^{n-2}} = b^2 = c^2 = [a,b] = [b,c] = e, a^c = ab \rangle$, where $n \geq 4$. This group has eleven types of subgroups as follows:

(1) Subgroups $G_1(i) = \langle a^i \rangle$, $i|2^{n-2}$.
(2) Subgroups $G_2(i) = \langle a^i, b \rangle$, $i|2^{n-2}$.
(3) Subgroups $G_3(i) = \langle a^i, c \rangle$, $i|2^{n-2}$.
(4) Subgroups $G_4(i) = \langle a^i, bc \rangle$, $i|2^{n-2}$.
(5) Subgroups $G_5(i) = \langle a^i b \rangle$, $i|2^{n-3}$.
(6) Subgroups $G_6(i) = \langle a^i c \rangle$, $i|2^{n-3}$.
(7) Subgroups $G_7(i) = \langle a^i bc \rangle$, $i|2^{n-3}$.

(8) Subgroups $G_8(i) = \langle a^i b, a^i c\rangle$, $i|2^{n-3}$.
(9) Subgroups $G_9(i) = \langle a^i b, a^i bc\rangle$, $i|2^{n-3}$.
(10) Subgroups $G_{10}(i) = \langle a^i c, a^i bc\rangle$, $i|2^{n-3}$.
(11) Subgroups $G_{11}(i) = \langle a^i, b, c\rangle$, $i|2^{n-2}$.

A similar calculations as other groups shows that the number of subgroups of a given order in $H(n)$ satisfies all information given Table 14.

Table 14. The number of subgroups of a given order in the group H_n.

j	1	2	3	4	...	$n-1$	n	$n+1$
$a_{1j}(H_n)$	1	2	2^2	2^3	...	2^{n-2}	2^{n-1}	2^n
$b_{1j}(H_n)$	1	7	11	11	...	11	3	1

The form of subgroups can be obtained in Table 15.

Table 15. The form of subgroups of a given order in the group H_n.

Order	1	2	2^2	2^3	...	2^{n-2}	2^{n-1}
	e	$\langle a^{2^{n-3}}\rangle$	$\langle a^{2^{n-4}}\rangle$	$\langle a^{2^{n-5}}\rangle$	...	$\langle a\rangle$	$\langle a, b\rangle$
		$\langle b\rangle$	$\langle a^{2^{n-3}}, b\rangle$	$\langle a^{2^{n-4}}, b\rangle$	...	$\langle a^2, b\rangle$	$\langle a, c\rangle$
		$\langle c\rangle$	$\langle a^{2^{n-3}}, c\rangle$	$\langle a^{2^{n-4}}, c\rangle$	...	$\langle a^2, b, c\rangle$	$\langle a^2, b, c\rangle$
		$\langle bc\rangle$	$\langle a^{2^{n-3}}, bc\rangle$	$\langle a^{2^{n-4}}, bc\rangle$	...	$\langle a^2, bc\rangle$	
		$\langle a^{2^{n-3}}b\rangle$	$\langle a^{2^{n-4}}b\rangle$	$\langle a^{2^{n-5}}b\rangle$	...	$\langle ab\rangle$	
		$\langle a^{2^{n-3}}c\rangle$	$\langle a^{2^{n-4}}c\rangle$	$\langle a^{2^{n-5}}c\rangle$	...	$\langle ac\rangle$	
		$\langle a^{2^{n-3}}bc\rangle$	$\langle a^{2^{n-4}}bc\rangle$	$\langle a^{2^{n-5}}bc\rangle$	...	$\langle abc\rangle$	
			$\langle a^{2^{n-3}}c, a^{2^{n-3}}b\rangle$	$\langle a^{2^{n-4}}c, a^{2^{n-4}}b\rangle$	...	$\langle a^2c, a^2b\rangle$	
			$\langle a^{2^{n-3}}bc, a^{2^{n-3}}b$	$\langle a^{2^{n-4}}bc, a^{2^{n-4}}b\rangle$	...	$\langle a^2bc, a^2b\rangle$	
			$\langle a^{2^{n-3}}bc, a^{2^{n-3}}c\rangle$	$\langle a^{2^{n-4}}bc, a^{2^{n-4}}c\rangle$	...	$\langle a^2bc, a^2c\rangle$	
			$\langle b, c\rangle$	$\langle a^{2^{n-3}}, b, c\rangle$	...	$\langle a^{2^2}, b, c\rangle$	
sum	1	7	11	11	...	11	3

Table 16. The number of normal subgroups of a given order in the group $H_n, n \geq 5$.

j	1	2	3	4	....	$n-2$	$n-1$	n	$n+1$
$a_{1j}(H_n)$	1	2	2^2	2^3	....	2^{n-3}	2^{n-2}	2^{n-1}	2^n
$n_{1j}(H_n)$	1	3	5	5	......	5	3	3	1

A similar calculation as other groups shows that

$$b_{1j} = \begin{cases} 1 & j = 1, n+1 \\ 3 & j = n \\ 7 & j = 2 \\ 11 & 3 \leq j \leq n-1 \end{cases}.$$

Thus $Sub(H_n) = 12 + 11(n-3)$. The number of normal subgroups of this group are recorded in Table 16. By information given this table, one can see that $NSub(H_n) = 11 + 5(n-4)$. The structure of normal subgroups are as Table 17.

Table 17. The normal subgroups of a given order in the group $H_n, n \geq 6$.

order	1	2	2^2	2^3	$\cdots$	2^{n-2}	2^{n-1}
	e	$\langle a^{2^{n-3}} \rangle$	$\langle a^{2^{n-4}} \rangle$	$\langle a^{2^{n-5}} \rangle$	$\cdots$	$\langle a^2, b \rangle$	$\langle a, b \rangle$
	-	$\langle b \rangle$	$\langle a^{2^{n-4}} b \rangle$	$\langle a^{2^{n-5}} b \rangle$	$\cdots$	$\langle a^2 c, a^2 bc \rangle$	$\langle a, bc \rangle$
	-	$\langle a^{2^{n-3}} b \rangle$	$\langle a^{2^{n-3}}, b \rangle$	$\langle a^{2^{n-4}}, b \rangle$	$\cdots$	$\langle a^4, b, c \rangle$	$\langle a^2, b, c \rangle$
	-	-	$\langle a^{2^{n-3}} c, a^{2^{n-3}} bc \rangle$	$\langle a^{2^{n-4}} c, a^{2^{n-4}} bc \rangle$	$\cdots$	-	-
	-	-	$\langle b, c \rangle$	$\langle a^{2^{n-3}}, b, c \rangle$	$\cdots$	-	-
sum	1	3	5	5	$\cdots$	3	3

If $n \geq 6$ then the number of characteristic subgroups and their structures are recorded in Tables 18 and 19, respectively.

Table 18. The number characterstic subgroups of a given order in the group H_n.

j	1	2	3	4	....	$n-3$	$n-2$	$n-1$	n	$n+1$
$a_{1j}(H_n)$	1	2	2^2	2^3	$\cdots$	2^{n-4}	2^{n-3}	2^{n-2}	2^{n-1}	2^n
$c_{1j}(H_n)$	1	3	3	5	$\cdots$	5	3	3	1	1

Table 19. The characterstic subgroups of a given order in the group H_n.

Order	1	2	2^2	2^3	$\cdots$	2^{n-4}
	e	$\langle a^{2^{n-3}} \rangle$	$\langle a^{2^{n-4}} \rangle$	$\langle a^{2^{n-5}} \rangle$	$\cdots$	$\langle a^{2^2} \rangle$
	-	$\langle a^{2^{n-3}} b \rangle$	$\langle a^{2^{n-4}} b \rangle$	$\langle a^{2^{n-5}} b \rangle$	$\cdots$	$\langle a^{2^2} b \rangle$
	-	$\langle b \rangle$	$\langle a^{2^{n-3}}, b \rangle$	$\langle a^{2^{n-4}}, b \rangle$	$\cdots$	$\langle a^{2^3}, b \rangle$
	-	-	-	$\langle a^{2^{n-4}} c, a^{2^{n-4}} bc \rangle$	$\cdots$	$\langle a^{2^3} c, a^{2^3} bc \rangle$
	-	-	-	$\langle a^{2^{n-3}}, b, c \rangle$	$\cdots$	$\langle a^{2^4}, b, c \rangle$
sum	1	3	3	5	$\cdots$	5

Table 19. Continued.

Order	2^{n-3}	2^{n-2}	2^{n-1}	2^n
	$\langle a^{2^2}, b \rangle$	$\langle a^2, b \rangle$	$\langle a^2, b, c \rangle$	G
	$\langle a^{2^2} c, a^{2^2} bc \rangle$	$\langle a^2 c, a^2 bc \rangle$	-	-
	$\langle a^{2^3}, b, c \rangle$	$\langle a^{2^2}, b, c \rangle$	-	-
	-	-	-	-
	-	-	-	-
sum	3	3	1	1

Therefore, we have proved the following theorem:

Theorem 2.6. *Suppose n is a given positive integer. Then the number of subgroups, normal subgroups and characteristic subgroups of the group* H_n *is computed as follows:*

- $Sub(H_n) = 12 + 11(n-3)$.

- $NSup(H_n) = 11 + 5(n-4), n \geq 5$.
- $CSup(H_n) = 5 + 5(n-4), n \geq 6$.

3. Concluding Remarks

In this paper the number of subgroups, normal subgroups and characteristic subgroups of some classes of finite groups are computed. We present a matrix model for description of subgroup lattices of these groups. We checked our results by computer algebra system GAP.

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