Left Jordan derivations on Banach algebras

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Abstract. In this paper we characterize the left Jordan derivations
on Banach algebras. Also, it is shown that every bounded linear map
d : \mathcal{A} → \mathcal{M} from a von Neumann algebra \mathcal{A} into a Banach \mathcal{A}−module
\mathcal{M} with property that d(p^2) = 2pd(d) for every projection p in \mathcal{A} is a left
Jordan derivation.

Keywords: Left Jordan derivation, von Neumann algebra.

2000 Mathematics subject classification: Primary 46L10; Secondary 46L05,
46H25, 46L57.

1. Introduction

Let \mathcal{A} be a unital Banach algebra. We denote the identity of \mathcal{A} by 1. A
Banach \mathcal{A}−module \mathcal{M} is called unital provided that 1x = x = x1 for each
x ∈ \mathcal{M}. A linear (additive) mapping d : \mathcal{A} → \mathcal{M} is called a left derivations
(left ring derivation) if
\[ d(ab) = ad(b) + bd(a) \quad (a, b ∈ \mathcal{A}). \tag{1} \]

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Received 21 July 2009; Accepted 12 April 2010

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Also, \( d \) is called a left Jordan derivation (or Jordan left derivation) if
\[
d(a^2) = 2ad(a) \quad (a \in \mathcal{A}).
\] (2)

Bresar, Vukman \[4\], Ashraf et al. \[1, 2\], Jung and Park \[8\], Vukman \[13, 14\] studied left Jordan derivations and left derivations on prime rings and semiprime rings, which are in a close connection with so-called commuting mappings (see also \[7, 10, 11, 12\]).

Suppose that \( \mathcal{A} \) is a Banach algebra and \( \mathcal{M} \) is an \( \mathcal{A}-\)module. Let \( S \) be in \( \mathcal{A} \). We say that \( S \) is a left separating point of \( \mathcal{M} \) if the condition \( Sm = 0 \) for \( m \in \mathcal{M} \) implies \( m = 0 \).

We refer to \[3\] for the general theory of Banach algebras.

**Theorem 1.1.** Let \( \mathcal{A} \) be a unital Banach algebra and \( \mathcal{M} \) be a Banach \( \mathcal{A}-\)module. Let \( S \) be in \( Z(\mathcal{A}) \) such that \( S \) is a left separating point of \( \mathcal{M} \). Let \( f : \mathcal{A} \to \mathcal{M} \) be a bounded linear map. Then the following assertions are equivalent

a) \( f(ab) = af(b) + bf(a) \) for all \( a, b \in \mathcal{A} \) with \( ab = ba = S \).

b) \( f \) is a left Jordan derivation which satisfies \( f(Sa) = Sf(a) + af(S) \) for all \( a \in \mathcal{A} \).

**Proof.** First suppose that (a) holds. Then we have
\[
f(S) = f(1S) = f(S)1 + f(1)S = f(S) + f(1)S
\]
hence, by assumption, we get that \( f(1) = 0 \). Let \( a \in \mathcal{A} \). For scalars \( \lambda \) with \( |\lambda| < \frac{1}{\|a\|} \), \( 1 - \lambda a \) is invertible in \( \mathcal{A} \). Indeed, \( (1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n \). Then
\[
f(S) = f((1 - \lambda a)(1 - \lambda a)^{-1}S) = ((1 - \lambda a)^{-1}S)f((1 - \lambda a))
+ (1 - \lambda a)f((1 - \lambda a)^{-1}S)
= -\lambda \sum_{n=0}^{\infty} \lambda^n a^n S)f(a) + (1 - \lambda a)f(\sum_{n=1}^{\infty} \lambda^n a^n S)
= f(S) + \sum_{n=1}^{\infty} \lambda^n [f(a^n S) - a^n-1Sf(a) - af(a^n-1 S)].
\]
So
\[
\sum_{n=1}^{\infty} \lambda^n [f(a^n S) - a^n-1Sf(a) - af(a^n-1 S)] = 0
\]
for all \( \lambda \) with \( |\lambda| < \frac{1}{\|a\|} \). Consequently
\[
f(a^n S) - a^n-1Sf(a) - af(a^n-1 S) = 0 \quad (3)
\]
for all \( n \in \mathbb{N} \). Put \( n = 1 \) in (3) to get
\[
f(Sa) = f(aS) = af(S) + Sf(a). \quad (4)
\]
for all \( a \in \mathcal{A} \).
Now, put \( n = 2 \) in (3) to get
\[
f(a^2S) = aSf(a) + af(aS) = aSf(a) + a(af(S) + Sf(a)) = a^2f(S) + 2Saf(a).
\] (5)
Replacing \( a \) by \( a^2 \) in (4), we get
\[
f(a^2S) = a^2f(S) + Sf(a^2).
\] (6)
It follows from (5), (6) that
\[
S(f(a^2) - 2af(a)) = 0.
\] (7)
On the other hand \( S \) is right separating point of \( \mathcal{M} \). Then by (7), \( f \) is a left Jordan derivation.
Now suppose that the condition (b) holds. We denote \( aob := ab + ba \) for all \( a, b \in \mathcal{A} \). It follows from left Jordan derivation identity that
\[
f(aob) = 2(bf(a) + af(b))
\] (8)
for all \( a, b \in \mathcal{A} \) (see proposition 1.1 of [4]). On the other hand, we have
\[
a \circ (a \circ b) = a \circ (ab + ba) = a^2 \circ b + 2aba
\]
for all \( a, b \in \mathcal{A} \). Then
\[
2f(aba) = f(a \circ (a \circ b)) - f(a^2 \circ b)
\]
\[
= 2[(a \circ b)f(a) + af(a \circ b)] - 2[bf(a^2) + a^2f(b)]
\]
\[
= 2[(ab + ba)f(a) + 2a(bf(a) + af(b))] - 2[2baf(a) + a^2f(b)]
\]
\[
= 6abf(a) + 2a^2f(b) - 2baf(a).
\]
Hence,
\[
f(aba) = 3abf(a) + a^2f(b) - baf(a)
\] (9)
for all \( a, b \in \mathcal{A} \). Now suppose that \( ab = ba = S \), then
\[
f(Sa) = 3abf(a) + a^2f(b) - baf(a) = 2abf(a) + a^2f(b).
\] (10)
On the other hand \( S \in \mathcal{Z}(\mathcal{A}) \). Then by multiplying both sides of (10) by \( b \) to get
\[
Sf(S) - Sbf(a) - Saf(b) = 0
\] (11)
since \( S \in \mathcal{Z}(\mathcal{A}) \), then it follows from (11) that
\[
[f(S) - f(a)b - f(b)a]S = 0
\]
then we have
\[
f(S) = f(a)b + f(b)a.
\]
\( \square \)
Now, we characterize the left Jordan derivations on von Neumann algebras.
Theorem 1.2. Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{M}$ be a Banach $\mathcal{A}$–module and $d: \mathcal{A} \to \mathcal{M}$ be a bounded linear map with property that $d(p^2) = 2pd(p)$ for every projection $p$ in $\mathcal{A}$. Then $d$ is a left Jordan derivation.

Proof. Let $p, q \in \mathcal{A}$ be orthogonal projections in $\mathcal{A}$. Then $p + q$ is a projection wherefore by assumption,

$$2pd(p) + 2qd(q) = d(p) + d(q) = d(p + q) = 2(p + q)d(p + q) = 2[pd(p) + pd(q) + qd(q) + qd(p)].$$

It follows that

$$pd(q) + qd(p) = 0. \quad (12)$$

Let $a = \sum_{j=1}^{n} \lambda_{j} p_{j}$ be a combination of mutually orthogonal projections $p_{1}, p_{2}, ..., p_{n} \in \mathcal{A}$. Then we have

$$p_{i}d(p_{j}) + p_{j}d(p_{i}) = 0 \quad (13)$$

for all $i, j \in \{1, 2, ..., n\}$ with $i \neq j$. So

$$d(a^2) = d(\sum_{j=1}^{n} \lambda_{j}^2 p_{j}) = \sum_{j=1}^{n} \lambda_{j}^2 d(p_{j}). \quad (14)$$

On the other hand by (13), we obtain that

$$ad(a) = \left( \sum_{i=1}^{n} \lambda_{i} p_{i} \right) \left( \sum_{j=1}^{n} \lambda_{j} d(p_{j}) \right) = \lambda_{1} p_{1} \sum_{j=1}^{n} \lambda_{j} d(p_{j})$$

$$+ \lambda_{2} p_{2} \sum_{j=1}^{n} \lambda_{j} d(p_{j}) + ... + \lambda_{n} p_{n} \sum_{j=1}^{n} \lambda_{j} d(p_{j})$$

$$= \sum_{j=1}^{n} \lambda_{j}^2 p_{j} d(p_{j}).$$

It follows from above equation and (14) that $d(a^2) = 2ad(a)$. By the spectral theorem (see Theorem 5.2.2 of [9]), every self adjoint element $a \in \mathcal{A}_{sa}$ is the norm–limit of a sequence of finite combinations of mutually orthogonal projections. Since $d$ is bounded, then

$$d(a^2) = 2ad(a) \quad (15)$$

for all $a \in \mathcal{A}_{sa}$. Replacing $a$ by $a + b$ in (15), we obtain

$$d(a^2 + b^2 + ab + ba) = 2(a + b)(d(a) + d(b))$$

$$= 2ad(a) + 2bd(b) + 2ad(b) + 2bd(a),$$

$$d(ab + ba) = 2ad(b) + 2bd(a) \quad (16)$$
for all $a, b \in A_{sa}$. Let $a \in A$. Then there are $a_1, a_2 \in A_{sa}$ such that $a = a_1 + ia_2$. Hence,

$$d(a^2) = d(a_1^2 + a_2^2 + i(a_1a_2 + a_2a_1))$$

$$= 2a_1d(a_1) + 2a_2d(a_2) + i[2a_1d(a_2) + 2a_2d(a_1)]$$

$$= 2ad(a).$$

This completes the proof of theorem. □

**Corollary 1.3.** Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{M}$ be a Banach $\mathcal{A}$–module and $d : \mathcal{A} \to \mathcal{M}$ be a bounded linear map. Then the following assertions are equivalent

a) $ad(a^{-1}) + a^{-1}d(a) = 0$ for all invertible $a \in \mathcal{A}$.

b) $d$ is a left Jordan derivation.

c) $d(p^2) = 2pd(p)$ for every projection $p$ in $\mathcal{A}$.

**Proof.** $(a) \iff (b)$ follows from Theorem 1.1, and $(b) \iff (c)$ follows from Theorem 1.2. □

In 1996, Johnson [6] proved the following theorem (see also Theorem 2.4 of [3]).

**Theorem 1.4.** Suppose $\mathcal{A}$ is a $C^*$–algebra and $\mathcal{M}$ is a Banach $\mathcal{A}$–module. Then each Jordan derivation $d : \mathcal{A} \to \mathcal{M}$ is a derivation.

We do not know whether or not every left Jordan derivation on a $C^*$–algebra is a left derivation.

**References**


