Properties of Central Symmetric $X$-Form Matrices

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Abstract. In this paper we introduce a special form of symmetric matrices that is called central symmetric $X$-form matrix and study some of their properties, the inverse eigenvalue problem and inverse singular value problem for these matrices.

Keywords: Inverse eigenvalue problem, Inverse singular value problem, eigenvalue, singular value.


1. Introduction

In [3,4] H. Pickman et. studied the inverse eigenvalue problem of symmetric tridiagonal and symmetric bordered diagonal matrices. In this paper we introduce the odd and even order central symmetric $X$-form matrix for an integer number $n$ respectively as below:
suppose

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\[ A_n = \begin{pmatrix} a_n & b_n \\ \vdots & \ddots & \ddots \\ a_2 & b_2 & a_1 \\ b_2 & a_1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ b_n & \cdots & \cdots & a_n \end{pmatrix}_{(2n-1) \times (2n-1)}, \]

\[ B_n = \begin{pmatrix} a_n & b_n \\ \vdots & \ddots & \ddots \\ a_2 & b_2 & a_1 \\ b_2 & a_1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ b_n & \cdots & \cdots & a_n \end{pmatrix}_{(2n) \times (2n)} \]

For a given \( 2n - 1 \) real numbers such as

\[
\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_2^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)},
\]

or for a given \( 2n \) real numbers such as

\[
\lambda_1^{(2n)} < \lambda_1^{(2(n-1))} < \ldots < \lambda_1^{(2)} < \lambda_2^{(2)} < \ldots < \lambda_2^{(2(n-1))} < \lambda_2^{(2n)},
\]

we construct a matrix \( A_n \) such that \( \lambda_1^{(j)} \) and \( \lambda_1^{(j)} \) are the maximal and minimal eigenvalues of submatrix \( A_j \) respectively for \( j = 1, 2, \ldots, n \) where \( A_j \) is defined by

\[ A_j = \begin{pmatrix} a_j & b_j \\ \vdots & \ddots & \ddots \\ a_2 & b_2 & a_1 \\ b_2 & a_1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ b_j & \cdots & \cdots & a_j \end{pmatrix}_{(2j-1) \times (2j-1)}. \]
matrix $B_j$ such that $\lambda_1^{(2j)}$ and $\lambda_2^{(2j)}$ are the maximal and minimal eigenvalues of submatrix $B_j$ respectively for $j = 1, 2, ..., n$ where $B_j$ is defined by

$$
B_j = \begin{pmatrix}
a_j & b_j \\
b_j & a_j \\
\vvdots & \vvdots \\
a_2 & b_2 \\
b_2 & a_2 \\
\vvdots & \vvdots \\
a_1 & b_1 \\
b_1 & a_1
\end{pmatrix}_{(2j) \times (2j)}.
$$

2. Properties of the matrices $A_n$ and $B_n$

Let $p_0(\lambda) = 1$, $q_0(\lambda) = 1$, $p_j(\lambda) = \text{det}(A_j - \lambda I_j)$ for $j = 1, 2, ..., n$ and $q_j(\lambda) = \text{det}(B_j - \lambda I_j)$ for $j = 1, 2, ..., n$.

**Lemma 1.** For a given matrix $A_j$ and $B_j$ the sequence $p_j(\lambda)$ and $q_j(\lambda)$ satisfy in the following recurrence relations:

a) $p_1(\lambda) = (a_1 - \lambda)$,

b) $p_j(\lambda) = [(a_j - \lambda)^2 - b_j^2]p_{j-1}(\lambda)$, $j = 2, 3, ..., n$.

c) $q_1(\lambda) = ((a_1 - \lambda)^2 - b_1^2)$,

d) $q_j(\lambda) = [(a_j - \lambda)^2 - b_j^2]q_{j-1}(\lambda)$, $j = 2, 3, ..., n$.

**Proof.** The proof is clear by extending determinants of $(A_j - \lambda I_j)$ and $(B_j - \lambda I_j)$ on their first columns.

2.1. LU factorization of central symmetric X-form matrix.

Let $A$ be a central symmetric X-form matrix in form (1) and $B$ be a central symmetric X-form matrix in form (2), then we see that the LU Doolitel factorization of $A$ and $B$ are given by

$$
L_A = \begin{pmatrix}
1 & & & \\
& 1 & & \\
& & \ell_{n+1,n-1} & 1 \\
& & & \ell_{2n-1,1}
\end{pmatrix}_{(2n-1) \times (2n-1)}.
$$
\[ U_A = \begin{pmatrix}
    u_{1,1} & \cdots & \cdots & u_{1,2n-1} \\
    \vdots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    u_{n-1,n-1} & \cdots & \cdots & u_{n-1,n+1} \\
    u_{n,n} & \cdots & \cdots & u_{n,n+1} \\
    \vdots & \ddots & \ddots & \vdots \\
    u_{n+1,n+1} & \cdots & \cdots & u_{n+1,n+1} \\
    \vdots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    u_{2n-1,2n-1} & \cdots & \cdots & u_{2n-1,2n-1} \\
\end{pmatrix}_{(2n-1) \times (2n-1)} \]

where the elements \( \ell_{i,j} \) and \( u_{i,j} \) are as following:

\[ \ell_{n+i,n-i} = \frac{b_{i+1}}{a_{i+1}} \quad i = 1, 2, \ldots, n-1, \]

Also

\[
\begin{aligned}
  u_{1,2n-i} &= b_{n+1-i} & i &= 1, 2, \ldots, n-1 \\
  u_{i+1} &= a_{n+1-i} & i &= 1, 2, \ldots, n \\
  u_{n+i,n+i} &= \frac{a_{2i}^2 - b_{2i}^2}{a_{i+1}} & i &= 1, 2, \ldots, n-1.
\end{aligned}
\]

and \( L_B \) and \( U_B \) in factorization of \( B = L_B U_B \) are as below:

\[
L_B = \begin{pmatrix}
    1 \\
    \vdots \\
    \vdots \\
    1 \\
    \vdots \\
    1 \\
    1 \\
\end{pmatrix}_{(2n) \times (2n)}
\]

\[
U_B = \begin{pmatrix}
    a_n & b_n \\
    \vdots & \vdots \\
    a_2 & b_2 \\
    a_1 & b_1 \\
    \frac{a_1^2 - b_1^2}{a_1} & \frac{a_2^2 - b_2^2}{a_2} \\
    \vdots & \vdots \\
    \frac{a_{n-1}^2 - b_{n-1}^2}{a_{n-1}} & \frac{a_n^2 - b_n^2}{a_n}
\end{pmatrix}_{(2n) \times (2n)}
\]

**Remark 1.** We observe that the matrices \( L_A \) and \( L_B \) in LU factorization of central symmetric X-form matrix has a unit \( \lambda \)-matrix.

**Corollary 1.** If \( A \) and \( B \) are odd-order and even-order of a central symmetric X-form matrices in form (1) and (2) respectively, then

\[
\det(A) = a_1 \prod_{i=2}^{n} (a_i^2 - b_i^2),
\]

\[
\det(B) = \prod_{i=1}^{n} (a_i^2 - b_i^2).
\]
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2.2. **Inverse of $A_n$ and $B_n$.** It is clear that the necessary and sufficient conditions for invertibility of $A_n$ are $a_1 \neq 0$ and $a_i \neq \pm b_i$ for $i = 2, 3, \ldots, n$. If the matrix $\Phi$ be the inverse of $A_n$, then we have $A_n \Phi = I$. If the elements of column $j$ of $\Phi$ be $(\Phi_{1j}, \Phi_{2j}, \ldots, \Phi_{nj})^T$ then we have the following linear system of equations,

$$
\begin{pmatrix}
    a_n & a_{n-1} & & & & b_n \\
    & a_2 & b_2 & & & \\
    & & a_1 & & & \\
    & & & b_2 & a_2 & \\
    & & & & b_{n-1} & a_{n-1} \\
    b_n & & & & & \\
\end{pmatrix}
\begin{pmatrix}
    \Phi_{1j} \\
    \Phi_{2j} \\
    \vdots \\
    \Phi_{j-1,j} \\
    \Phi_{j+1,j} \\
    \Phi_{2n-2,j} \\
    \Phi_{2n-1,j} \\
\end{pmatrix} = 
\begin{pmatrix}
    0 \\
    0 \\
    \vdots \\
    0 \\
    0 \\
    0 \\
    0 \\
\end{pmatrix}
$$

With solving the above linear system for all column of $\Phi$ we have

$$
\begin{align*}
\Phi_{ii} &= \Phi_{2n-i,2n-i} = -\frac{a_{n+1-i}}{a_{n+1-i} - a_{n+1-i}} & i = 1, 2, \ldots, n-1, \\
\Phi_{nn} &= \frac{1}{a_n} \\
\Phi_{2n-i,i} &= \Phi_{i,2n-i} = \frac{-a_{n+1-i}}{a_{n+1-i} - a_{n+1-i}} & i = 1, 2, \ldots, n-1,
\end{align*}
$$

and this shows that $\Phi$ is also the central symmetric $X$-form matrix. For the inverse of $B_n$ we also have similar relations.

3. **Inverse eigenvalue problem**

**Theorem 1.** Assume $\lambda_1^{(j)}, \lambda_2^{(j)}$ for $j = 1, \ldots, n$ are the $2n - 1$ distinct real numbers, then there exist a central symmetric $X$-form matrix in form (1) such that $\lambda_1^{(j)}$ and $\lambda_2^{(j)}$ are the minimal and maximal eigenvalues of submatrix $A_j$ respectively in form (3) if and only if

$$
\lambda_1^{(n)} < \lambda_2^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_2^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_2^{(n)}.
$$

**Proof.** Existence of matrices $A_n$ such that $\lambda_1^{(j)}, \lambda_2^{(j)}$ are the its maximal and minimal eigenvalues respectively of its submatrix for $j = 1, 2, \ldots, n$ is equivalence to finding the solution for the following linear system of equations:

$$
\begin{align*}
p_j(\lambda_1^{(j)}) &= [(a_j - \lambda_1^{(j)})^2 - b_j^2] p_{j-1}(\lambda_1^{(j)}) = 0, \\
p_j(\lambda_2^{(j)}) &= [(a_j - \lambda_2^{(j)})^2 - b_j^2] p_{j-1}(\lambda_2^{(j)}) = 0,
\end{align*}
$$

$$
\begin{align*}
\Rightarrow & \quad [(a_j - \lambda_1^{(j)})^2 - b_j^2] [(a_{j-1} - \lambda_1^{(j)})^2 - b_{j-1}^2] \cdots [(a_2 - \lambda_1^{(j)})^2 - b_2^2] [a_1 - \lambda_1^{(j)}] = 0, \\
\Rightarrow & \quad [(a_j - \lambda_1^{(j)})^2 - b_j^2] [(a_{j-1} - \lambda_2^{(j)})^2 - b_{j-1}^2] \cdots [(a_2 - \lambda_2^{(j)})^2 - b_2^2] [a_1 - \lambda_2^{(j)}] = 0.
\end{align*}
$$

Thus

$$
\begin{align*}
(a_j - \lambda_1^{(j)})^2 - b_j^2 &= 0 \quad \text{for} \quad j = 2, 3, \ldots, n \\
(a_j - \lambda_2^{(j)})^2 - b_j^2 &= 0 \quad \text{for} \quad j = 2, 3, \ldots, n.
\end{align*}
$$
Then $a_1 = \lambda_1^{(1)}$ and whereas $\lambda_1^{(j)} \neq \lambda_j^{(j)}$, we have $a_j = \frac{\lambda_1^{(j)} + \lambda_j^{(j)}}{2}$ and $b_j^2 = \left(\frac{\lambda_1^{(j)} - \lambda_j^{(j)}}{2}\right)^2$ for $j = 2, 3, \ldots, n$, therefore we can find all entries of matrix $A_n$.

Conversely since $p_j(\lambda) = [(a_j - \lambda)^2 - b_j^2] p_{j-1}(\lambda)$, then each root of $p_{j-1}$ is a root of $p_j$, and we know that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix $A_j$ in form (3) respectively, thus $\lambda_1^{(j-1)}$ and $\lambda_j^{(j-1)}$ are in between $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, i.e.

\[
\lambda_1^{(j)} < \lambda_1^{(j-1)} < \lambda_j^{(j-1)} < \lambda_j^{(j)}.
\]

and so on we can write

\[
\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}.
\]

So the proof is completed. □

**Theorem 2.** Assume $\lambda_1^{(2j)}, \lambda_2^{(2j)}$ for $j = 1, \ldots, n$ are the $2n$ distinct real numbers, then there exist an even-order central symmetric X-form matrix in form (2) such that $\lambda_1^{(2j)}$ and $\lambda_2^{(2j)}$ are the minimal and maximal eigenvalues of submatrix $B_j$ respectively, if and only if

\[
\lambda_1^{(2n)} < \lambda_1^{(2n-2)} < \ldots < \lambda_1^{(2)} < \lambda_2^{(2)} < \ldots < \lambda_2^{(2n)}.
\]

**Proof.** Proof is similar to proof of Theorem 1. □

**Remark 2.** If $b_j$ for $j = 2, 3, \ldots, n$, are positive, then the matrix $A_n$ is unique.

**Remark 3.** Whereas all eigenvalues of $A_j$ are the subset of eigenvalues $A_j$ then all eigenvalues relation (4) are all eigenvalues of $A_n$.

**Lemma 2.** If $A_n$ is a central symmetric X-form matrix in form (1), then we have

(a) $\lambda_1^{(j)} < a_j < \lambda_j^{(j)}$ where $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of submatrix of $A_n$, for $j = 2, 3, \ldots, n$.

(b) If $b_i > 0$ for $i = 2, \ldots, j$, then $\| A_j \|_\infty = \| A_j \|_1 = \lambda_j^{(j)}$, and if $b_i < 0$ for $i = 2, \ldots, j$, then $\| A_j \|_\infty = \| A_j \|_1 = \lambda_1^{(j)}$.

**Proof.** (a) According to the previous theorem 2, we have $a_j = \frac{\lambda_1^{(j)} + \lambda_j^{(j)}}{2}$ and also we have $\lambda_1^{(j)} < \lambda_j^{(j)}$ for $j = 2, 3, \ldots, n$, then

\[
\frac{\lambda_1^{(j)} + \lambda_1^{(j)}}{2} < \frac{\lambda_1^{(j)} + \lambda_j^{(j)}}{2} < \frac{\lambda_j^{(j)} + \lambda_j^{(j)}}{2} \quad \Rightarrow \quad \lambda_1^{(j)} < a_j < \lambda_j^{(j)}.
\]

(b) Case (I): $b_i > 0$ for $i = 2, \ldots, n$, then

\[
\| A_j \|_\infty = \| A_j \|_1 = \max \{ a_i + b_i = \lambda_1^{(i)} + \lambda_j^{(j)} + \lambda_j^{(i)} - \lambda_1^{(i)} = \lambda_1^{(i)} \quad i = 2, \ldots, j \} = \lambda_j^{(j)}
\]

for $j=2,\ldots,n$.

Case (II): $b_i < 0$ for $i = 2, \ldots, n$, then

\[
\| A_j \|_\infty = \| A_j \|_1 = \max \{ a_i + b_i = \lambda_1^{(i)} + \lambda_j^{(j)} + \lambda_j^{(i)} - \lambda_1^{(i)} = \lambda_1^{(i)} \quad i = 2, \ldots, j \} = \lambda_1^{(j)}
\]

for $j=2,\ldots,n$. 


so that proof is completed. □

4. INVERSE SINGULAR VALUE PROBLEM

In this section we study two inverse singular value problems as below:

**problem I.** Given $2n - 1$ nonnegative real numbers $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$. We find $(2n - 1) \times (2n - 1)$ central symmetric X-form matrix $A_n$ in form (1), such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$, are minimal and maximal singular value of submatrix $A_j$ of $A_n$ in form (3), and for given $2n$ nonnegative real numbers $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, similarly we find $(2n) \times (2n)$ central symmetric X-form matrix $B_n$ in form (2), such that $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, are minimal and maximal singular value of submatrix $B_j$ of $B_n$.

**problem II.** Given $2n - 1$ nonnegative real numbers $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$, we find the $\lambda$-matrix $\Lambda_n$ in form (9) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$, are the minimal and maximal singular values of submatrix $\Lambda_j$ from $\Lambda_n$, where

$$ (9)\Lambda_n = \begin{pmatrix} \alpha_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \beta_n & \cdots & \sqrt{\alpha_n^2 - \beta_n^2} \end{pmatrix} (2n-1) \times (2n-1) $$

Furthermore for $2n$ given nonnegative real numbers $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, we find the $\lambda$-matrix $\Gamma_n$ in form (10) such that $\sigma_1^{(2j)}$ and $\sigma_{2j}^{(2j)}$ for $j = 1, 2, ..., n$, are the minimal and maximal singular values of submatrix $\Gamma_j$ from $\Gamma_n$, where

$$ (10)\Gamma_n = \begin{pmatrix} \alpha_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \beta_n & \cdots & \sqrt{\alpha_n^2 - \beta_n^2} \end{pmatrix} (2n) \times (2n) $$

**Theorem 3.** Assume $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$ are the $(2n-1)$ real nonnegative numbers, then there exist a central symmetric X-form matrix in form (1) such that $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ are the minimal and maximal singular values of submatrix $A_j$ respectively in form (3) if and only if $\sigma_1^{(j)}$ and $\sigma_j^{(j)}$ for $j = 1, 2, ..., n$ satisfy in the following
relation:

\[
\sigma_1^{(n)} < \sigma_1^{(n-1)} < \cdots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_1^{(2)} < \cdots < \sigma_n^{(1)}
\]  

**Proof.** Let \(\sigma_1^{(j)}\) and \(\sigma_j^{(j)}\) for \(j = 1, 2, \ldots, n\) be the real nonnegative number that satisfy in (11). It is clear that \((\sigma_1^{(j)})^2\) and \((\sigma_j^{(j)})^2\) for \(j = 1, 2, \ldots, n\), satisfy in there relations, this means

\[
(\sigma_1^{(n)})^2 < (\sigma_1^{(n-1)})^2 < \cdots < (\sigma_1^{(2)})^2 < (\sigma_1^{(1)})^2 < (\sigma_2^{(2)})^2 < \cdots < (\sigma_n^{(n)})^2.
\]

By Theorem 1 there exist an odd-order central symmetric X-form matrix that \((\sigma_1^{(j)})^2\) and \((\sigma_j^{(j)})^2\) are the minimal and maximal eigenvalues of its submatrices respectively. We show this matrix by \(A_n\) as follows:

\[
A_n = \begin{pmatrix} a_n & b_n \\ \vdots & \ddots & & \vdots \\ a_2 & b_2 & a_1 \\ b_2 & a_2 \\ \vdots & \ddots & \ddots & \ddots \\ b_n & \cdots & & a_n \end{pmatrix}_{(2n-1) \times (2n-1)}
\]

where

\[
a_i = \frac{\left(\sigma_i^{(1)}\right)^2 + \left(\sigma_i^{(i)}\right)^2}{2}, \quad i = 1, 2, \ldots, n
\]

and

\[
b_i = \frac{\left(\sigma_i^{(1)}\right)^2 - \left(\sigma_i^{(i)}\right)^2}{2}, \quad i = 2, 3, \ldots, n
\]

On the other hand if \(C_n\) be an odd-order central symmetric X-form matrix as follows

\[
C_n = \begin{pmatrix} \alpha_n & \beta_n \\ \vdots & \ddots & \vdots \\ \alpha_2 & \beta_2 & \alpha_1 \\ \beta_2 & \alpha_2 \\ \vdots & \ddots & \ddots & \ddots \\ \beta_n & \cdots & \cdots & \alpha_n \end{pmatrix}_{(2n-1) \times (2n-1)}
\]

then
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\[ C_nC_n^T = \begin{pmatrix} 
\alpha_1^2 + \beta_1^2 & 2\alpha_1\beta_1 \\
\alpha_2^2 + \beta_2^2 & 2\alpha_2\beta_2 \\
2\alpha_2\beta_2 & \alpha_2^2 + \beta_2^2 \\
\vdots & \ddots & \ddots & \ddots \\
2\alpha_n\beta_n & \cdots & 2\alpha_n\beta_n & \alpha_n^2 + \beta_n^2
\end{pmatrix}_{(2n-1) \times (2n-1)} \]

(14)

Since \( A_n = C_nC_n^T \), we can find the all elements of matrix \( C_nC_n^T \) as the following:

\[
\begin{align*}
    a_1 &= \alpha_1^2, \\
    a_i &= \alpha_i^2 + \beta_i^2, \quad i = 2, 3, \ldots, n \\
    b_i &= 2\alpha_i\beta_i, \quad i = 2, 3, \ldots, n
\end{align*}
\]

by combination of the above relations we have

\[
\begin{align*}
    (\alpha_i + \beta_i)^2 &= a_i + b_i \\
    (\alpha_i - \beta_i)^2 &= a_i - b_i \\
    \Rightarrow \quad \begin{cases} 
      \alpha_i = \frac{\sqrt{(a_i + b_i) + \sqrt{(a_i - b_i)^2}}}{2} \\
      \beta_i = \frac{\sqrt{(a_i + b_i) - \sqrt{(a_i - b_i)^2}}}{2}
    \end{cases} \quad i = 2, 3, \ldots, n,
\end{align*}
\]

Therefore the matrix \( C_n \) is solution of our problem. Conversely at first, assume \( C_n \) is a matrix of form (1) of order \((2n-1) \times (2n-1)\) such that \( \sigma_1^j \) and \( \sigma_2^j \) are the minimal and maximal singular values of submatrix \( C_j \) in form (3) respectively. Then \((\sigma_1^j)^2\) and \((\sigma_2^j)^2\) are the minimal and maximal eigenvalues of submatrices \((C_nC_n^T)_j\) of \( C_nC_n^T \) respectively. By Theorem 1 we have

\[
(15) \quad (\sigma_1^{(n)})^2 < (\sigma_1^{(n-1)})^2 < \cdots < (\sigma_1^{(2)})^2 < (\sigma_1^{(1)})^2 < (\sigma_2^{(2)})^2 < \cdots < (\sigma_2^{(n)})^2,
\]

consequently we have

\[
\sigma_1^{(n)} < \sigma_1^{(n-1)} < \cdots < \sigma_1^{(2)} < \sigma_1^{(1)} < \sigma_2^{(2)} < \cdots < \sigma_2^{(n)},
\]

and proof will be completed. □

**Remark 4.** There is a similar result for even-order of above Theorem.

**Theorem 4.** Assume \( \sigma_1^{(j)} \) and \( \sigma_2^{(j)} \) for \( j = 1, 2, \ldots, n \) are the \((2n-1)\) positive real numbers, then there exist a matrix in form (9) such that \( \sigma_1^{(j)} \) and \( \sigma_2^{(j)} \) are the minimal and maximal singular values of submatrix \( \Lambda_j \) of \( \Lambda_n \) respectively, if and only if \( \sigma_1^{(j)} \) and \( \sigma_2^{(j)} \) for \( j = 1, 2, \ldots, n \) satisfy in the following relation

\[
(16) \quad (\sigma_1^{(n)})^2 < (\sigma_1^{(n-1)})^2 < \cdots < (\sigma_1^{(2)})^2 < (\sigma_1^{(1)})^2 < (\sigma_2^{(2)})^2 < \cdots < (\sigma_2^{(n)})^2
\]

**Proof.** Assume \( \sigma_1^{(j)} \) and \( \sigma_2^{(j)} \), are \( 2n - 1 \) positive real numbers which satisfy in the relation (11), consider the squares \((\sigma_1^{(j)})^2\) and \((\sigma_2^{(j)})^2\) for \( j = 1, \ldots, n \), it is clear that
these satisfy in relation (15), then from Theorem 1 there exist a central symmetric X-form matrix

\[
A = \begin{pmatrix}
a_n & b_n \\
& \ddots & \ddots \\
a_2 & a_1 & b_2 & b_1 \\
& \ddots & \ddots & \ddots & \ddots \\
& & b_n & a_n \\
\end{pmatrix}_{(2n-1) \times (2n-1)}
\]

such that \((\sigma_j^{(j)})^2\) and \((\sigma_j^{(j)})^2\) for \(j = 1, 2, ..., n\), are the minimal and maximal eigenvalues of \(A_j\) from \(A\) respectively. We observe that if matrix \(\Lambda\) has form (9) then \(\Lambda\Lambda^T\) has form (1) as follows

\[
\Lambda\Lambda^T = \begin{pmatrix}
a_n^2 & a_n b_n & \alpha_{n-1}^2 & \alpha_{n-1} \beta_{n-1} \\
a_n b_n & a_n^2 & \alpha_{n-1} \beta_{n-1} & \alpha_{n-1} \beta_{n-1} \\
\alpha_{n-1}^2 & \alpha_{n-1} \beta_{n-1} & \alpha_{n-1}^2 & \alpha_{n-1} \beta_{n-1} \\
\alpha_{n-1} b_n & \alpha_{n-1} \beta_{n-1} & \alpha_{n-1} \beta_{n-1} & \alpha_{n-1}^2 \\
\end{pmatrix}_{(2n-1) \times (2n-1)}
\]

Now we set \(\alpha_j^2 = a_j, \beta_j^2 = b_j\), \(j = 1, ..., n\), to compute the entries of an \((2n-1) \times (2n-1)\) matrix \(\Lambda\) of the form (9) with the prescribed extremal singular values for the submatrices \(\Lambda_j\).

The proof of the second part is similar to the proof of inverse Theorem 3.

**Remark 5.** There is a similar result for even-order of above Theorem.

5. **Examples**

**Example 1.** Assume \(n = 5\) and given 9 real numbers as below

\[
\lambda_4^{(5)} \lambda_4^{(4)} \lambda_4^{(3)} \lambda_4^{(2)} \lambda_4^{(1)} \lambda_5^{(2)} \lambda_3^{(3)} \lambda_2^{(4)} \lambda_5^{(5)},
\]

\[-5 -3 \quad 0 \quad 2 \quad 6 \quad 9 \quad 10 \quad 12 \quad 23,
\]

find the central symmetric X-form matrix such that \(\lambda_4^{(j)}\) and \(\lambda_5^{(j)}\) for \(j = 1, 2, 3, 4, 5\) are the eigenvalues of submatrix \(A_j\) respectively.

**Solution.** By theorem 1 and some simple calculations, the solution of problem obtain
in the following form
\[
\begin{pmatrix}
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\
0 & 4.5 & 0 & 0 & 0 & 0 & 7.5 & 0 \\
0 & 0 & 5 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 5.5 & 0 & 3.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 3.5 & 0 & 5.5 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 5 & 0 & 0 \\
0 & 7.5 & 0 & 0 & 0 & 0 & 4.5 & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
\end{pmatrix}
\]

**Example 2.** Assume \( n = 5 \), given 9 real numbers as below
\[
\sigma(5) \quad \sigma(4) \quad \sigma(3) \quad \sigma(2) \quad \sigma(1)
\]

find the central symmetric \( X \)-form matrix \( C_n \) and \( \lambda \)-matrix \( \Lambda_n \) such that \( \sigma_j \) and \( \sigma_j \) for \( j = 1, 2, 3, 4, 5 \) are the singular values of submatrices \( \Lambda_j \) for \( j = 1, 2, 3, 4, 5 \) respectively such that \( \Lambda_j \) has form (9) and \( C_j \) has form (3).

**Solution.** At first we find \( X \)-form matrix \( A \) by Theorem 1 as below
\[
A = \begin{pmatrix}
6.452214 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.188655 \\
0 & 4.858691 & 0 & 0 & 0 & 0 & 0 & 0 & 4.536147 \\
0 & 0 & 3.529162 & 0 & 0 & 0 & 2.937148 & 0 & 0 \\
0 & 0 & 0 & 2.000078 & 0 & 1.414287 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.414287 & 0 & 2.000078 & 0 & 0 & 0 \\
0 & 0 & 2.937148 & 0 & 0 & 0 & 3.529162 & 0 & 0 \\
0 & 4.536147 & 0 & 0 & 0 & 0 & 0 & 4.858691 & 0 \\
6.188655 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.452214 \\
\end{pmatrix}
\]

such that \( \sigma^2_1 \) and \( \sigma^2_j \) for \( j = 1, 2, 3, 4, 5 \) are the minimal and maximal eigenvalues of submatrices \( A \) respectively in form (3). Then by Theorem 3 we find \( X \)-form matrix \( C_n \), such that \( \sigma^2_1 \) and \( \sigma^2_j \) for \( j = 1, 2, 3, 4, 5 \) are the minimal and maximal singular values of submatrices \( C_n \) respectively in form (3)
\[
C_n = \begin{pmatrix}
-2.03369 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.52101 \\
0 & 1.816515 & 0 & 0 & 0 & 0 & 0 & 0 & 1.248585 \\
0 & 0 & -1.5764 & 0 & 0 & 0 & 0 & 0 & -0.9316 \\
0 & 0 & 0 & -1.306585 & 0 & -0.541215 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.541215 & 0 & -1.306585 & 0 & 0 & 0 \\
0 & 0 & -0.9316 & 0 & 0 & 0 & -1.5764 & 0 & 0 \\
0 & 1.248585 & 0 & 0 & 0 & 0 & 0 & 1.816515 & 0 \\
-1.52101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.03369 \\
\end{pmatrix}
\]

Then by Theorem 4 we find the \( \lambda \)-form matrix \( \Lambda_n \) such that \( \sigma^2_1 \) and \( \sigma^2_j \) for \( j = 1, 2, 3, 4, 5 \) are the minimal and maximal singular values of submatrices \( \Lambda_n \) respectively in form (9)
\[
\Lambda_n = \begin{pmatrix}
2.540121 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.204244 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.831097 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.414241 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.000032 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.604038 & 0 & 0 & 0 & 0.883164 & 0 & 0 \\
0 & 2.057915 & 0 & 0 & 0 & 0 & 0 & 0.789732 & 0 \\
2.436362 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.718577 \\
\end{pmatrix}
\]
References


