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Convergence of an Approach for Solving Fredholm Functional Integral Equations

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Abstract. In this work, we give a product Nyström method for solving a Fredholm functional integral equation (FIE) of the second kind. With this method solving FIE reduce to solving an algebraic system of equations. Then we use some theorems to prove the existence and uniqueness of the system. Finally we investigate the convergence of the method.

Keywords: Functional integral equation, Fredholm, Product Nyström method, Lagrange interpolation, Convergence.

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1. Introduction

Functional integral equations have a significant role in important branches of linear and nonlinear functional analysis and their applications. Equations of such a type are often arise in physics, mechanics, control theory, economics and engineering, for instance [16]-[22]. Functional integral equations have been studied widly in several papers and monographs [23]-[28].

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Consider the following Fredholm functional integral equation of the second kind

$$y(x) - p(x)y(h(x)) - \lambda \int_a^b k(x,t)y(t)dt = g(x), \quad a \le x \le b, \tag{1.1}$$

where $p(x), h(x), g(x)$ and $k(x,t)$ are known functions and λ is known parameter and $y(x)$ is the unknown function to be determined. Here, we suppose that the kernel $k(x,t)$ is discontinuous at finite points and the unknown function $y(x)$ is continuous. Usually we can write the discontinuous kernel $k(x,t)$ as $k(x,t) = p(x,t)\bar{k}(x,t)$, where $p(x,t)$ and $\bar{k}(x,t)$ are ill-posed and well-posed functions with respect to their arguments, respectively. In the sequel, we suppose that we have such representation.

2. The Method

We divide the interval $[a,b]$ into N subinterval such that

$$h = \frac{b-a}{N}, \quad x_i = t_i = a + ih, \quad i = 0, 1, \ldots, N,$$

and N is multiplication of integer $s \ge 1$. The integral part of (1.1) can be write as

$$\begin{aligned} \int_a^b k(x,t)y(t)dt &= \int_a^b p(x,t)\bar{k}(x,t)y(t)dt \\ &= \sum_{j=0}^{\frac{N-s}{s}} \int_{t_{sj}}^{t_{sj+s}} p(x,t)\bar{k}(x,t)y(t)dt, \end{aligned} \tag{2.1}$$

where choosing s is depend upon the used integration method, e.g. $s = 1$ in the Trapezoidal rule and $s = 2$ in the Simpson rule.

In the product Nyström method, the well-posed part of integration over every subinterval

$$I_j = [t_{sj}, t_{sj+s}], \quad j = 0, 1, \ldots, \frac{N-s}{s},$$

approximated by using Lagrange polynomials of degree s which interpolates at points

$$t_{sj}, t_{sj+1}, \ldots, t_{sj+s}.$$

If we use the notation $L_{N,j}$ for the Lagrange polynomial at the subinterval I_j, we have:

$$\bar{k}(x,t)y(t)|_{I_j} \simeq L_{N,j} = \sum_{i=sj}^{sj+s} l_{i,j}(t)\bar{k}(x,t_i)y(t_i), \quad j = 0, 1, \ldots, \frac{N-s}{s}, \tag{2.2}$$

where $l_{i,j}(t)$ denote the Lagrange polynomial of degree s at the interval I_j, and is defined as

$$l_{i,j}(t) = \prod_{k=sj,\ k\neq i}^{sj+s} \frac{t - t_k}{t_i - t_k}, \quad i = sj, sj+1, \ldots, sj+s; \quad j = 0, 1, \ldots, \frac{N-s}{s},$$

so, for every subinterval I_j,

$$\int_{t_{sj}}^{t_{sj+s}} k(x,t)y(t)dt \simeq \int_{t_{sj}}^{t_{sj+s}} L_{N,j}p(x,t)dt,$$

and the approximation error can be find by

$$e = \Big| \int_{t_{sj}}^{t_{sj+s}} k(x,t)y(t)dt - \int_{t_{sj}}^{t_{sj+s}} L_{N,j}p(x,t)dt \Big|.$$

By substituting the interpolation polynomial $L_{N,j}$ in the relation (2.1), the approximate value of integral part of the equation (1.1) reduce to

$$\begin{aligned}
\int_a^b k(x,t)y(t)dt &\simeq \sum_{j=0}^{\frac{N-s}{s}} \int_{t_{sj}}^{t_{sj+s}} L_{N,j}p(x,t)dt \\
&= \sum_{j=0}^{\frac{N-s}{s}} \sum_{i=sj}^{sj+s} \left(\bar{k}(x,t_i)y(t_i) \int_{t_{sj}}^{t_{sj+s}} l_{i,j}(t)p(x,t)dt \right) \\
&= \sum_{i=0}^{s} \left(\bar{k}(x,t_i)y(t_i) \int_{t_0}^{t_s} l_{i,0}(t)p(x,t)dt \right) \\
&+ \sum_{i=s}^{2s} \left(\bar{k}(x,t_i)y(t_i) \int_{t_s}^{t_{2s}} l_{i,1}(t)p(x,t)dt \right) \\
&+ \cdots + \sum_{i=N-s}^{N} \left(\bar{k}(x,t_i)y(t_i) \int_{t_{N-s}}^{t_N} l_{i,\frac{N-s}{s}}(t)p(x,t)dt \right),
\end{aligned}$$

thus, for $i = sj$ $(j = 1, \ldots, \frac{N-s}{s})$, we have two integral and one for other is. After collecting we can rewrite the above integral as

$$\int_a^b k(x,t)y(t)dt \simeq \sum_{i=0}^{N} w_i(x)\bar{k}(x,t_i)y(t_i), \tag{2.3}$$

where

$$
\begin{aligned}
w_0(x) &= \int_{t_0}^{t_s} \prod_{k=1}^{s} \frac{t-t_k}{t_0-t_k} p(x,t)dt \\
w_{sj}(x) &= \int_{t_{sj-s}}^{t_{sj}} \prod_{k=sj-s}^{sj-1} \frac{t-t_k}{t_{sj}-t_k} p(x,t)dt + \int_{t_{sj}}^{t_{sj+s}} \prod_{k=sj+1}^{sj+s} \frac{t-t_k}{t_{sj}-t_k} p(x,t)dt, \\
&\qquad \text{for } \; j = 1,2,\ldots,\frac{N-s}{s} \\
w_{sj+m}(x) &= \int_{t_{sj}}^{t_{sj+s}} \prod_{k=sj,\; k\neq sj+m}^{sj+s} \frac{t-t_k}{t_{sj+m}-t_k} p(x,t)dt, \\
&\qquad \text{for } \; j = 1,2,\ldots,\frac{N-s}{s}, \quad m = 1,2,\ldots,s-1 \\
w_N(x) &= \int_{t_{N-s}}^{t_N} \prod_{k=N-s}^{N-1} \frac{t-t_k}{t_N-t_k} p(x,t)dt. \qquad (2.4)
\end{aligned}
$$

Now, we approximate $y(h(x))$ as

$$y(h(x)) \simeq \sum_{i=0}^{N} l_{i,N}(h(x)) y(x_i),$$

where $l_{i,N}(h(x))$ is defined as the following

$$l_{i,N}(h(x)) = \prod_{k=0,\; k\neq i}^{N} \frac{h(x)-x_k}{x_i-x_k}.$$

Substituting these relations in (1.1), we have an approximate to the integral equation (1.1) as the following

$$y_N(x) - p(x) \sum_{i=0}^{N} l_{i,N}(h(x)) y_N(x_i) - \lambda \sum_{i=0}^{N} w_i(x) \bar{k}(x,t_i) y_N(t_i) = g(x), \quad a \le x \le b, \tag{2.5}$$

where $y_N(x)$ shows the approximate solution from product Nyström method for $y(x)$. From $x_i = t_i = a + ih$, (2.5) can be rewritten as the following

$$y_N(x) - \sum_{i=0}^{N} \Big\{ p(x) l_{i,N}(h(x)) + \lambda w_i(x) \bar{k}(x,t_i) \Big\} y_N(t_i) = g(x), \quad a \le x \le b. \tag{2.6}$$

Theorem 2.1. *For $x = x_j = t_j, \quad j = 0,1,\ldots,N$, solving (2.6) is equal to solving the following system of linear algebraic equation*

$$y_N(t_j) - \sum_{i=0}^{N} \Big\{ p(t_j) l_{i,N}(h(t_j)) + \lambda w_{ij} \bar{k}(t_j,t_i) \Big\} y_N(t_i) = g(t_j), \;\; j = 1,2,\ldots,N, \tag{2.7}$$

where $w_{ij} = w_i(t_j)$, and the vector $Y_N = [y_N(t_0),\ldots,y_N(t_N)]^T$ is unknown.

Proof. See [2]. □

The unknown function $Y_N(x)$ can be calculated from (2.6) by having approximate values of $y_N(t_i)$ as following

$$y_N(x) = g(x) + \sum_{i=0}^{N} \Big\{ p(x) l_{i,N}(h(x)) + \lambda w_i(x) \bar{k}(x, t_i) \Big\} y_N(t_i). \tag{2.8}$$

The (2.8) is called Nyström interpolation formula.

In the next section we prove the existence and uniqueness of the system of linear algebraic equations (2.7).

3. System of Linear Algebraic Equations

In this section we discuss the necessary conditions for existence and uniqueness of the system of linear algebraic equations (2.7) in the Banach space L^∞. For easy discussing, consider the following functional integral equation of the second kind

$$y(x) - p(x)y(h(x)) - \lambda \int_{-1}^{1} p(x,t)y(t)dt = g(x), \quad -1 \le x \le 1, \tag{3.1}$$

where $g(x)$ is a known continuous function and $p(x,t)$ is a weakly singular kernel. By using the product Nyström method for equation (3.1) at nodes $\{x_j\}_{j=0}^{N}$, we have the following linear algebraic system

$$y_N(t_j) = g(t_j) + \sum_{i=0}^{N} \Big\{ p(t_j) l_{i,N}(h(t_j)) + w_{ij} \Big\} y_N(t_i), \quad j = 0, 1, \ldots, N.$$

Theorem 3.1. *Suppose a function $f(x)$ is interpolated on the interval $[a,b]$ by a polynomial $p_n(x)$ whose degree does not exceed n. Suppose further that f is arbitrarily often differentiable on $[a,b]$ and there exists M such that $|f^{(i)}(x)| \le M$ for $i = 0, 1, 2, \ldots$ and any $x \in [a,b]$. It can be shown without additional hypotheses about the location of the support abscissas $x_i \in [a,b]$, that $p_n(x)$ converges uniformly on $[a,b]$ to $f(x)$ as $n \to \infty$.*

Proof. See [8]. □

Theorem 3.2. *Let $\{x_i\}_{i=sj}^{sj+s}$, $j = 0, 1, \ldots, \frac{N-s}{s}$ be the $s+1$ support points of Lagrange polynomial of degree s on subinterval $[t_{sj}, t_{sj+s}]$. Moreover suppose that the weakly singular kernel $p(x,t)$ satisfies the condition $p(x,t) \in L_q$ for $q > 1$ and let $l_{N,j}(f,t)$ denotes the interpolating Lagrange polynomial of degree $\le s$ that interpolate function f at the nodes $\{x_i\}_{i=sj}^{sj+s}$. Then, for every function $f \in C[-1,1]$ which satisfies the hypothesis of theorem (3.1), we have*

$$\lim_{N \to \infty} \left\| \int_{-1}^{1} p(x,t)f(t)dt - \int_{-1}^{1} p(x,t) l_{N,j}(f,t)dt \right\|_\infty = 0 \tag{3.2}$$

Proof.

$$\begin{aligned}\Big\|\int_{-1}^{1} p(x,t)f(t)dt - \int_{-1}^{1} p(x,t)l_{N,j}(f,t)dt\Big\|_{\infty} &= \sup_x \Big|\int_{-1}^{1} p(x,t)\Big(f(t)-l_{N,j}(f,t)\Big)dt\Big| \\ &= \sup_x \Big|\sum_{j=0}^{\frac{N-s}{s}}\int_{t_{sj}}^{t_{sj+s}} p(x,t)\Big(f(t)-l_{N,j}(f,t)\Big)dt\Big| \\ &\le \sup_x \sum_{j=0}^{\frac{N-s}{s}}\Big(\int_{t_{sj}}^{t_{sj+s}} |p(x,t)|\cdot|f(t)-l_{N,j}(f,t)|dt\Big).\end{aligned}$$

Applying Hölder inequality for $q, q' > 1$, $(\frac{1}{q}+\frac{1}{q'}=1)$, we have

$$\sup_x \sum_{j=0}^{\frac{N-s}{s}}\Big(\int_{t_{sj}}^{t_{sj+s}} |p(x,t)|\cdot|f(t)-l_{N,j}(f,t)|dt\Big) \le \sup_x \sum_{j=0}^{\frac{N-s}{s}} \|p\|_{L_q}\cdot\|f-l_{N,j}\|_{L_{q'}}.$$

Also from theorem 3.1 $\lim_{N\to\infty} l_{N,j}(f,t) = f(t)$. Thus $\|f-l_{N,j}\|_{L_{q'}} \to 0$ as $N\to\infty$. Also according to the assumption $p\in L_q$, we obtain that

$$\lim_{N\to\infty}\sup_x \Big|\int_{-1}^{1} p(x,t)f(t)dt - \int_{-1}^{1} p(x,t)l_{N,j}(f,t)dt\Big| = 0,$$

and this complete the proof. □

For proving the existence and uniqueness of the solution of the linear algebraic system (2.7), we use the Banach fixed point theorem. For providing the conditions of the Banach fixed point theorem, we define the operator $\bar{T}$ as following

$$\bar{T}(y_j)_N = T(y_j)_N + g_j,$$

where, $g_j = g(t_j)$, $(y_j)_N = y_N(t_j)$,

$$T(y_j)_N = \sum_{i=0}^{N}\Big\{p(t_j)l_{i,N}(h(t_j)) + w_{ij}\Big\}(y_i)_N, \quad j = 0,1,\ldots,N.$$

We will show that $\bar{T}$ is a contraction in Banach space L^∞. For this, we need the following lemmas.

Lemma 3.3. *For a given set of nodes $\{x_i\}_{i=0}^N$ defined as in theorem 3.2, let $l_{i,j}(t)$ denotes the corresponding Lagrange polynomial on subinterval $[t_{sj}, t_{sj+s}]$. Then $\sup_N \sum_{j=0}^N |w_{ij}|$ exists for all functions $p\in L_q$, $(q>1)$ with $\|p\|_{L_q} = \Big\{\int_{-1}^{1}|p(x,t)|^q dt\Big\}^{\frac{1}{q}}$.*

Proof. Applying Hölder inequality for every $p(x,t) \in L_q$ $(q, q' > 1,\ \frac{1}{q} + \frac{1}{q'} = 1)$, we have

$$\begin{aligned} \left| \int_{t_{sj}}^{t_{sj+s}} p(x,t) l_{i,j}(t) dt \right| &\leq \int_{t_{sj}}^{t_{sj+s}} |p(x,t)| \cdot |l_{i,j}(t)| dt \\ &\leq \|p\|_{L_q} . \|l_{i,j}\|_{L_{q'}}. \end{aligned}$$

Since $p \in L_q$ and also for all j, $l_{i,j}$ is a polynomial of degree s and hence belong to $L_{q'}$ so

$$\exists E_1 > 0, \qquad \left| \int_{t_{sj}}^{t_{sj+s}} p(x,t) l_{i,j}(t) dt \right| \leq E_1,$$

so,

$$\exists E_2 > 0, \qquad \sum_{j=0}^{\frac{N-s}{s}} \Big| \int_{t_{sj}}^{t_{sj+s}} p(x,t) l_{i,j}(t) | dt \Big| \leq E_2.$$

Therefore from the relation (2.4), we get

$$\exists E > 0, \qquad \sum_{j=0}^{N} |w_{ij}| \leq E.$$

Since this inequality satisfies for all N, thus $\sup_N \sum_{j=0}^{N} |w_{ij}|$ exists. □

Lemma 3.4. *Assume that we have the same assumptions of the lemma 3.3, and let the kernel p satisfies the conditions*

$$\begin{cases} p \in L_q, & q > 1; \\ \lim_{x_j \to x_k} \|p(x_j,t) - p(x_k,t)\|_{L_q} = 0, & \forall x_j, x_k \in [-1,1], \end{cases} \tag{3.3}$$

then

$$\lim_{x_j \to x_k} \sup_N \sum_{i=0}^{N} |w_i(x_j) - w_i(x_k)| = 0. \tag{3.4}$$

Proof. Suppose that $x_j, x_k \in [-1,1]$ are arbitrary points of partition points set, then for all functions $p(x,t) \in L_q$ we have

$$\begin{aligned} &\sup_N \sum_{r=0}^{\frac{N-s}{s}} \left| \int_{t_{sr}}^{t_{sr+s}} l_{i,r}(t) \Big(p(x_j,t) - p(x_k,t) \Big) dt \right| \\ &\leq \sup_N \sum_{r=0}^{\frac{N-s}{s}} \int_{t_{sr}}^{t_{sr+s}} |p(x_j,t) - p(x_k,t)| . |l_{i,r}(t)| dt. \end{aligned}$$

By using Hölder inequality for $q, q' > 1(\frac{1}{q} + \frac{1}{q'} = 1)$, we get

$$\sup_N \sum_{r=0}^{\frac{N-s}{s}} \left| \int_{t_{sr}}^{t_{sr+s}} l_{i,r}(t)\Big(p(x_j,t) - p(x_k,t)\Big)dt \right|$$
$$\leq \sup_N \sum_{r=0}^{\frac{N-s}{s}} \Big\{ \int_{t_{sr}}^{t_{sr+s}} |p(x_j,t) - p(x_k,t)|^q dt \Big\}^{\frac{1}{q}} . \|l_{i,r}\|_{L'_q}.$$

Since $l_{i,r}$ is a polynomial of degree s for every r, thus $l_{i,r} \in L_{q'}$. Also we have $\lim_{x_j \to x_k} \|p(x_j,t) - p(x_k,t)\|_{L_q} = 0$. Therefore the relation (2.4) completes the proof. □

Lemma 3.5. *If* $\sup_j |g(t_j)|, \sup_j |p(t_j)|$ *and* $\sup_N \sum_{j=0}^N |w_{ij}|$ *exist, then* $\bar{T}$ *is an operator from* L^∞ *into itself.*

Proof. Let U be the set of all functions $y_N = (y_j)_N$ in L^∞ such that

$$\forall y_N, \quad \|y_N\|_{L^\infty} = \sup_j |(y_j)_N| \leq \beta,$$

where β is constant. We define operator norm in Banach space L^∞ as

$$\|\bar{T}y_N\|_{L^\infty} = \sup_j |\bar{T}(y_j)_N|. \tag{3.5}$$

From the definition of the operator $\bar{T}$ we have

$$|\bar{T}(y_j)_N| \leq |p_j| \sum_{i=0}^N |l_{i,N}(h_j)| \sup_i |(y_i)_N| + \sum_{i=0}^N |w_{ij}| \sup_i |(y_i)_N| + \sup_j |g_j|,$$

from the lemma assumptions

$$\exists H_1, \ \sup_j |g_j| \leq H_1,$$
$$\exists H_2, \ \sup_j |p_j| \leq H_2,$$

and

$$\exists E_1, \ \sup_N \sum_{i=0}^N |w_{ij}| \leq E_1.$$

Since $l_{i,N}$ is a polynomial of degree N for every i, thus

$$\exists E_2; \ \sup_j \sum_{i=0}^N |l_{i,N}(h_j)| \leq E_2.$$

So

$$\sup_j |\bar{T}(y_j)_N| \leq H_2E_2\|(y_j)_N\|_{L^\infty} + E_1\|(y_j)_N\|_{L^\infty} + H_1.$$

Since this inequality satisfies for all j, therefore

$$\|\bar{T}(y_j)_N\|_{L^\infty} \leq \sigma_1\|(y_j)_N\|_{L^\infty} + H_1,$$

where $\sigma_1 = H_2E_2 + E_1$. So $\bar{T}y_N \in L^\infty$ i.e. $\|\bar{T}y_N\|_{L^\infty} \leq \beta$. Also $\|\bar{T}y_N\|_{L^\infty} \leq \sigma_1\beta + H_1$. By comparing two last relations we have:

$$\sigma_1\beta + H_1 \leq \beta \Rightarrow (H_2E_2 + E_1)\beta + H_1 \leq \beta \Rightarrow \frac{H_1}{\beta} \leq 1 - H_2E_2 - E_1.$$

Since $H_1 > 0$ and $\beta > 0$ thus $H_2E_2 + E_1 < 1$, that is $\sigma_1 < 1$. Furthermore the operator T is bounded because $|Ty_N|_{L^\infty} \leq \sigma_1\|y_N\|_{L^\infty}$. Therefore from definition of $\bar{T}$ we conclude that $\bar{T}$ is a bounded operator. □

Lemma 3.6. *With the conditions of lemma 3.5, $\bar{T}$ is a contraction operator in Banach space L^∞.*

Proof. According to the definition of operator $\bar{T}$, for functions $y_N = (y_j)_N$ and $z_N = (z_j)_N$ from L^∞ we have:

$$|\bar{T}(y_j)_N - \bar{T}(z_j)_N| \leq \sum_{i=0}^{N} |p_j||l_{i,N}(h_j)| + |w_{ij}| \sup_j |(y_j)_N - (z_j)_N|.$$

By using the conditions of lemma 3.5, we get

$$|\bar{T}(y_j)_N - \bar{T}(z_j)_N| \leq \big(H_2E_2 + E_1\big)\|(y_j)_N - (z_j)_N\|_{L^\infty}.$$

This inequality satisfies for all j, so

$$\|\bar{T}y_N - \bar{T}z_N\|_{L^\infty} \leq \sigma_1\|y_N - z_N\|_{L^\infty}.$$

Consequently under the condition of $\sigma_1 < 1$, $\bar{T}$ is a contraction operator in Banach space L^∞. □

Theorem 3.7. *With the assumptions of lemma 3.5, the system of equations (2.7) has a unique solution in Banach space L^∞.*

Proof. According to the Banach fixed point theorem, since $\bar{T}$ is a contraction operator, thus the system of equations (2.7) has a unique solution in L^∞. □

4. Convergence of the Method

By applying the product Nyström method for solving the integral equation (3.1), we obtain the approximate solution $y_N(x)$ as follows:

$$y_N(x) = g(x) + \sum_{i=0}^{N} \Big\{p(x)l_{i,N}(h(x)) + w_i(x)\Big\}y_N(x_i),$$

where $w_i(x)$ can be obtain from relation (2.4).

Definition 4.1. The product Nyström method is convergent of order r in $[-1, 1]$, if and only if for sufficiently large N, there is a constant $c > 0$ independent from N such that

$$\|y(x) - y_N(x)\|_{L^\infty} \leq cN^{-r}.$$

Local approximate error obtains from

$$y(x)-y_N(x)=\sum_{i=0}^{N}\Big[w_i(x)+p(x)l_{i,N}(h(x))\Big]\Big[y(x_i)-y_N(x_i)\Big]+e_N(x).$$

So

$$(y-y_N)-\sum_{i=0}^{N}\Big[w_i(x)+p(x)l_{i,N}(h(x))\Big]\Big[y(x_i)-y_N(x_i)\Big]=e_N(x). \qquad (4.1)$$

Now we define linear operator A_N as

$$\begin{aligned} A_N &: C[-1,1]\to C[-1,1]\\ A_Nf(x) &= \sum_{i=0}^{N}\Big[w_i(x)+p(x)l_{i,N}(h(x))\Big]f(x_i),\quad f\in C[-1,1],\quad x\in[-1,1]. \end{aligned}$$

So we can rewrite the relation (4.1) as follow

$$\big(I-A_N\big)\Big(y(x)-y_N(x)\Big)=e_N(x),$$

thus

$$\big(y-y_N(x)\big)=\Big(I-A_N\Big)^{-1}e_N(x),$$

since this satisfies for every x, therefore

$$\sup_x\Big|\big(y-y_N\big)(x)\Big|\le\sup_x\Big|\big(I-A_N\big)^{-1}e_N(x)\Big|,$$

so

$$\big\|y-y_N\big\|_\infty\le\Big\|\big(I-A_N\big)^{-1}\Big\|_\infty.\|e_N\|_\infty. \qquad (4.2)$$

Theorem 4.2. *If we define integral operator A' as follow*

$$\begin{aligned} A' &: C[-1,1]\to C[-1,1],\\ A'f(x) &= \int_{-1}^{1}p(x,t)f(t)dt,\quad f\in C[-1,1],\quad x\in[-1,1], \end{aligned}$$

then the integral operator A' with weakly singular kernel of $p(x,t)$ is a compact operator on $C[-1,1]$.

Proof. See [1]. □

Now we define operator A as follow

$$\begin{aligned} A &: C[-1,1]\to C[-1,1]\\ Af(x) &= \int_{-1}^{1}p(x,t)f(t)dt+\sum_{i=0}^{N}p(x)l_{i,N}(h(x))f(x_i), \end{aligned}$$

Since operator A' is compact, thus the operator A is compact too. Also according to the definition of operators A and A_N we have

$$\|A-A_N\|_\infty=\left\|\int_{-1}^{1}p(x,t)f(t)dt-\int_{-1}^{1}p(x,t)L_{N,j}(f,t)dt\right\|_\infty, \qquad (4.3)$$

where $L_{N,j}$ is the Lagrange interpolation polynomial of the continues function f. Also from (4.2) the right hand side of relation (4.3) converges to zero, when $N \to \infty$. So $\lim_{N\to\infty} \|A - A_N\|_\infty = 0$.

For studying behavior of $\|\big(I - A_N\big)^{-1}\|_\infty$, we have the following theorem from [7].

Theorem 4.3. *Suppose that $A : C[-1,1] \to C[-1,1]$ is a linear, compact operator and A_N is a sequence of linear, bounded operators such that $\lim_{N\to\infty} \|A - A_N\|_\infty = 0$, then the inverse operator $\big(I - A_N\big)^{-1} : C[-1,1] \to C[-1,1]$ exists for all sufficiently large N, and there exist constant $c > 0$ independent of N such that $\|\big(I - A_N\big)^{-1}\|_\infty \leq c$.*

From (4.2), $\lim_{N\to\infty} e_N = 0$, so we have the following theorem for convergence of product Nyström method:

Theorem 4.4. *Under the conditions of theorem 4.3 the approximate solution $y_N(x)$ from product Nyström method is uniformly convergent to exact solution $y(x)$.*

Proof. The proof follows from (4.3) and theorem 4.3. □

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