On the Decomposition of Hilbert Spaces

H.R. Afshin* and M.A. Ranjbar
Department of Mathematics, Vali-e-Asr university, Rafsanjan, Iran
E-mail: afshin@vru.ac.ir
E-mail: mranj_ma@yahoo.com

Abstract. A basic relation between numerical range and Davis-Wielandt shell of an operator $A$ acting on a Hilbert space with orthonormal basis $\xi = \{e_i | i \in I\}$ and its conjugate $\bar{A}$ which is introduced in this paper are obtained. The results are used to study the relation between point spectrum, approximate spectrum and residual spectrum of $A$ and $\bar{A}$. A necessary and sufficient condition for $A$ to be self-conjugate ($A = \bar{A}$) is given using a subgroup of $H$.

Keywords: Numerical range, Davis-Wielandt shell, Spectra, Conjugate of an operator.


1. Introduction

Let $H$ be a Hilbert space and $B(H)$ be the algebra of bounded linear operators acting on $H$. The numerical range of $A \in B(H)$ is defined by

$$W(A) = \{\langle Ax, x \rangle | x \in H, \|x\| = 1\};$$

see [4, 5, 6, 7]. The numerical range is useful to study matrices, operators and to classify them. For example $W(A) = \{\mu\}$ if and only if $A = \mu I$ and $W(A) \subseteq \mathbb{R}$ if and only if $A = A^*$. Also, there are nice connections between $W(A)$ and spectrum $\sigma(A)$. For instance $\sigma(A) \subseteq \text{cl} W(A)$ where $\text{cl} W(A)$ is the closure of $W(A)$ and $\text{cl} W(A) = \text{conv} \sigma(A)$ if $A$ is normal ($\text{conv} \sigma(A)$ is the

*Corresponding Author

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convex hull of $\sigma(A)$). The Davis-Wielandt shell of an operator $A \in B(H)$ is a generalization of the classical numerical range and it is defined by

$$DW(A) = \{(\langle Ax, x \rangle, \langle Ax, Ax \rangle) : x \in H, \|x\| = 1\};$$

see [1, 2, 9]. In fact $W(A)$ is the projection of $DW(A)$ on the first coordinate.

In this paper, we introduce the conjugate of $A \in B(H)$ and obtain the relation between $DW(A)$ and $DW(A)$. Also we obtain some relations between point spectrum, approximate spectrum and residual spectrum of $A$ and $\tilde{A}$. In section 3 we obtain a necessary and sufficient condition for $A$ to be self-conjugate ($A = \tilde{A}$) by a subgroup of $H$. Also we find the form of an operator $D \in B(H)$ such that $W(D) \subseteq \{(x, x) | x \in \mathbb{R}\}$.

2. The Conjugate of an Operator

In [8], for $A \in B(H), A^t$ is defined. In the following we introduce a new operator that has similar properties.

**Definition 2.1.** Let $H$ be a Hilbert space, $A \in B(H)$ and $\xi = \{e_i | i \in I\}$ be an orthonormal basis of $H$. The conjugate of $A$ with respect to $\xi$ is the operator $\tilde{A}_t \in B(H)$ defined by

$$\langle \tilde{A}_t e_i, e_j \rangle = \langle A e_i, e_j \rangle,$$

for every $i, j \in I$.

Let $x = \sum_{i \in I} x_i e_i \in H$. Define $\bar{x} = \sum_{i \in I} \bar{x}_i e_i$, where $\bar{x}_i$ is the complex conjugate of $x_i$. Let $y = \sum_{j \in I} y_j e_j \in H$, we have

$$\langle \tilde{A}_t x, y \rangle = \sum_{i} \sum_{j} x_i \bar{y}_j \langle \tilde{A}_t e_i, e_j \rangle = \sum_{i} \sum_{j} \bar{x}_i y_j \langle A e_i, e_j \rangle = \langle A \bar{x}, \bar{y} \rangle.$$

By the above discussion we see that the definition of $\tilde{A}_t$ is independent of $\xi$, so we denote the conjugate of $A$ by $\tilde{A}$.

Also we know that $\langle \bar{x}, \bar{y} \rangle = \overline{\langle x, y \rangle}$ and we have $\overline{\langle A \bar{x}, \bar{y} \rangle} = \langle \tilde{A}_t x, y \rangle = \langle \tilde{A}_t y, \tilde{A}_t x \rangle$ for every $x, y \in H$. Hence $\overline{\tilde{A}_t x} = \tilde{A}_t \bar{x}$. Furthermore, we recall that

$$\langle A^t x, y \rangle = \langle \tilde{A}_t \tilde{A}_t x, \tilde{A}_t \tilde{A}_t y \rangle = \langle A \bar{x}, \bar{y} \rangle,$$

for every $x, y \in H$ [8].

**Lemma 2.2.** Let $A, B \in B(H)$ and $\alpha \in \mathbb{C}$ be an arbitrary constant. Then:

a) $\tilde{A} = (A^t)^*$;

b) $\overline{\alpha A + B} = \bar{\alpha} \tilde{A} + \bar{B}$ and $\overline{AB} = \tilde{A} \tilde{B}$.

**Proof.** a) We have

$$\overline{\langle (A^t)^* x, y \rangle} = \overline{\langle A^t y, x \rangle} = \langle \bar{y}, \overline{Ax} \rangle = \overline{\langle A \bar{x}, \bar{y} \rangle} = \langle \tilde{A}_t x, y \rangle,$$

for every $x, y \in H$. Thus the assertion follows.

b) Let $x, y \in H$ be arbitrary, then

$$\langle \overline{(A + B)x, y} \rangle = \langle \overline{(A + \alpha B)x, \bar{y}} \rangle = \langle A \bar{x}, \bar{y} \rangle + \langle \alpha B \bar{x}, \bar{y} \rangle = \langle \tilde{A}_t x, y \rangle + \bar{\alpha} \langle \tilde{B} x, y \rangle = \langle A \bar{x}, \bar{y} \rangle + \overline{\alpha \langle B \bar{x}, \bar{y} \rangle}.$$
and hence
\[ \langle A \sigma \rangle = \langle B \sigma \rangle. \]

Recall that the point spectrum of \( A \in B(H) \) is the set \( \sigma_p(A) \) of eigenvalues of \( A \). The residual spectrum of \( A \) is the set \( \sigma_r(A) \) of complex numbers \( \lambda \) such that the range of \( \lambda I - A \) is not dense in \( H \). The approximate spectrum of \( A \) is the set \( \sigma_a(A) \) of complex numbers \( \lambda \) such that there exists a sequence of unit vectors \( \{x_n\}_{n=1}^{\infty} \in H \) such that \( \lim_{n \to \infty} \| (\lambda I - A)x_n \| = 0 \). It is well known that \( \sigma_p(A) \subseteq \sigma_a(A) \) and \( \sigma(A) = \sigma_a(A) \cup \sigma_r(A) \); see [3].

In the following theorem, we obtain some relations between the Davis-Wielandt shell and the spectrum of \( A \) and \( \bar{A} \).

**Theorem 2.3.** Let \( A \in B(H) \). Then:

a) \( DW(A) = \{(\bar{\mu}, r) \mid (\mu, r) \in DW(A)\} \)

b) \( \sigma_p(A) = \sigma_p(\bar{A}) \), \( \sigma_a(A) = \sigma_a(\bar{A}) \) and \( \sigma_r(\bar{A}) = \sigma_r(A) \).

Proof. a) \( (\mu, r) \in DW(A) \) if and only if \( \langle Ax, x \rangle = \mu, \langle \bar{A}x, \bar{x} \rangle = r \) for some unit vector \( x \). Since \( \langle \bar{A}x, x \rangle = \langle Ax, \bar{x} \rangle \) and \( \|x\| = 1 \), then \( \bar{\mu} = \langle \bar{A}x, \bar{x} \rangle \in W(A) \).

Also
\[ \langle Ax, \bar{A}x \rangle = \langle A\bar{x}, \bar{x} \rangle = \langle A\bar{x}, \bar{A}\bar{x} \rangle = \langle A\bar{x}, A\bar{x} \rangle. \]

Therefore
\( (\mu, r) = (\langle Ax, \bar{A}x \rangle, \langle \bar{A}x, \bar{A}x \rangle) \in DW(\bar{A}) \) if and only if \( (\langle A\bar{x}, \bar{x} \rangle, \langle A\bar{x}, A\bar{x} \rangle) = (\bar{\mu}, \bar{r}) \in DW(A) \), and the result follows.

b) \( \mu \in \sigma_p(\bar{A}) \) if and only if \( \langle (\bar{A} - \mu I)x, y \rangle = 0 \) for all \( y \in H \) and some nonzero vector \( x \). We have \( \langle (\bar{A} - \mu I)x, y \rangle = 0 \) if and only if \( \langle (\bar{A} - \mu I)x, \bar{y} \rangle = \langle (\lambda I - A)x, y \rangle = 0 \). Thus the first assertion holds.

We know that \( \mu \in \sigma_a(\bar{A}) \) if and only if there is a sequence \( \{x_n\}_{n=1}^{\infty} \) of unit vectors such that \( \lim_{n \to \infty} ((\mu I - \bar{A})x_n) = 0 \). We have
\[ 0 = \lim_{n \to \infty} (\mu x_n - \bar{A}x_n) = \lim_{n \to \infty} (\mu x_n - A\bar{x}_n), \]

which implies that \( 0 = \lim_{n \to \infty} (\mu x_n - A\bar{x}_n) = \lim_{n \to \infty} (\mu x_n - A\bar{x}_n) \), and the second assertion holds. If \( \mu \in \sigma_r(\bar{A}) \), then \( \text{cl}(\text{Im}(\mu I - \bar{A})) \neq H \) and vice versa.

This holds if and only if there exists a nonzero vector \( v \in H \) which is orthogonal to \( \text{Im}(\mu I - \bar{A}) \). Hence for any \( x \in H \) we have
\[ 0 = \langle (\mu I - \bar{A})x, z \rangle = \langle x, (\mu I - A^t)^*z \rangle = \langle x, (\mu I - A^t)z \rangle = \langle x, \bar{\mu}z \rangle - \langle x, A^t z \rangle = \mu(x, z) - \bar{\mu}(\bar{z}, \bar{x}) = \bar{\mu}(\bar{z}, \bar{x}) - \bar{z}, A\bar{x} = \langle \bar{z}, (\mu I - A)\bar{x} \rangle. \]

Since \( x \) is arbitrary, thus \( \bar{z} \in \text{Im}(\mu I - A)^\perp \) and \( \text{cl}(\text{Im}(\mu I - A)) \neq H \). Therefore \( \bar{\mu} \in \sigma_r(A) \) and vice versa. \( \square \)
3. Decomposition of a Hilbert space

Let $H$ be a Hilbert space and $\xi = \{e_i|i \in I\}$ be an orthonormal basis of $H$. Define $H^\xi_R = \{x \in H|x = \bar{x}\}$. Since $0 = \bar{0}$, $0 \in H^\xi_R$. It is clear that $H^\xi_R$ is a subgroup of $H$.

**Theorem 3.2.** Let $H$ be a Hilbert space with orthonormal basis $\xi = \{e_i|i \in I\}$. Then $H \cong H^\xi_R \times H^\xi_R$ and each $x \in H$ can be uniquely written as $x = a + ib$, where $a, b \in H^\xi_R$.

**Proof.** Let $x \in H$ be arbitrary. We have $\bar{x} + x = \bar{x} + x$. Thus $\bar{x} + x \in H^\xi_R$. Also $i(\bar{x} - x) = i(\bar{x} - x)$. Then $i(\bar{x} - x) \in H^\xi_R$. Therefore $x = a + ib$, where $a = \frac{x + \bar{x}}{2}$, $b = \frac{i(\bar{x} - x)}{2}$ and $a, b \in H^\xi_R$. Now we show that the above decomposition is unique. Let $x = a + ib = a' + ib'$ where $a', b' \in H^\xi_R$ and

$$a = \sum_{i \in I} a_i e_i, b = \sum_{i \in I} b_i e_i, a' = \sum_{i \in I} a'_i e_i, b' = \sum_{i \in I} b'_i e_i.$$ 

Since $a, b, a', b' \in H^\xi_R$, then for every $i \in I$, $a_i, b_i, a'_i, b'_i \in \mathbb{R}$,

$$\sum_{i \in I} ((a_i - a'_i) + i(b_i - b'_i))e_i = 0.$$ 

So we have $a_i = a'_i$ and $b_i = b'_i$ for every $i \in I$, which proves the uniqueness. \[\square\]

We remark that if $x = a + ib \in H$ and $a, b \in H^\xi_R$, then $\bar{x} = a - ib$.

**Theorem 3.2.** Let $A \in B(H)$ with orthonormal basis $\xi = \{e_i|i \in I\}$. The following conditions are equivalent:

a) $A\bar{x} = \overline{Ax}$ for all $x \in H$;

b) $A = \bar{A}$;

c) $H^\xi_R$ is $A$-invariant ($A(H^\xi_R) \subseteq H^\xi_R$).

**Proof.** $a \Rightarrow b$ and $b \Rightarrow c$ are trivial, we prove $c \Rightarrow a$.

Let $x = \alpha + i\beta \in H$ be arbitrary and $\alpha, \beta \in H^\xi_R$. We have $A\bar{x} = A(\alpha - i\beta) = A\alpha - iA\beta$. Since we assumed that $H^\xi_R$ is $A$-invariant, then by previous remark we have

$$A\alpha - iA\beta = A\alpha + iA\beta = A(\alpha + i\beta) = A\bar{x}.$$ 

\[\square\]

**Corollary 3.3.** Let $H$ be a Hilbert space with orthonormal basis $\xi = \{e_i|i \in I\}$. Then for all $x = a + ib \in H$ (where $a, b \in H^\xi_R$), $\|x\|^2 = \|a\|^2 + \|b\|^2$.

**Proof.** We have

$$\|x\|^2 = \langle x, x \rangle = \langle a + ib, a + ib \rangle = \|a\|^2 + \|b\|^2 + i\langle b, a \rangle - i\langle a, b \rangle.$$ 

We show that for all $a, b \in H^\xi_R$, $\langle b, a \rangle = \langle a, b \rangle$.

Let $a = \sum_{i \in I} a_i e_i, b = \sum_{j \in I} b_j e_j$. (Since $a, b \in H^\xi_R$, we have $a_i, b_j \in \mathbb{R}$, for any
Therefore $T$ exists a Hermitian operator

Conversely let

Since

Note that for elements of a Hilbert space the concept of conjugate depends on the notation of orthonormal basis when we refer to the conjugate of Hilbert space elements.

**Corollary 3.5.** Assume that $A \in B(H)_\mathbb{R}$. Then $A^t, A^* \in B(H)_\mathbb{R}$ and $A^{-1} \in B(H)_\mathbb{R}$ if $A$ is invertible.

**Proof.** Let $A \in B(H)_\mathbb{R}$ and $x, y \in H$ be arbitrary. We have

$$\langle (A^t)x, y \rangle = \langle A^t \bar{x}, \bar{y} \rangle = \langle Ay, x \rangle = \langle \bar{Ay}, \bar{x} \rangle = \langle A^t x, y \rangle.$$ 

Since $x, y$ are arbitrary, we have $A^t \in B(H)_\mathbb{R}$. Similarly, $A^* \in B(H)_\mathbb{R}$. Now let $A$ be invertible and $x, y \in H$ be arbitrary. Consider $A^{-1} \bar{y} = y_0, A^{-1} \bar{x} = x_0$. By Theorem 3.2, $A \bar{x}_0 = \bar{Ax}_0 = x$ and $A \bar{y}_0 = y$. Thus,

$$\langle (A^{-1})x, y \rangle = \langle (A^{-1}) \bar{x}, \bar{y} \rangle = \langle x_0, A \bar{y}_0 \rangle = \langle \bar{x}_0, A \bar{y}_0 \rangle = \langle A^{-1}(A \bar{x}_0), A \bar{y}_0 \rangle = \langle A^{-1} \bar{x}_0, \bar{y} \rangle = \langle A^{-1} x, y \rangle.$$ 

Hence, our claim follows.

In the following theorem we deduce a necessary and sufficient condition for $W(D)$ to be a subset of $\{(r, r) : r \in \mathbb{R}\}$ when $D \in B(H)$.

**Theorem 3.6.** Let $D = A + iB \in B(H)$ and $A, B \in B(H)_\mathbb{R}$. Then $W(D) \subseteq \{(r, r) : r \in \mathbb{R}\}$ if and only if $B = A^t$.

**Proof.** Let $H$ be a Hilbert space and $\xi = \{e_i | i \in I\}$ be an orthonormal basis. Let $x = a + ib \in H$ be arbitrary where $a, b \in H^*_\mathbb{R}$ and let $D = A + iA^t$. Then

$$\langle Dx, x \rangle = \langle (A + iA^t)(a + ib), a + ib \rangle = \langle (Aa, a) - \langle Ab, b \rangle + (A^t a, b) + i((Ab, a) + (A^t a, a) - (Ab, b)) \rangle = (Aa, a) - (Ab, b) + (A^t a, b) + i((Ab, a) + (A^t a, a) - (Ab, b)).$$

Since $(Aa, a) - (Ab, b) + (A^t a, b) + (Ab, a) \in \mathbb{R}$, the result follows.

Conversely let $W(D) \subseteq \{(r, r) : r \in \mathbb{R}\}$. Then $D$ is essentially self adjoint[4,5,6]. Therefore $D = \alpha H + \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $H$ is Hermitian and there exists a Hermitian operator $T$ such that $D = e^{itT}$. Set $T = T_1 + iT_2$ where $T_1, T_2 \in B(H)_\mathbb{R}$. Since $T = T^*$, thus $T_1 + iT_2 = T_1^* - iT_2^*$. By Corollary 3.5,
Let $T_1^*, T_2^* \in B(H)_R$ and $T_1 = T_1^*, T_2 = -T_2^*$. Then $T_1 = T_1^1, T_2 = -T_2^1$. Now we have
\[ D = e^{\frac{i\pi}{2}}(T_1 + iT_2) = \frac{\sqrt{2}}{2}(T_1 - T_2) + i\frac{\sqrt{2}}{2}(T_1 - T_2)^t. \]
If $A = \frac{\sqrt{2}}{2}(T_1 - T_2)$, the proof is complete. \qed

**Theorem 3.7.** Let $A \in B(H)$ and $Ax = \bar{A}x$ for some $0 \neq x \in H$. Then
\[ W(A) \cap W(\bar{A}) \cap \mathbb{R} \neq \emptyset. \]

**Proof.** We may assume that $x$ is a unit vector. Then $\mu = \langle Ax, x \rangle = \langle \bar{A}x, x \rangle \in W(\bar{A})$. Hence $\mu \in W(A) \cap W(\bar{A})$. Since $W(A) = W(\bar{A})$ by Theorem 2.3, we have $\mu \in W(A) \cap W(\bar{A})$. By convexity of numerical range the line segment joins $\mu$ and $\bar{\mu}$ lies in $W(A)$ and $W(\bar{A})$. This line segment intersects the real line and the result follows. \qed

**Theorem 3.8.** Let $H$ be a Hilbert space and $A \in B(H)$, $A = \lambda I$ for some $\lambda \in \mathbb{C}$ if and only if for every $\alpha \in \mathbb{C}$, there exists a scalar $\beta \in \mathbb{C}$ such that $\alpha A + \beta I$ is self-conjugate.

**Proof.** The implication ($\Rightarrow$) is clear. Suppose that for every $\alpha \in \mathbb{C}$, there exists $\beta \in \mathbb{C}$ such that $\alpha A + \beta I$ is self-conjugate. If $H$ is one dimensional the result holds. Let $\xi = \{e_i\}_{i \in I}$ be an orthonormal basis for $H$ and $e_j, e_k \in \xi$ be arbitrary and distinct. First we prove that $A$ is diagonal. Suppose $Ae_j = \sum_{i \in I} \alpha_i e_i$ and $Ae_k = \sum_{i \in I} \beta_i e_i$. For every $\alpha \in \mathbb{C}$, there exists $\beta \in \mathbb{C}$ such that $(\alpha A + \beta I)e_j = \sum_{i \in I} \alpha r_i e_i, (\alpha A + \beta I)e_k = \sum_{i \in I} s_i e_i$, and $r_i, s_i \in \mathbb{R}$ for every $i \in I$. Hence, for every $i \neq j$, $\alpha \alpha_i = r_i \in \mathbb{R}$ and for every $i \neq k$, $\alpha \beta_i = s_i \in \mathbb{R}$. Since $\alpha$ is arbitrary, we must have $\alpha_i = 0$ for every $i \neq j$ and $\beta_i = 0$ for every $i \neq k$. Since $e_j, e_k \in \xi$, $A$ is diagonal. Now, suppose that $Ae_j = \lambda_j e_j$ and $Ae_k = \lambda_k e_k$. It is enough to show that $\lambda_j = \lambda_k$. But for every $\alpha \in \mathbb{C}$, there exists $\beta \in \mathbb{C}$, such that $\alpha \lambda_j e_j + \beta e_j = r_j e_j$ and $r_j \in \mathbb{R}$, and we have $\alpha \lambda_k e_k + \beta e_k = s_k e_k$ where $s_k \in \mathbb{R}$. Thus for every $\alpha \in \mathbb{R}$, $\alpha(\lambda_j - \lambda_k) \in \mathbb{R}$. Hence $\lambda_j - \lambda_k = 0$ and the result follows. \qed

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**References**


