Generalized Douglas-Weyl Finsler Metrics

Mohammad Hosein Emamian, Akbar Tayebi∗

Department of Mathematics, Faculty of Science University of Qom, Qom, Iran.
E-mail: hosein.emamian@gmail.com
E-mail: akbar.tayebi@gmail.com

Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl \((\alpha, \beta)\)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.


1. Introduction

Let \((M, F)\) be a Finsler manifold. In local coordinates, a curve \(c(t)\) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \(\ddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the spray coefficients [10]. \(F\) is called a Berwald metric, if \(G^i\) are quadratic in \(y \in T_x M\) for any \(x \in M\) or equivalently \(G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^j y^k\). As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. \(F\) is called a Douglas metric if \(G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^j y^k + P(x, y)y^i\).

A Finsler metric \(F\) is called generalized Douglas-Weyl metric (briefly, GDW-metric) if \(D^i_{jkl}|m y^m = T_{jkl} y^i\) holds for some tensor \(T_{jkl}\), where \(D^i_{jkl}|m\) denotes the horizontal covariant derivatives of \(D^i_{jkl}\) with respect to the Berwald...
connection of $F$ [8][18]. For a manifold $M$, let $G_{DW}(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $G_{DW}(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $H$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ is said to be of isotropic S-curvature if $S = (n + 1)cF$, where $c = c(x)$ is a scalar function on $M$. In [14], it is shown that every isotropic Berwald metric has isotropic S-curvature. In [4], Cheng-Shen show that every $(\alpha, \beta)$-metric with constant Killing 1-form has vanishing S-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha \pm \beta^2/\alpha + \epsilon \beta$ has vanishing S-curvature if and only if $\beta$ is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on $M$ [6]. In this paper, we are going to study generalized Douglas-Weyl $(\alpha, \beta)$-metrics with vanishing S-curvature.

**Theorem 1.1.** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that

$$F \neq c_3\alpha \left( \frac{\beta}{\alpha} \right)^{\frac{c_1}{c_2 + 1}} \left( \frac{\beta}{\alpha} \right)^{c_1}^{\frac{c_2}{c_2 + 1}} + c_2 + 1 \right)^{\frac{c_1}{c_2 + 1}} \quad \text{and} \quad F \neq d_1 \sqrt{\alpha^2 + d_2 \beta^2} + d_3 \beta,$$

where $c_1$, $c_2$, $d_3$, $d_1$, $d_2$ and $d_3$ are real constants. Let $F$ has vanishing S-curvature. Then $F$ is a GDW-metric if and only if it is a Berwald metric.

**2. Preliminary**

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{ij} \left\{ \left[ F^2 \right]_{x^i y^j} - \left[ F^2 \right]_{x^j} \right\}, \quad y \in T_x M.$$

The $G$ is called the spray associated to $F$.

Define $B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $E_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $B_y(u, v, w) := B^i_{jkl}(y)u^jv^kw^l \frac{\partial}{\partial x^i}$ and $E_y(u, v) := E_{jk}(y)u^jv^k$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^i_{jkm}.$$
Let 
\[ D^j_{ji} := \frac{\partial^3}{\partial y^j \partial y^i \partial y^k} (G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i). \]
It is easy to verify that 
\[ D := D^j_{ji} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l \]
is a well-defined tensor on slit tangent bundle \( TM_0 \). We call \( D \) the Douglas tensor. A Finsler metric with \( D = 0 \) is called a Douglas metric. The notion of Douglas metrics was proposed by Bácso-Matsumoto as a generalization of Berwald metrics [2].

The Douglas tensor \( D \) is a non-Riemannian projective invariant, namely, if two Finsler metrics \( F \) and \( \bar{F} \) are projectively equivalent, 
\[ G^i = \bar{G}^i + P y^i, \]
where \( P = P(x, y) \) is positively \( y \)-homogeneous of degree one, then the Douglas tensor of \( F \) is same as that of \( \bar{F} \). Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \), the Busemann-Hausdorff volume form 
\[ dV_F = \sigma_F(x) dx^1 \cdots dx^n \]
is defined by 
\[ \sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}([y^i] \in \mathbb{R}^n | F(y^i \frac{\partial}{\partial x^i}|_x) < 1)}. \]
Let \( G^i \) denote the geodesic coefficients of \( F \) in the same local coordinate system. The S-curvature is defined by 
\[ S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[ \ln \sigma_F(x) \right], \]
where \( y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M \). \( S \) is said to be isotropic if there is a scalar functions \( c = c(x) \) on \( M \) such that \( S = (n + 1)cF \).

For an \((\alpha, \beta)\)-metric \( F = \alpha \phi(s), s = \beta/\alpha \), put 
\[ \Phi := -(q - sq')[n \Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'', \]
where 
\[ q := \frac{\phi'}{\phi - s \phi'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'. \]
In [4], Cheng-Shen characterize \((\alpha, \beta)\)-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let \( F = \alpha \phi(s), s = \beta/\alpha \), be a non-Riemannian \((\alpha, \beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Suppose that \( \phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s \) for any constant \( c_1 > 0, c_2 \) and \( c_3 \). Then \( F \) is of isotropic S-curvature \( S = (n + 1)cF \) if and only if one of the following holds (a) \( \beta \) satisfies 
\[ r_{ij} = \varepsilon (b^2 a_{ij} - b_i b_j), \quad s_j = 0, \] (2.1)
where \( \varepsilon = \varepsilon(x) \) is a scalar function, \( b := \| \beta \|_\alpha \) and \( \phi = \phi(s) \) satisfies
\[
\Phi = -2(n + 1)k \frac{\phi \Delta^2}{b^2 - s^2},
\]
where \( k \) is a constant. In this case, \( S = (n + 1)cF \) with \( c = k\varepsilon \).

(b) \( \beta \) satisfies
\[
r_{ij} = 0, \quad s_j = 0
\]
In this case, \( S = 0 \).

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen’s paper is not complete \[4\]. Their result is correct for dimension \( n \geq 3 \). For the case dimension \( \text{dim}(M) = 2 \), see \[16\].

3. Proof of Main Results

Let \( F := \alpha \phi(s) \), \( s = \beta/\alpha \), be an \((\alpha, \beta)\)-metric on a manifold \( M \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) and \( \beta(y) = b_i(x)y^i \). Define \( b_{ij} \) by \( b_{ij} \beta^i := db_i - b_j \theta^j_i \), where \( \theta^i := dx^i \) and \( \theta^i_j := \tilde{\Gamma}^j_{ik} dx^k \) denote the Levi-Civita connection forms of \( \alpha \). Let
\[
r_{ij} := \frac{1}{2} \left[ b_{ij} + b_{ji} \right], \quad s_{ij} := \frac{1}{2} \left[ b_{ij} - b_{ji} \right],
\]
\[
r_{00} := r_{ij} y^j, \quad r_{00} := r_{ij} y^j y^i, \quad r_j := b^i r_{ij}, \quad t^i_j := s^i_l s^m_j
\]
\[
s_{00} := s_{ij} y^j, \quad s_j := b^i s_{ij}, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.
\]
Then \( \beta = b_i(x)y^i \) is a constant Killing one-form on \( M \) if \( r_{ij} = s_j = 0 \) hold. By definition, we have
\[
b_{ij} = s_{ij} + r_{ij}.
\]
Since \( y^i|_s = 0 \), then for a constant Killing 1-form \( \beta \) we have
\[
r_{00} = 0, \quad r_i + s_i = 0.
\]
For an \((\alpha, \beta)\)-metric \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), the following hold.

**Proposition 3.1.** Let \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), be an \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold \( M \) of dimension \( n \geq 3 \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a one-form on \( M \). Suppose that \( F \) is of vanishing S-curvature. Then \( F \) is a GDW-metric if and only if the following holds
\[
C_1 s_{j0|0} y^i - (C_2 y_j + C_3 b_j) y^i t_{00} = C_4 y_j s^i_{0|0} + C_5 (b_j s^i_{0|0} + s_{j0} s^i_0) + C_6 s^i_{j0} + C_7 (y_j t^i_0 + s_{j0} s^i_0) + C_8 b_j t^i_0,
\]
\[
(3.1)
\]
where

\[
C_1 := -(n+1)Q_\alpha + 2\beta Q_{\alpha\beta} \alpha^{-3} - [Q_{\alpha\alpha} + b^2 Q_{\beta\beta}] \alpha^{-2},
\]

\[
C_2 := (n+1) \left[ Q^2_{\alpha} + Q Q_{\alpha\alpha} - \alpha^{-1} Q_{\alpha\alpha} \right] \alpha^{-4} - 2 \left[ Q_{\alpha} Q_{\beta} + Q Q_{\alpha\beta} \right] \beta \alpha^{-5} + 2 \left[ 2Q_{\alpha} Q_{\alpha\beta} + Q_{\alpha\alpha} Q_{\beta} + Q Q_{\alpha\alpha\beta} \right] \beta \alpha^{-4} + b^2 \left[ 2Q_{\alpha} Q_{\beta} + Q_{\alpha} Q_{\beta\beta} \right] \alpha^{-3} + b^2 Q_{\alpha\beta\beta} + 3Q_{\alpha} Q_{\alpha\alpha} + Q Q_{\alpha\alpha\alpha} \alpha^{-3},
\]

\[
C_3 := (n+3) \left[ Q_{\alpha} Q_{\beta} + Q Q_{\alpha\beta} \right] \alpha^{-3} + 2 \left[ Q_{\alpha} Q_{\beta\beta} + Q Q_{\alpha\beta\beta} \right] \beta \alpha^{-3} + 2Q_{\alpha} Q_{\beta} + Q_{\beta} Q_{\alpha\alpha} + Q Q_{\alpha\alpha\beta} + 4\beta \alpha^{-1} Q_{\beta\beta} Q_{\alpha\beta} \alpha^{-2} + b^2 \left[ 3Q_{\beta\beta} Q_{\beta\beta} + Q Q_{\beta\beta\beta} \right] \alpha^{-2},
\]

\[
C_4 := -(n+1)Q_{\alpha} + 2\beta Q_{\beta\beta} \alpha^{-3} + 2 \left[ Q_{\alpha\alpha\beta} + Q_{\alpha\beta} \right] \alpha^{-2} + b^2 Q_{\alpha\beta\beta} + Q_{\alpha\alpha\alpha} \alpha^{-1},
\]

\[
C_5 := (n+3) \alpha^{-1} Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + 2\beta \alpha^{-1} Q_{\beta\beta} + b^2 Q_{\beta\beta\beta},
\]

\[
C_6 := (n+1) \alpha^{-1} Q_{\alpha} + Q_{\alpha\alpha} + 2\beta \alpha^{-1} Q_{\beta\beta} + b^2 Q_{\beta\beta\beta},
\]

\[
C_7 := (n+1) \alpha^{-3} Q_{\alpha\alpha} - (n+1) \alpha^{-2} (Q^2_{\alpha} + Q Q_{\alpha\alpha}) - 2\beta \alpha^{-2} Q_{\alpha\alpha\beta} + 2 \left[ Q_{\alpha} Q_{\beta} + Q_{\beta} Q_{\alpha\alpha} \right] \beta \alpha^{-3} - b^2 \left[ Q_{\alpha} Q_{\beta\beta} + 2Q_{\alpha\beta} Q_{\beta} \right] \alpha^{-1}
\]

\[
- 2 \left[ 2Q_{\alpha} Q_{\beta} + Q_{\beta} Q_{\alpha\alpha} \right] \beta \alpha^{-2} - b^2 \alpha^{-1} Q_{\alpha} Q_{\beta\beta} - 3\alpha^{-1} Q_{\alpha} Q_{\alpha\alpha} - 2\alpha^{-1} Q Q_{\alpha\alpha\alpha},
\]

\[
C_8 := -(n+3) \left[ Q Q_{\alpha\beta} + Q_{\alpha} Q_{\beta} \right] \alpha^{-1} - 2 \left[ Q_{\beta\beta} Q_{\alpha\beta} + Q Q_{\alpha\beta\beta} + Q_{\alpha} Q_{\beta\beta} \right] \beta \alpha^{-1} - b^2 \left[ Q Q_{\beta\beta\beta} + 3Q_{\beta\beta} Q_{\beta\beta} \right] - Q_{\beta\beta} Q_{\alpha\alpha} - Q Q_{\alpha\alpha\beta} - 2Q_{\alpha} Q_{\alpha\beta}.
\]

\[
Proof. \text{ Let } G^i \text{ and } G^i_{\alpha} \text{ denote the spray coefficients of } F \text{ and } \alpha, \text{ respectively, in the same coordinate system. Then, we have}
\]

\[
G^i = G^i_{\alpha} + Py^i + Q^i,
\]

where

\[
Q := \alpha q = \frac{\alpha \phi'}{\phi - s \phi'},
\]

\[
P := \alpha^{-1} \Theta (r_{00} - 2Qs_0), \quad Q^i := Q s^i_0 + \Psi (r_{00} - 2Qs_0)b^i,
\]

\[
\Theta = \frac{q - sq'}{2\Delta} = \frac{\phi' - s(\phi'' + \phi' \phi')}{2\phi (\phi - s \phi') + (b^2 - s^2) \phi''}
\]

\[
\Psi := \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{(\phi - s \phi') + (b^2 - s^2) \phi''}.
\]
By Lemma 2.1, we have $r_{00} = s_0 = 0$. Then (3.2) reduces to following
\begin{equation}
G^i = G^i_\alpha + Q s^i_0. \tag{3.3}
\end{equation}
Let $\nabla$ and $\partial$ denote the covariant differentiations with respect to $G^i$ and $G^i_\alpha$ respectively. Then by (3.3), we have
\begin{equation}
D^i_{jkl|m}y^m = D^i_{jkl|m}y^m - 2Q s^i_0 \frac{\partial D^i_{jkl}}{\partial y^p} + D^p_{jkl} \tilde{N}^i_p - D^i_{jkl} \tilde{N}^i_p
- D^i_{jlp} \tilde{N}^p_k - D^i_{jkp} \tilde{N}^p_l, \tag{3.4}
\end{equation}
where
\begin{equation}
D^i_{jkl|m}y^m = \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1}Q_{\alpha})(A_{jk}y_l + A_{kl}y_j + A_{jl}y_k)s^i_{0|0}
+ \alpha^{-3}Q_{\alpha}(A_{jks} s^i_{l|0} + A_{kl}s^i_{j|0} + A_{jl}s^i_{k|0})
+ \alpha^{-3}Q_{\alpha\beta}[A_{jk}b_l + A_{kl}b_j + A_{jl}b_k)s^i_{0|0}
+ (A_{jk}s_0 + A_{kl}s_{j0} + A_{jl}s_{k0})s^i_0
+ \alpha^{-2}Q_{\alpha\beta}[y_jy_kb_l + y_kb_jb_l + y_kb_kb_j)s^i_{0|0}
+ (y_jy_kb_0 + y_kb_jb_0 + y_kb_kb_0)s^i_0
+ \alpha^{-1}Q_{\alpha\beta}[y_jb_l + y_kb_lb_j]s^i_{0|0}
+ (y_jb_l + y_kb_l)s_k0 + (y_kb_jb_0)s_{l0}
+ (y_kb_0 + y_kb_k) [s^i_{0|0}]
+ \alpha^{-2}Q_{\alpha\alpha}(y_jy_k s^i_{l|0} + y_kb_0 s^i_{j|0} + y_kb_0 s^i_{k|0})
+ Q_{\beta\beta}(b_kb_0 s_{j0} + b_kb_0 s_{k0} + b_kb_0 s_{l0})s^i_0
+ \alpha^{-3}Q_{\alpha\alpha\alpha}y_jy_kb_0 s^i_{0|0}
+ \alpha^{-1}Q_{\alpha\beta}[y_kb_l + y_kb_j]s^i_{0|0}
+ (y_kb_l + y_kb_k)s_j0 + (y_kb_0 + y_kb_k) s^i_{j|0} + (y_kb_j + y_kb_l) s^i_{k|0}
+ (y_kb_l + y_kb_k) s^i_{l|0} + (y_kb_0 + y_kb_k) s^i_{j|0} + (y_kb_j + y_kb_l) s^i_{k|0}
+ (y_kb_l + y_kb_k) [s^i_{l|0}]
+ Q_{\beta\beta}[b_kb_k s^i_{l|0} + b_kb_k s^i_{j|0} + b_kb_k s^i_{k|0} + (s_{j0}b_k + b_kb_0) s^i_{l}]
+ (s_{k0}b_l + b_kb_0) s^i_{j} + (b_kb_0 + b_kb_0) s^i_{k} + Q_{\beta\beta}b_k b_l s^i_{0|0} \tag{3.5}
\end{equation}
and
\begin{align}
A_{ij} &= \alpha^2 a_{ij} - y_{ij}, \tag{3.6}
\tilde{N}^i_p &= Q s^i_{0|0} + [\alpha^{-1}Q_{\alpha}y_p + Q_{\beta}b_p] s^i_{0,} \tag{3.7}
\frac{\partial D^i_{jkl}}{\partial y^p} &= Q_{ijkl} s^i_{0} + Q_{jkl} s^i_{0|0} + Q_{jl} s^i_{k|0} + Q_{jkl} s^i_{0|0}. \tag{3.8}
\end{align}
Let $F$ be a GDW-metric. Then there exists a tensor $D^i_{jkl}$ such that
\begin{equation}
D^i_{jkl|m}y^m = D^i_{jkl}y^i. \tag{3.9}
\end{equation}
By (3.4), we have

\[
D_{jkl} y^i = D'_{jkl|m} y^m - 2Q \frac{\partial D'_{jkl}}{\partial y^p} s^p_0 + D'_{jkl} N^p_i - D'_{pkl} \tilde{N}^p_j - D'_{jpl} \tilde{N}^p_k - D'_{jkl} \tilde{N}^p_l. \tag{3.9}
\]

By contracting (3.9) with \( y_i \) and using (3.5), (3.7) and (3.8) we get the following

\[
D_{jkl} = D_1 \left[ A_{jkl} s_{i0} + A_{kl} s_{j0} + A_{jl} s_{k0} \right] \\
+ D_2 \left[ y_j y_k s_{i0} + y_k y_j s_{i0} + y_j y_l s_{k0} \right] \\
+ D_3 \left[ (y_j b_k + y_k b_j) s_{i0} + (y_j b_l + y_l b_j) s_{k0} \right] \\
+ D_4 \left[ b_j b_k s_{i0} + b_k b_j s_{k0} + b_j b_k s_{l0} \right] \\
+ D_5 \left[ A_{jkl} y_i + A_{kl} y_j + A_{jl} y_k \right] t_{00} \\
+ D_6 \left[ A_{jkl} b_i + A_{kl} b_j + A_{jl} b_k \right] t_{00} \\
+ D_7 \left[ y_j y_k b_i + y_k y_j b_i + y_j y_l b_i \right] t_{00} \\
+ D_8 \left[ y_j b_k b_i + y_k b_j b_i + y_j b_l b_i \right] t_{00} \\
+ D_9 y_j y_k y_l t_{00} + D_{10} b_j b_k b_l t_{00} \\
+ D_{11} \left[ y_j s_{j0} s_{k0} + y_j s_{k0} s_{l0} + y_k s_{j0} s_{l0} \right] \\
+ D_{12} \left[ b_j s_{j0} s_{k0} + b_j s_{k0} s_{l0} + b_k s_{j0} s_{l0} \right]. \tag{3.10}
\]

where

\[
D_1 := -\alpha^{-5} Q_\alpha, \\
D_2 := -\alpha^{-4} Q_{\alpha\alpha}, \\
D_3 := -\alpha^{-3} Q_{\alpha\beta}, \\
D_4 := -\alpha^{-2} Q_{\beta\beta}, \\
D_5 := -\alpha^{-6} Q_\alpha^2 - \alpha^{-6} Q_{\alpha\alpha} + \alpha^{-7} Q_\alpha, \\
D_6 := -\alpha^{-5} Q_\alpha Q_\beta - \alpha^{-5} Q_{\alpha\beta}, \\
D_7 := -\alpha^{-4} Q_{\alpha\alpha} Q_\beta - 2\alpha^{-4} Q_{\alpha\beta} Q_\alpha - \alpha^{-4} Q_{\alpha\alpha\beta}, \\
D_8 := -\alpha^{-3} Q_{\beta\beta} Q_\alpha - 2\alpha^{-3} Q_{\alpha\beta} Q_\beta - \alpha^{-3} Q_{\alpha\beta\beta}, \\
D_9 := -3\alpha^{-3} Q_{\alpha\alpha} Q_\alpha - \alpha^{-5} Q_{\alpha\alpha\alpha}, \\
D_{10} := -3\alpha^{-2} Q_{\beta\beta} Q_\beta - \alpha^{-2} Q_{\beta\beta\beta}, \\
D_{11} := -2\alpha^{-3} Q_{\alpha\beta} + 2\alpha^{-3} Q_\alpha^2 + 2\alpha^{-4} Q_{\alpha\alpha} - 2\alpha^{-5} Q_\alpha, \\
D_{12} := -2\alpha^{-2} Q_{\beta\beta} + 2\alpha^{-3} Q_{\alpha\beta} + 2\alpha^{-3} Q_\alpha Q_\beta.
\]
Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a^{kl}$, we get (3.1). □

**Proof of Theorem 1.1:** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y^j$, we get

$$-\alpha QQ_{\alpha\alpha}t^{00} = 0.$$  \hspace{1cm} (3.11)

If $Q_{\alpha\alpha\alpha} = 0$ then

$$Q = c_1 \alpha + c_2 \frac{\alpha^2}{\beta},$$

where $c_1$ and $c_2$ are real constants. Thus, we get

$$F = c_3 \alpha \left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{c_2 + 1}} \left(c_1 \frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{c_2}}.$$

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t^{00} = 0$ which results that $s_i^0 = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric. □

**Acknowledgments**

The authors are very grateful to the anonymous referee for his or her comments and suggestions.

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