Energy of Graphs, Matroids and Fibonacci Numbers

Saeid Alikhani and Mohammad A. Iranmanesh
Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
E-mail: alikhani@yazduni.ac.ir
E-mail: iranmanesh@yazduni.ac.ir

Abstract. The energy $E(G)$ of a graph $G$ is the sum of the absolute values of the eigenvalues of $G$. In this article we consider the problem whether generalized Fibonacci constants $\varphi_n$ ($n \geq 2$) can be the energy of graphs. We show that $\varphi_n$ cannot be the energy of graphs. Also we prove that all natural powers of $\varphi_{2n}$ cannot be the energy of a matroid.

Keywords: Graph energy, Fibonacci numbers, Matroid.


1. Introduction

Let $G = (V, E)$ be a simple and finite graph of order $n$ where $V$ and $E$ be vertex and edge sets of $G$, respectively. If $A$ is the adjacency matrix of $G$, then the eigenvalues of $A$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are said to be the eigenvalues of the graph $G$. These are the roots of the characteristic polynomial $\phi(G, \lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$. An interval $I$ is called a zero-free interval for a characteristic polynomial $\phi(G, \lambda)$ if $\phi(G, \lambda)$ has no root in $I$.

The energy of the graph $G$ is defined as $E = E(G) = \sum_{i=1}^{n} |\lambda_i|$. This definition was put forward by I. Gutman [6] and was motivated by earlier

*Corresponding author

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results in theoretical chemistry [7]. It is easy to see that if a undirected graph $G$ has positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then $E = 2\Sigma_{i=1}^{m} \lambda_i$.

A matroid $M$ consists of a non-empty finite set $E$ and a non-empty collection $I$ of subsets of $E$, called independent sets, satisfying the following properties:

(i) any subset of an independent set is independent,
(ii) if $I$ and $J$ are independent sets with $|J| > |I|$ then there is an element $e$, contained in $J$ but not in $I$ such that $I \cup \{e\}$ is independent.

Let $M = (E, I)$ be a matroid defined in terms of its independent sets. Then a subset of $E$ is dependent if it is not independent and a minimal dependent set is called a cycle. If $M(G)$ is the cycle matroid of a graph $G$ then the cycles of $M(G)$ are precisely the cycles of $G$. A graphic matroid is a matroid $M(G)$ on the set of edges of a graph $G$ by taking the cycles of $G$ as the cycles of the matroid. For a subset $A$ of $E$, the rank of $A$ denoted by $r(A)$, is the size of the largest independent set contained in $A$. Note that the rank of $M$ is equal to $r(E)$ since a subset $A$ of $E$ is independent if and only if $r(A) = |A|$. Recall that a complex number $\zeta$ is called an algebraic number (respectively, algebraic integer) if it is a zero of some monic polynomial with rational (respectively, integer) coefficients (see [13]). Corresponding to any algebraic number $\zeta$, there is a unique monic polynomial $p$ with rational coefficients, called the minimal polynomial of $\zeta$ (over the rationals), with the property that $p$ divides every polynomial with rational coefficients having $\zeta$ as a zero. (The minimal polynomial of $\zeta$ has integer coefficients if and only if $\zeta$ is an algebraic integer.) Since the characteristic polynomial is a monic polynomial in $\lambda$ with integer coefficients, its zeros are, by definition, algebraic integers. This naturally raises the question: Which algebraic integers can occur as the energy of a graph?

In 2004 Bapat and Pati [2] obtained the following result:

**Theorem 1.1.** The energy of a graph cannot be an odd integer.

In 2008 Pirzada and Gutman communicated an interesting result:

**Theorem 1.2.** ([10]) The energy of a graph cannot be the square root of an odd integer.

Also [1] and [11] contribute to the question of which numbers can be graph energies.

In this paper we prove some further results of this kind.

In Section 2, we prove that $\tau$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, cannot be the energy of a graph. Also we generalize this result and prove that all $n$-anacci numbers cannot be energy of graphs. In Section 3, we study natural powers of $2n$-anacci constants as the energy of a matroid. We show that all natural powers of $2n$-anacci constants cannot be the energy of a matroid.
2. ENERGY OF GRAPH AND THE GOLDEN RATIO

In this section, we investigate the quantity $\tau$, where $\tau$ is the golden ratio as a graph energy. We show that $\tau$ cannot be a graph energy. Also we prove that all $n$-anacci constants cannot be a graph energy. We need the following theorem:

**Theorem 2.1.** ([3]) If graph $G$ with order $n$ has no isolated vertices, then $E(G) \geq 2\sqrt{n-1}$, with equality for stars.

The following theorem is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** The golden ratio cannot be the energy of a graph.

*Proof.* Since for every $n \geq 2$, $2\sqrt{n-1} > \frac{1+\sqrt{5}}{2}$, the result is true for every graphs of order $n \geq 2$. Since $\tau$ is not the energy of $K_1$, therefore we have the result by Theorem 2.1. \hfill \Box

Fibonacci numbers are terms of the sequence defined in a quite simple recursive fashion.

An $n$-step ($n \geq 2$) Fibonacci sequence $F^{(n)}_k$, $k = 1, 2, 3, \ldots$ is defined by letting $F^{(n)}_1 = F^{(n)}_2 = \ldots = F^{(n)}_n = 1$ and other terms according to the linear recurrence equation $F^{(n)}_k = \sum_{i=1}^{k-1} F^{(n)}_{k-i}$, $(k > 2)$. The limit $\varphi_n = \lim_{k \to \infty} \frac{F^{(n)}_k}{F^{(n)}_{k-1}}$ is called the $n$-anacci constant.

It is easy to see that $\varphi_n$ is the real positive zero of $f_n(x) = x^n - x^{n-1} - \ldots - x - 1$, and this polynomial is the minimal polynomial of $\varphi_n$ over $\mathbb{Z}[x]$. It is obvious that $\varphi_n$ is a zero of $g_n(x) = x^n(2-x) - 1$. Note that $\varphi_2 = \tau$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $\lim_{n \to \infty} \varphi_n = 2$ (see [9]).

**Theorem 2.3.** For every integer $n \geq 2$, the $n$-anacci numbers $\varphi_n$ cannot be the energy of a graph.

*Proof.* By above statements, for every $n \geq 2$, $\{\varphi_n\}$ is an increasing sequence and $\varphi_n < 2$. Therefore we have the result similar to the proof of Theorem 2.2. \hfill \Box

3. $2n$-ANACCI AND ENERGY OF MATROID

In this section we shall study natural powers of $2n$-anacci numbers as the energy of a matroid. Characteristic polynomials of matroids were first studied by Rota [12]. Heron [8] defined chromatic polynomials of matroids and showed that they are equivalent to characteristic polynomials.

We need the following theorem which is about the zeros of characteristic polynomials of matroids

**Theorem 3.1.** ([5]) Let $M$ be a loopless matroid with rank $r$ and characteristic polynomial $P(M, t)$. Then
Proof. Let $\alpha$ be the energy of a graph.

(i) $P(M,t) = t^r - |M|t^{r-1} + k_{r-2}t^{r-2} - \cdots + (-1)^r k_0$ where $k_0, \cdots, k_{r-2}$ are positive integers.

(ii) $(-1)^r P(M,t) > (1 - t)^r$ for $t \in (-\infty, 1)$.

(iii) $P(M,1) = 0$, and the multiplicity of 1 as a zero of $P(M,t)$ is equal to the number of components of $M$.

(iv) if $r(M)$ and $c(M)$ be rank and the number of components of $M$ respectively, then for $t \in (1, \frac{32}{27})$, we have $(-1)^{r(M) - c(M)}P(M,t) \geq (t-1)^{r(M)}$.

By Theorem 3.1, we deduce that the maximal zero-free intervals for characteristic polynomials of loopless matroids are precisely $(-\infty, 1)$ and $(1, \frac{32}{27})$.

Using the terminology and notation from the book [4], we define two operations with graphs. By $V(G)$ and $E(G)$ are denoted the vertex and edge sets, respectively, of the graph $G$. Let $G_1$ and $G_2$ be two graphs with disjoint vertex sets of orders $n_1$ and $n_2$, respectively. The direct product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices $(x_1, x_2) \in V(G_1 \times G_2)$ and $(y_1, y_2) \in V(G_1 \times G_2)$ are adjacent if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$. The sum of $G_1$ and $G_2$, (or Cartesian product) denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices $(x_1, x_2) \in V(G_1 + G_2)$ and $(y_1, y_2) \in V(G_1 + G_2)$ are adjacent if and only if either $(x_1, y_1) \in E(G_1)$ and $x_2 = y_2$ or $(x_2, y_2) \in E(G_2)$ and $x_1 = y_1$. The above specified two graph products have the following spectral properties (see [4], p.70). Let $\lambda_i^{(1)}$, $i = 1, \cdots, n_1$, and $\lambda_j^{(2)}$, $j = 1, \cdots, n_2$, be, respectively, the eigenvalues of the graphs $G_1$ and $G_2$.

Lemma 3.1. The eigenvalues of $G_1 \times G_2$ are $\lambda_i^{(1)} \lambda_j^{(2)}$, $i = 1, \cdots, n_1$; $j = 1, \cdots, n_2$.

Lemma 3.2. The eigenvalues of $G_1 + G_2$ are $\lambda_i^{(1)} + \lambda_j^{(2)}$, $i = 1, \cdots, n_1$; $j = 1, \cdots, n_2$.

Now, we state and prove the following theorem:

Theorem 3.2. If $\alpha$ is not a root of any characteristic polynomial of graph, then $\alpha$ cannot be the energy of a graph.

Proof. Suppose that there exist a graph $G$ such that $E(G) = \alpha$. Let $\lambda_1, \lambda_2, \cdots, \lambda_m$ be positive eigenvalues of $G$. Then in view of the fact that the sum of all eigenvalues of any graph is equal to zero, $E(G) = 2 \sum_{i=1}^{m} \lambda_i$. Denote $\lambda_1 + \lambda_2 + \cdots + \lambda_m$ by $\lambda$. By Lemma 3.1 $\lambda$ is an eigenvalue of some graph $H$ isomorphic to the sum of $m$ disjoint copies of the graph $G$. By Lemma 3.2, $2\lambda$ is an eigenvalues of the product of $P_2$ and $H$. Therefore $\alpha$ is an eigenvalue of $H \times P_2$, a contradiction. Hence we have the result.

We need the following theorem to show our main results in this section.

Theorem 3.3. ([9]) The polynomial $f_n(x) = x^n - x^{n-1} - \cdots - x - 1$ is an irreducible polynomial over $\mathbb{Q}$. 
Here, we may prove that all natural powers of $2n$-anacci constants cannot be the energy of a matroid.

**Theorem 3.4.** All natural powers of $\phi_{2n}$ cannot be energy of matroids.

**Proof.** By Theorem 3.2, it suffices to prove that $\phi_{2n}^m (m \in \mathbb{N})$ cannot be a characteristic zero. Suppose that $\phi_{2n}^m (m \in \mathbb{N})$ is a characteristic zero, that is there exists a characteristic polynomial

$$P(G, \lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_1\lambda$$

such that $P(G, \phi_{2n}^m) = 0$. Therefore,

$$\phi_{2n}^{mk} + a_{k-1}\phi_{2n}^{m(k-1)} + \cdots + a_1\phi_{2n}^m = 0.$$ 

Hence $\phi_{2n}$ is a zero of the polynomial,

$$Q(\lambda) = \lambda^{mk} + a_{k-1}\lambda^{mk-m} + \cdots + a_1\lambda^m$$

in $\mathbb{Z}[x]$. But $f_{2n}(\lambda) = \lambda^{2n} - \lambda^{2n-1} - \cdots - \lambda - 1$ is the minimal polynomial of $\phi_{2n}$ over $\mathbb{Z}[x]$. Therefore $f_{2n}(\lambda)$ divides $Q(\lambda)$. Since $f_{2n}(0) = -1 < 0$ and $f_{2n}(-1) = 1 > 0$, $f_{2n}(\lambda)$ and so $Q(\lambda)$ must have a zero say $\alpha$, in $(-1, 0)$. Therefore, $\alpha^m$ is a root of $P(G, \lambda)$. Since $\alpha^m \in (-1, 0) \cup (0, 1)$, we have a contradiction. □

**Conjecture 3.1.** Let $n \in \mathbb{N}$. Then all natural powers of $2n+1$-anacci numbers cannot be the energy of a graph and a matroid.

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**References**


