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A Topology Generated by ψ-Operation and Ideal Spaces

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Abstract. In this paper, we introduce the notion of ψ-operation on an m-space (X, m). By using the ψ-operation, for every subset A of X, we define the ψ-operation $A_\psi(\mathcal{I}, m)$ with respect to an ideal $\mathcal{I}$ and an m-structure m and investigate their properties. Moreover, on the m we introduce and investigate the notion of ψ-compatibility with an ideal $\mathcal{I}$.

Keywords: Ideal m-space, ψ-operation, ψ-open, m-open, m^*_ψ-open, ψ-compatible.

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1. Introduction and preliminaries

An ideal $\mathcal{I}$ on a space X is a non-empty collection of subsets of X which satisfies the following properties:

(1) $A \in \mathcal{I}$ and $B \subseteq A$ implies that $B \in \mathcal{I}$.
(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X and is denoted by $(X, \tau, \mathcal{I})$ (see [9, 10]).

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Definition 1.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m of $\mathcal{P}(X)$ is called a minimal structure (briefly m-structure) on X if m satisfies the following conditions:

(1) $\emptyset, X \in m$.
(2) The union of any family of subsets belonging to m belongs to m.

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in m\}$ is denoted by $m(x)$.

First we state the following: in [11], a minimal structure m is defined as follows: m is called a minimal structure if $\emptyset, X \in m$. Several characterizations of minimal stracture were provided in [1, 2, 3, 4, 5, 6]. However, in this paper, we define as follows:

Definition 1.2. Let (X, m) be an m-space. A function $\psi : m \to \mathcal{P}(X)$ is called a ψ-operation on m if $\psi(U) \subseteq U$ for every a proper subset $U \in m$ and $\psi(X) = X$. A subset A of X is said to be ψ-open if there exists a proper subset $U \in m$ such that $A \subseteq \psi(U)$ or $A = \psi(X) = X$. We put $\Psi = \{A \subseteq X : A \subseteq \psi(U)$ for some a proper subset $U \in m$ or $A = X\}$. Then Ψ is the family of all ψ-open sets. The complement of a ψ-open set is said to be ψ-closed.

Lemma 1.3. *Let (X, m) be an m-space. For Ψ the following properties hold:*

(1) $\emptyset, X \in \Psi$.
(2) *If $A_\alpha \in \Psi$ for each $\alpha \in \Lambda$, then $\cap_{\alpha\in\Lambda} A_\alpha \in \Psi$.*

Proof. (1): $\psi(\emptyset) = \emptyset$ and $\psi(X) = X$ are always true.
(2): If $A_\alpha \in \Psi$ for each $\alpha \in \Lambda$, then there exists $U_\alpha \in m$ such that $A_\alpha \subseteq \psi(U_\alpha)$. Since $\cap_{\alpha\in\Lambda} A_\alpha \subseteq A_\alpha$, $\cap_{\alpha\in\Lambda} A_\alpha \subseteq \psi(U_\alpha)$. Therefore, $\cap_{\alpha\in\Lambda} A_\alpha \in \Psi$. □

We set $Int_\psi(A) = \cup\{U : U \subseteq A, U \in \Psi\}$ and $Cl_\psi(A) = \cap\{F : A \subseteq F, X - F \in \Psi\}$.

2. ψ-local functions

Definition 2.1. Let $(X, m, \mathcal{I})$ be an ideal m-space. For a subset A of X, we define the following set: $A_\psi(\mathcal{I}, m) = \{x \in X : A \cap U \notin \mathcal{I}$ for every $U \in \Psi(x)\}$, where $\Psi(x) = \{U \in \Psi : x \in U\}$. In case there is no confusion $A_\psi(\mathcal{I}, m)$ is briefly denoted by A_ψ and is called the ψ-local function of A with respect to $\mathcal{I}$ and m.

Lemma 2.2. *Let (X, m) be an m-space, $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and let A and B be subsets of X. Then the following properties hold:*

(1) *If $A \subseteq B$, then $A_\psi \subseteq B_\psi$.*
(2) *If $\mathcal{I} \subseteq \mathcal{J}$, then $A_\psi(\mathcal{I}) \supseteq A_\psi(\mathcal{J})$.*
(3) $A_\psi = Cl_\psi(A_\psi) \subseteq Cl_\psi(A)$.

(4) *If* $A \subseteq A_\psi$, *then* $A_\psi = Cl_\psi(A_\psi) = Cl_\psi(A)$.
(5) *If* $A \in \mathcal{I}$, *then* $A_\psi = \emptyset$.
(6) $(A \cap B)_\psi \subseteq A_\psi \cap B_\psi$.

Proof. (1) Suppose that $x \notin B_\psi$. Then there exists some $U \in \Psi(x)$ such that $U \cap B \in \mathcal{I}$. Since $U \cap A \subseteq U \cap B$, $U \cap A \in \mathcal{I}$. Hence $x \notin A_\psi$. Thus $X \setminus B_\psi \subseteq X \setminus A_\psi$ or $A_\psi \subseteq B_\psi$.
(2) Suppose that $x \notin A_\psi(\mathcal{I})$. There exists some $U \in \Psi(x)$ such that $U \cap A \in \mathcal{I}$. Since $\mathcal{I} \subseteq \mathcal{J}$, $U \cap A \in \mathcal{J}$ and $x \notin A_\psi(\mathcal{J})$. Therefore, $A_\psi(\mathcal{J}) \subseteq A_\psi(\mathcal{I})$.
(3) We have $A_\psi \subseteq Cl_\psi(A_\psi)$ in general. Let $x \in Cl_\psi(A_\psi)$. Then $A_\psi \cap U \neq \emptyset$ for every $U \in \Psi(x)$. Therefore, there exists some $y \in A_\psi \cap U$ and $U \in \Psi(y)$. Since $y \in A_\psi$, $A \cap U \notin \mathcal{I}$ and hence $x \in A_\psi$. Hence we have $Cl_\psi(A_\psi) \subseteq A_\psi$ and hence $A_\psi = Cl_\psi(A_\psi)$. Again, let $x \in Cl_\psi(A_\psi) = A_\psi$, then $U \cap A \notin \mathcal{I}$ for every $U \in \Psi(x)$. This implies $U \cap A \neq \emptyset$ for every $U \in \Psi(x)$. Therefore, $x \in Cl_\psi(A)$. This shows that $A_\psi(\mathcal{I}) = Cl_\psi(A_\psi) \subseteq Cl_\psi(A)$.
(4) For any subset A of X, by (3) we have $A_\psi = Cl_\psi(A_\psi) \subseteq Cl_\psi(A)$. Since $A \subseteq A_\psi$, $Cl_\psi(A) \subseteq Cl_\psi(A_\psi)$ and hence $A_\psi = Cl_\psi(A_\psi) = Cl_\psi(A)$.
(5) Suppose that $x \in A_\psi$. Then for any $U \in \Psi(x)$, $U \cap A \notin \mathcal{I}$. But since $A \in \mathcal{I}$, $U \cap A \in \mathcal{I}$ for any $U \in \Psi(x)$. This is a contradiction. Hence $A_\psi = \emptyset$.
(6) The proof is obvious by (1). □

Lemma 2.3. *Let* $(X, m, \mathcal{I})$ *be an ideal* m*-space and* $A \subseteq B \subseteq X$. *Then,* $A_\psi(\mathcal{I}_B, \Psi|_B) = A_\psi(\mathcal{I}, m) \cap B$.

Proof. First we prove the following implication: $A_\psi(\mathcal{I}_B, \Psi|_B) \subseteq A_\psi(\mathcal{I}, m) \cap B$ Let $x \notin A_\psi(\mathcal{I}, m) \cap B$. We consider the following two cases:
Case 1. $x \notin B$: Since $A_\psi(\mathcal{I}_B, \Psi|_B) \subseteq B$, then $x \notin A_\psi(\mathcal{I}_B, \Psi|_B)$.
Case 2. $x \in B$. In this case $x \notin A_\psi(\mathcal{I}, m)$. There exists a set $V \in \Psi(x)$ and $V \cap A \in \mathcal{I}$. Since $x \in B$, we have a set $B \cap V \in \Psi|_B$ such that $x \in B \cap V$ and $(B \cap V) \cap A \in \mathcal{I}$ and hence $(B \cap V) \cap A \in \mathcal{I}_B$. Consequently, $x \notin A_\psi(\mathcal{I}_B, \Psi|_B)$. Both cases show the implication. The converse implication: $A_\psi(\mathcal{I}, m) \cap B \subseteq A_\psi(\mathcal{I}_B, \Psi|_B)$. Let $x \notin A_\psi(\mathcal{I}_B, \Psi|_B)$. Then, for some ψ-open subset $U \cap B$ of $(B, \Psi|_B)$ containing x, we have $(U \cap B) \cap A \in \mathcal{I}_B$. Since $A \subseteq B$, then $U \cap A \in \mathcal{I}_B \subseteq \mathcal{I}$ i.e., $U \cap A \in \mathcal{I}$ for some $U \in \Psi(x)$. This shows that $x \notin A_\psi(\mathcal{I}, m)$. □

3. A topology associated with ideal m-spaces

Theorem 3.1. *Let* $(X, m, \mathcal{I})$ *be an ideal* m*-space and* A, B *any subsets of* X. *Then the following properties hold:*

(1) $(\emptyset)_\psi = \emptyset$.
(2) $(A_\psi)_\psi \subseteq A_\psi$.
(3) $A_\psi \cup B_\psi = (A \cup B)_\psi$.

Proof. (1) The proof is obvious.
(2) Let $x \in (A_\psi)_\psi$. Then for every $U \in \Psi(x)$, $U \cap A_\psi \notin \mathcal{I}$ and hence $U \cap A_\psi \neq \emptyset$. Let $y \in U \cap A_\psi$. Then $U \in \Psi(y)$ and $y \in A_\psi$. Hence we have $U \cap A \notin \mathcal{I}$ and $x \in A_\psi$. This shows that $(A_\psi)_\psi \subseteq A_\psi$.
(3) It follows from Lemma 2.2 that $(A \cup B)_\psi \supseteq A_\psi \cup B_\psi$. To prove the reverse inclusion, let $x \in (A \cup B)_\psi$. Then for every $U \in \Psi(x)$, $U \cap (A \cup B) = (U \cap A) \cup (U \cap B) \notin \mathcal{I}$. Therefore, $U \cap A \notin \mathcal{I}$ or $U \cap B \notin \mathcal{I}$. (Because if $U \cap A \in \mathcal{I}$ and $U \cap B \in \mathcal{I}$, then $(U \cap A) \cup (U \cap B) = U \cap (A \cup B) \in \mathcal{I}$. This is a contradiction). Hence $x \in A_\psi$ or $x \in B_\psi$. And hence $x \in A_\psi \cup B_\psi$. Therefore, $(A \cup B)_\psi \subseteq A_\psi \cup B_\psi$. Hence we obtain $A_\psi \cup B_\psi = (A \cup B)_\psi$. □

Theorem 3.2. *Let $(X, m, \mathcal{I})$ be an ideal m-space, $Cl^*_\psi(A) = A_\psi \cup A$ and A, B be subsets of X. Then*

(1) $Cl^*_\psi(\emptyset) = \emptyset$.
(2) $A \subseteq Cl^*_\psi(A)$.
(3) $Cl^*_\psi(A \cup B) = Cl^*_\psi(A) \cup Cl^*_\psi(B)$.
(4) $Cl^*_\psi(A) = Cl^*_\psi(Cl^*_\psi(A))$.
(5) *If* $A \subseteq B$, *then* $Cl^*_\psi(A) \subseteq Cl^*_\psi(B)$.

Proof. By Theorem 3.1, we obtain
(1) $Cl^*_\psi(\emptyset) = (\emptyset)_\psi \cup \emptyset = \emptyset$.
(2) $A \subseteq A \cup A_\psi = Cl^*_\psi(A)$.
(3) $Cl^*_\psi(A \cup B) = (A \cup B)_\psi \cup (A \cup B) = (A_\psi \cup B_\psi) \cup (A \cup B) = Cl^*_\psi(A) \cup Cl^*_\psi(B)$.
(4) $Cl^*_\psi(Cl^*_\psi(A)) = Cl^*_\psi(A_\psi \cup A) = (A_\psi \cup A)_\psi \cup (A_\psi \cup A) = ((A_\psi)_\psi \cup A_\psi) \cup (A_\psi \cup A) = A_\psi \cup A = Cl^*_\psi(A)$.
(5) Since $A \subseteq B$, we have $Cl^*_\psi(A) = A_\psi \cup A \subseteq B_\psi \cup B = Cl^*_\psi(B)$. □

By Theorem 3.2, we obtain that $Cl^*_\psi(A) = A \cup A_\psi$ is a Kuratowski closure operator. We will denote by $m^*_\psi(\mathcal{I}) = m^*_\psi$ the topology generated by Cl^*_ψ, that is, $m^*_\psi = \{U \subseteq X : Cl^*_\psi(X - U) = X - U\}$.

Lemma 3.3. *Let $(X, m, \mathcal{I})$ be an ideal m-space and A, B be subsets of X. Then $A_\psi - B_\psi = (A - B)_\psi - B_\psi$.*

Proof. We have by Theorem 3.1 $A_\psi = [(A - B) \cup (A \cap B)]_\psi = (A - B)_\psi \cup (A \cap B)_\psi \subseteq (A - B)_\psi \cup B_\psi$. Thus $A_\psi - B_\psi \subseteq (A - B)_\psi - B_\psi$. By Lemma 2.2, $(A - B)_\psi \subseteq A_\psi$ and hence $(A - B)_\psi - B_\psi \subseteq A_\psi - B_\psi$. Hence $A_\psi - B_\psi = (A - B)_\psi - B_\psi$. □

Corollary 3.4. *Let $(X, m, \mathcal{I})$ be an ideal m-space and A, B be subsets of X with $B \in \mathcal{I}$. Then $(A \cup B)_\psi = A_\psi = (A - B)_\psi$.*

Proof. Since $B \in \mathcal{I}$, by Lemma 2.2 $B_\psi = \emptyset$. By Lemma 3.3, $A_\psi = (A - B)_\psi$ and by Theorem 3.1 $(A \cup B)_\psi = A_\psi \cup B_\psi = A_\psi$. □

Lemma 3.5. *Let $(X, m, \mathcal{I})$ be an ideal m-space. If $U \in \Psi$, then $U \cap A_\psi = U \cap (U \cap A)_\psi \subseteq (U \cap A)_\psi$ for any subset A of X.*

Proof. Suppose that $U \in \Psi$ and $x \in U \cap A_\psi$. Then $x \in U$ and $x \in A_\psi$. Let $V \in \Psi(x)$. Then $V \cap U \in \Psi(x)$ and $V \cap (U \cap A) = (V \cap U) \cap A \notin \mathcal{I}$. This shows that $x \in (U \cap A)_\psi$ and hence we obtain $U \cap A_\psi \subseteq (U \cap A)_\psi$. Moreover, $U \cap A_\psi \subseteq U \cap (U \cap A)_\psi$ and by Lemma 2.2 $(U \cap A)_\psi \subseteq A_\psi$ and $U \cap (U \cap A)_\psi \subseteq U \cap A_\psi$. Therefore, $U \cap A_\psi = U \cap (U \cap A)_\psi$. □

Lemma 3.6. *Let $(X, m, \mathcal{I})$ be an ideal m-space and A, B be subsets of X. Then*

(1) *If $A \subseteq B$, then $Cl^*_\psi(A) \subseteq Cl^*_\psi(B)$.*
(2) *$Cl^*_\psi(A \cap B) \subseteq Cl^*_\psi(A) \cap Cl^*_\psi(B)$.*
(3) *If $U \in \Psi$, then $U \cap Cl^*_\psi(A) \subseteq Cl^*_\psi(U \cap A)$.*

Proof. (1) Since $A \subseteq B$, by Lemma 2.2 we have $Cl^*_\psi(A) = A \cup A_\psi \subseteq B \cup B_\psi = Cl^*_\psi(B)$.
(2) This is obvious by (1).
(3) Since $U \in \Psi$, by Lemma 3.5 we have $U \cap Cl^*_\psi(A) = U \cap (A \cup A_\psi) = (U \cap A) \cup (U \cap A_\psi) \subseteq (U \cap A) \cup (U \cap A)_\psi = Cl^*_\psi(U \cap A)$. □

Corollary 3.7. *Let $(X, m, \mathcal{I})$ be an ideal m-space and $A \subseteq X$. If $A \subseteq A_\psi$, then*

(1) *$Cl_\psi(A) = Cl^*_\psi(A)$.*
(2) *$Int_\psi(X - A) = Int^*_\psi(X - A)$.*

Proof. The proof follows from Lemma 2.2. □

Theorem 3.8. *Let $(X, m, \mathcal{I})$ be an ideal m-space. Then $\beta(\Psi, \mathcal{I}) = \{V - I : V \in \Psi, I \in \mathcal{I}\}$ is a basis for m^*_ψ.*

Proof. Let $(X, m, \mathcal{I})$ be an ideal m-space. It is obvious that A is m^*_ψ-closed if and only if $A_\psi \subseteq A$. Now we have $U \in m^*_\psi$ if and only if $(X - U)_\psi \subseteq X - U$ if and only if $U \subseteq X - (X - U)_\psi$. At first, we shall show that every member of $\beta(\Psi, \mathcal{I})$ is an open set in m^*_ψ. Let $V - I \in \beta(\Psi, \mathcal{I})$, where $V \in \Psi$ and $I \in \mathcal{I}$. For each $x \in V - I$, $x \in V$ and $V \cap (X - (V - I)) \subseteq I \in \mathcal{I}$. Hence, $x \notin (X - (V - I))_\psi$, and hence $V - I \subseteq X - (X - (V - I))_\psi$. Therefore, by the fact above, $V - I \in m^*_\psi$. Now, let $U \in m^*_\psi$. By the fact above $x \in U$ implies that $x \notin (X - U)_\psi$. This implies that there exists some $V \in \Psi(x)$ such that $V \cap (X - U) \in \mathcal{I}$. Now let $I = V \cap (X - U)$ and we have $x \in V - I \subseteq U$, $I \in \mathcal{I}$. Now we need only show that $\beta(\Psi, \mathcal{I})$ is closed under finite intersection. Let $A, B \in \beta(\Psi, \mathcal{I})$, then $A = H - I$ and $B = K - J$, where $H, K \in \Psi$ and

$I, J \in \mathcal{I}$. Now, we have

$$\begin{aligned}(H-I)\cap(K-J) &= (H\cap(X-I))\cap(K\cap(X-J))\\ &= (H\cap K)\cap((X-I)\cap(X-J))\\ &= (H\cap K)\cap(X-(I\cup J))\\ &= (H\cap K)-(I\cup J).\end{aligned}$$

Since $I\cup J\in\mathcal{I}$ and $H\cap K\in\Psi$, $A\cap B\in\beta(\Psi,\mathcal{I})$. Therefore $\beta(\Psi,\mathcal{I})$ is closed under finite intersection. Thus $\beta(\Psi,\mathcal{I})=\{V-I : V\in\Psi, I\in\mathcal{I}\}$ is a basis for m^*_ψ. □

Remark 3.9. Let $(X,m,\mathcal{I})$ be an ideal m-space. Since $\emptyset\in\mathcal{I}$, we have $m\nsubseteq\Psi\subseteq\beta(\Psi,\mathcal{I})\subseteq m^*_\psi$ and hence every ψ-open set is m^*_ψ-open.

The following examples show that $\beta(\Psi,\mathcal{I})$ is not a topology in general.

Example 3.10. Let $X=\{a,b,c,d\}$ and $m=\{\emptyset,X,\{a,b\},\{a,c,d\}\}$ with $\mathcal{I}=\{\emptyset,\{a\}\}$. And a function $\psi: m\to\mathcal{P}(X)$ is defined as $\psi(X)=X$, $\psi(\{a,b\})=\{a\}$, $\psi(\{a,c,d\})=\{a\}$ and $\psi(\emptyset)=\emptyset$. Then $\Psi=\{\emptyset,X,\{a\}\}$ and $\beta(\Psi,\mathcal{I})=\{\emptyset,X,\{a\},\{b,c,d\}\}$ and $m^*_\psi=\{\emptyset,X,\{a\},\{b,c,d\}\}$.

Example 3.11. Let $X=\{a,b,c,d\}$ and $m=\{\emptyset,X,\{a,b\},\{a,c,d\}\}$ with $\mathcal{I}=\{\emptyset,\{a\}\}$. And a function $\psi: m\to\mathcal{P}(X)$ is defined as $\psi(X)=X$, $\psi(\{a,b\})=\{c,d\}$, $\psi(\{a,c,d\})=\{a\}$ and $\psi(\emptyset)=\emptyset$. Then $\Psi=\{\emptyset,X,\{a\},\{c\},\{d\},\{c,d\}\}$ and $\beta(\Psi,\mathcal{I})=\{\emptyset,X,\{a\},\{c\},\{d\},\{c,d\},\{b,c,d\}\}$ and $m^*_\psi=\{\emptyset,X,\{a\},\{c\},\{d\},\{c,d\},\{a,c\},\{a,d\},\{a,c,d\},\{b,c,d\}\}$.

Note that if $\mathcal{I}$ and $\mathcal{J}$ are ideals on X, then $\mathcal{I}\vee\mathcal{J}=\{I\cup J : I\in\mathcal{I}$ and $J\in\mathcal{J}\}$ is also an ideal.

Theorem 3.12. *Let $(X,m,\mathcal{I})$ and $(X,m,\mathcal{J})$ be two ideal m-spaces and $A\subseteq X$. Then $A_\psi(\mathcal{I}\vee\mathcal{J},m)=A_\psi(\mathcal{I},m^*_\psi(\mathcal{J}))\cap A_\psi(\mathcal{J},m^*_\psi(\mathcal{I}))$.*

Proof. Assume that $x\notin A_\psi(\mathcal{I}\vee\mathcal{J},m)$. Then there exists a ψ-open set U containing x such that $U\cap A\in\mathcal{I}\vee\mathcal{J}$. Let $I\in\mathcal{I}$ and $J\in\mathcal{J}$ such that $U\cap A=I\cup J$. Because of the heredity of $\mathcal{I}$, we may assume $I\cap J=\phi$. Thus we have $(U\cap A)-I=J$ and $(U\cap A)-J=I$ which implies that $(U-I)\cap A=J\in\mathcal{J}$ and $(U-J)\cap A=I\in\mathcal{I}$. Therefore $x\notin A_\psi(\mathcal{I},m^*_\psi(\mathcal{J}))$ and $x\notin A_\psi(\mathcal{J},m^*_\psi(\mathcal{I}))$. Hence $A_\psi(\mathcal{I},m^*_\psi(\mathcal{J}))\cap A_\psi(\mathcal{J},m^*_\psi(\mathcal{I}))\subseteq A_\psi(\mathcal{I}\vee\mathcal{J},m)$.

Conversely, if $x\notin A_\psi(\mathcal{I},m^*_\psi(\mathcal{J}))$. This implies there exist an m-open set U containing x and $J\in\mathcal{J}$ such that $(U-J)\cap A\in\mathcal{J}$. We may assume, because of the heredity of $\mathcal{J}$, that $J\subseteq A$. Now define $I=(U-J)\cap A$, and we have $(U-J)\cap A=I\cup J\in\mathcal{I}\vee\mathcal{J}$, hence $x\notin A_\psi(\mathcal{I}\vee\mathcal{J},m)$. Therefore $A_\psi(\mathcal{I}\vee\mathcal{J},m)\subseteq A_*(\mathcal{I},m^*_\psi(\mathcal{J}))$. Similarly, we have that $A_\psi(\mathcal{I}\vee\mathcal{J},m)\subseteq A_\psi(\mathcal{J},m^*_\psi(\mathcal{I}))$. Hence $A_\psi(\mathcal{I}\vee\mathcal{J},m)=A_\psi(\mathcal{I},m^*_\psi(\mathcal{J}))\cap A_\psi(\mathcal{J},m^*_\psi(\mathcal{I}))$ and the proof is complete. □

Theorem 3.13. *Let (X, m) be an m-space with $\mathcal{I}, \mathcal{J}$ ideals on X, then $m^*_\psi(\mathcal{I}\cap \mathcal{J}) = m^*_\psi(\mathcal{I}) \cap m^*_\psi(\mathcal{J})$.*

Proof. Since the intersection of two ideals is an ideal, $m^*_\psi(\mathcal{I}\cap\mathcal{J})$ is meaningful. As $\mathcal{I}\cap\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{I}\cap\mathcal{J} \subseteq \mathcal{J}$ we have $m^*_\psi(\mathcal{I}\cap\mathcal{J}) \subseteq m^*_\psi(\mathcal{I})$ and $m^*_\psi(\mathcal{I}\cap\mathcal{J}) \subseteq m^*_\psi(\mathcal{J})$. Therefore, $m^*_\psi(\mathcal{I}\cap\mathcal{J}) \subseteq m^*_\psi(\mathcal{I}) \cap m^*_\psi(\mathcal{J})$.

Conversely, let us assume that $V \in m^*_\psi(\mathcal{I}) \cap m^*_\psi(\mathcal{J})$ and let $A = X - V$. Then A is m^*_ψ-closed in $m^*_\psi(\mathcal{I})$ and A is m^*_ψ-closed in $m^*_\psi(\mathcal{J})$. This implies that $A_\psi(\mathcal{I}) \subseteq A$ and $A_\psi(\mathcal{J}) \subseteq A$. We aim to prove that A is m^*_ψ-closed in $m^*_\psi(\mathcal{I}\cap\mathcal{J})$. It is enough to show that $A_\psi(I\cap J) \subseteq A$. Assume that $x \notin A$. This implies that $x \notin A_\psi(\mathcal{I})$ and $x \notin A_\psi(J)$. Then there exist ψ-open sets U and V in $\Psi(x)$ such that $U\cap A \in \mathcal{I}$ and $V \cap A \in \mathcal{J}$. Let us take $G = U\cap V$. Clearly $x \in G$ and $G \in \Psi$. Also

$$G\cap A = (U\cap V)\cap A = (U\cap A)\cap(V\cap A) \in \mathcal{I}\cap\mathcal{J}.$$

Thus there exists a ψ-open set $G \in \Psi(x)$ such that $G\cap A \in \mathcal{I}\cap\mathcal{J}$. Therefore $x \notin A_\psi(\mathcal{I}\cap\mathcal{J})$ and hence $A_\psi(\mathcal{I}\cap\mathcal{J}) \subseteq A$. This implies that A is m^*_ψ-closed in $m^*_\psi(\mathcal{I}\cap\mathcal{J})$. Thus $m^*_\psi(\mathcal{I}) \cap m^*_\psi(\mathcal{J}) \subseteq m^*_\psi(\mathcal{I}\cap\mathcal{J})$. Hence we have $m^*_\psi(\mathcal{I}\cap\mathcal{J}) = m^*_\psi(\mathcal{I}) \cap m^*_\psi(\mathcal{J})$. □

4. ψ-compatible in ideal m-spaces

Definition 4.1. Let $(X, m, \mathcal{I})$ be an ideal m-space. The m-structure m is said to be ψ-compatible with the ideal $\mathcal{I}$, denoted $\tau \sim_\psi \mathcal{I}$, if the following holds for every $A \subseteq X$: if for every $x \in A$, there exists $U \in \Psi(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$.

Theorem 4.2. *Let $(X, m, \mathcal{I})$ be an ideal m-space. Then the following properties are equivalent:*

1. *$\tau \sim_\psi \mathcal{I}$;*
2. *If a subset A of X has a cover of ψ-open sets each of whose intersection with A is in $\mathcal{I}$, then $A \in \mathcal{I}$;*
3. *For every $A \subseteq X$, $A\cap A_\psi = \emptyset$ implies that $A \in \mathcal{I}$;*
4. *For every $A \subseteq X$, $A - A_\psi \in \mathcal{I}$;*
5. *For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B_\psi$, then $A \in \mathcal{I}$.*

Proof. (1) ⇒ (2): The proof is obvious.
(2) ⇒ (3): Let $A \subseteq X$ and $x \in A$. Then $x \notin A_\psi$ and there exists some $V_x \in \Psi(x)$ such that $V_x \cap A \in \mathcal{I}$. Therefore, we have $A \subseteq \cup\{V_x : x \in A\}$ and by (2) $A \in \mathcal{I}$.
(3) ⇒ (4): For any $A \subseteq X$, $A - A_\psi \subseteq A$ and $(A - A_\psi) \cap (A - A_\psi)_\psi \subseteq (A - A_\psi) \cap A_\psi = \emptyset$. By (3), $A - A_\psi \in \mathcal{I}$.

(4) $\Rightarrow$ (5): By (4), for every $A \subseteq X$, $A - A_\psi \in \mathcal{I}$. Let $A - A_\psi = J \in \mathcal{I}$, then $A = J \cup (A \cap A_\psi)$ and by Theorem 3.1(3) and Lemma 2.2, $A_\psi = J_\psi \cup (A \cap A_\psi)_\psi = (A \cap A_\psi)_\psi$. Therefore, we have $A \cap A_\psi = A \cap (A \cap A_\psi)_\psi \subseteq (A \cap A_\psi)_\psi$ and $A \cap A_\psi \subseteq A$. By the assumption $A \cap A_\psi = \emptyset$ and hence $A = A - A_\psi \in \mathcal{I}$.
(5) $\Rightarrow$ (1): Let $A \subseteq X$ and assume that for every $x \in A$, there exists some $U \in \Psi(x)$ such that $U \cap A \in \mathcal{I}$. Then $A \cap A_\psi = \emptyset$. Since $(A - A_\psi) \cap (A - A_\psi)_\psi \subseteq (A - A_\psi) \cap A_\psi = \emptyset$, $A - A_\psi$ contains no nonempty subset B with $B \subseteq B_\psi$. By (5), $A - A_\psi \in \mathcal{I}$ and hence $A = A \cap (X - A_\psi) = A - A_\psi \in \mathcal{I}$. □

Theorem 4.3. *Let $(X, m, \mathcal{I})$ be an ideal m-space. Then the following properties are equivalent:*

(1) $\tau \sim_\psi \mathcal{I}$;
(2) *For every m^*_ψ-closed subset A, $A - A_\psi \in \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2): It is clear by Theorem 4.2.
(2) $\Rightarrow$ (1): Let $A \subseteq X$ and assume that for every $x \in A$, there exists some $U \in \Psi(x)$ such that $U \cap A \in \mathcal{I}$. Then $A \cap A_\psi = \emptyset$. Since $Cl^*_\psi(A) = A \cup A_\psi$ is m^*_ψ-closed, we have $(A \cup A_\psi) - (A \cup A_\psi)_\psi \in \mathcal{I}$. Moreover, $(A \cup A_\psi) - (A \cup A_\psi)_\psi = (A \cup A_\psi) - (A_\psi \cup (A_\psi)_\psi) = (A \cup A_\psi) - A_\psi = A$. Therefore $A \in \mathcal{I}$. □

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If m is ψ-compatible with $\mathcal{I}$, then the following equivalent properties hold:*

(1) *For every $A \subseteq X$, $A \cap A_\psi = \emptyset$ implies that $A_\psi = \emptyset$;*
(2) *For every $A \subseteq X$, $(A - A_\psi)_\psi = \emptyset$;*
(3) *For every $A \subseteq X$, $(A \cap A_\psi)_\psi = A_\psi$.*

Proof. First, we show that (1) holds if m is ψ-compatible with $\mathcal{I}$. Let A be any subset of X and $A \cap A_\psi = \emptyset$. By Theorem 4.2, $A \in \mathcal{I}$ and by Lemma 2.2 (5) $A_\psi = \emptyset$.
(1) $\Rightarrow$ (2): Assume that for every $A \subseteq X$, $A \cap A_\psi = \emptyset$ implies that $A_\psi = \emptyset$. Let $B = A - A_\psi$, then

$$\begin{aligned} B \cap B_\psi &= (A - A_\psi) \cap (A - A_\psi)_\psi \\ &= (A \cap (X - A_\psi)) \cap (A \cap (X - A_\psi))_\psi \\ &\subseteq (A \cap (X - A_\psi)) \cap (A_\psi \cap (X - A_\psi)_\psi) = \emptyset. \end{aligned}$$

By (1), we have $B_\psi = \emptyset$. Hence $(A - A_\psi)_\psi = \emptyset$.
(2) $\Rightarrow$ (3): Assume for every $A \subseteq X$, $(A - A_\psi)_\psi = \emptyset$.

$$\begin{aligned} A &= (A - A_\psi) \cup (A \cap A_\psi) \\ A_\psi &= ((A - A_\psi) \cup (A \cap A_\psi))_\psi \\ &= (A - A_\psi)_\psi \cup (A \cap A_\psi)_\psi \\ &= (A \cap A_\psi)_\psi. \end{aligned}$$

(3) $\Rightarrow$ (1): Let $A \subseteq X$ such that $A \cap A_\psi = \emptyset$. Then by the assumption, $(A \cap A_\psi)_\psi = A_\psi$. This implies that $\emptyset = \emptyset_\psi = (A \cap A_\psi)_\psi = A_\psi$. □

Corollary 4.5. *Let $(X, m, \mathcal{I})$ be an ideal m-space. If m is ψ-compatible with $\mathcal{I}$, then $()_\psi$ is an idempotent operator i.e. $A_\psi = (A_\psi)_\psi$ for any subset A of X.*

Proof. By Theorems 4.4 and 3.1, we obtain $A_\psi = (A \cap A_\psi)_\psi \subseteq A_\psi \cap (A_\psi)_\psi = (A_\psi)_\psi$ and by Theorem 3.1, we have $A_\psi = (A_\psi)_\psi$ for any subset A of X. □

Theorem 4.6. *Let $(X, m, \mathcal{I})$ be an ideal m-space. Then the following properties are equivalent:*

(1) *$\Psi \cap \mathcal{I} = \{\emptyset\}$;*
(2) *If $I \in \mathcal{I}$, then $Int_\psi(I) = \{\emptyset\}$;*
(3) *For every $G \in \Psi$, $G \subseteq G_\psi$;*
(4) *$X = X_\psi$.*

Proof. (1) $\Rightarrow$ (2): Let $\Psi \cap \mathcal{I} = \{\emptyset\}$ and $I \in \mathcal{I}$. Suppose that $x \in Int_\psi(I)$. Then there exists some $U \in \Psi(x)$ such that $x \in U \subseteq I$. Since $I \in \mathcal{I}$ and hence $\emptyset \neq \{x\} \subseteq U \in \Psi \cap \mathcal{I}$. This contradicts with that $\Psi \cap \mathcal{I} = \{\emptyset\}$. Therefore, $Int_\psi(I) = \{\emptyset\}$.
(2)$\Rightarrow$ (3): Let $x \in G \in \Psi$. Assume $x \notin G_\psi$ then there exists some $U_x \in \Psi(x)$ such that $G \cap U_x \in \mathcal{I}$. By (2), $x \in G \cap U_x = Int_\psi(G \cap U_x) = \{\emptyset\}$. Hence $x \in G_\psi$ and $G \subseteq G_\psi$.
(3)$\Rightarrow$ (4): Since $X \in \Psi$, then $X = X_\psi$.
(4)$\Rightarrow$ (1): $X = X_\psi = \{x \in X : U \cap X = U \notin \mathcal{I}$ for each $U \in \Psi(x)\}$. Hence $\Psi \cap \mathcal{I} = \{\emptyset\}$. □

Definition 4.7. Let $(X, m, \mathcal{I})$ be an ideal m-space and $A \subseteq X$. Then, A is said to be ψ_m-open in X if $A \subseteq int_\psi(A_\psi)$. Let τ_ψ denote the collection of all ψ_m-open sets in $(X, m, \mathcal{I})$.

Lemma 4.8. *Any element of $\mathcal{I} \setminus \{\emptyset\}$ can not be a member of τ_ψ.*

Proof. Let U be any element of $\mathcal{I} \setminus \{\emptyset\}$. Then $U_\psi = \emptyset$ which further implies $U \nsubseteq int_\psi(U_\psi)$. Hence $U \notin \tau_\psi$. □

Lemma 4.9. *Let $(X, m, \mathcal{I})$ be an ideal m-space, if $X = X_\psi$, then every ψ-open set is ψ_m-open.*

Proof. Let $(X, m, \mathcal{I})$ be an ideal m-space such that $X = X_\psi$. If U is any ψ-open set, then by Theorem 4.6 (3) $U \subseteq U_\psi$. This shows that $U \subseteq int_\psi(U_\psi)$. Hence U is ψ_m-open. □

Theorem 4.10. *Let $(X, m, \mathcal{I})$ be an ideal m-space, m be ψ-compatible with $\mathcal{I}$ and $\Psi \cap \mathcal{I} = \{\emptyset\}$. Let G be a m^*_ψ-open set such that $G = U - A$, where $U \in \Psi$ and $A \in \mathcal{I}$. Then $Cl_\psi(G_\psi) = Cl_\psi(G) = G_\psi = U_\psi = Cl_\psi(U) = Cl_\psi(U_\psi)$.*

Proof. (1) Let $G = U - A$, where $U \in \Psi$ and $A \in \mathcal{I}$. Since $\Psi \cap \mathcal{I} = \{\emptyset\}$, by Theorem 4.6 we have $U \subseteq U_\psi$. Hence by Lemma 2.2, $U_\psi = Cl_\psi(U_\psi) = Cl_\psi(U)$.

(2) Since G is m^*_ψ-open, $X - G = Cl^*_\psi(X - G)$ and hence $(X - G)_\psi \subseteq X - G$. By Lemma 3.3, $X_\psi - G_\psi \subseteq (X - G)_\psi$. But $\Psi \cap \mathcal{I} = \emptyset$ and by Theorem 4.6, $X_\psi = X$ and hence $X - G_\psi \subseteq (X - G)_\psi \subseteq X - G$. Therefore, $G \subseteq G_\psi$. Hence, $Cl_\psi(G) \subseteq Cl_\psi(G_\psi)$. Hence by Lemma 2.2, $G_\psi = Cl_\psi(G) = Cl_\psi(G_\psi)$.
(3) Again, $G \subseteq U$ implies that $G_\psi \subseteq U_\psi$. By Lemma 3.3, $G_\psi = (U - A)_\psi \supseteq U_\psi - A_\psi = U_\psi$ since $A \in \mathcal{I}$. Thus $U_\psi = G_\psi$.
By (1), (2) and (3), we obtain the result. □

Theorem 4.11. *Let $(X, m, \mathcal{I})$ be an ideal m-space and m be ψ-compatible with $\mathcal{I}$. Then for any $G \in \Psi$ and any subset A of X, $(G \cap A)_\psi = (G \cap A_\psi)_\psi = Cl_\psi(G \cap A_\psi)$.*

Proof. (1) Let $G \in \Psi$. Then by Lemma 3.5, $G \cap A_\psi = G \cap (G \cap A)_\psi \subseteq (G \cap A)_\psi$ and hence $(G \cap A_\psi)_\psi \subseteq ((G \cap A)_\psi)_\psi = (G \cap A)_\psi$ by Theorem 3.1 and Corollary 4.5.
(2) Now by using Lemma 2.2 and Theorem 4.4, we obtain $(G \cap (A - A_\psi))_\psi \subseteq G_\psi \cap (A - A_\psi)_\psi = G_\psi \cap \emptyset = \emptyset$. Moreover, $(G \cap A)_\psi - (G \cap A_\psi)_\psi \subseteq ((G \cap A) - (G \cap A_\psi))_\psi = (G \cap (A - A_\psi))_\psi = \emptyset$, which implies that $(G \cap A)_\psi \subseteq (G \cap A_\psi)_\psi$.
By (1) and (2), we obtain $(G \cap A)_\psi = (G \cap A_\psi)_\psi$.
By Lemma 2.2, $(G \cap A)_\psi = (G \cap A_\psi)_\psi \subseteq Cl_\psi(G \cap A_\psi)$. Also, in view of Lemma 3.5, we have $G \cap A_\psi \subseteq (G \cap A)_\psi$ and hence $Cl_\psi(G \cap A_\psi) \subseteq Cl_\psi((G \cap A)_\psi) = (G \cap A)_\psi$ by Lemma 2.2(3). Consequently, we obtain $(G \cap A_\psi)_\psi = (G \cap A)_\psi = Cl_\psi(G \cap A_\psi)$. □

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