Clifford Wavelets and Clifford-valued MRAs

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\textbf{Abstract.} In this paper using the Clifford algebra over $\mathbb{R}^4$ and its matrix representation, we construct Clifford scaling functions and Clifford wavelets. Then we compute related mask functions and filters, which arise in many applications such as quantum mechanics.

\textbf{Keywords:} Clifford Wavelets, Clifford algebra, Multiresolution Analysis, Wavelets.


1. \textbf{Introduction}

A complex-valued representation of a real 1-dimensional signal is an important tool in analysis of signal processing. The reason is that in its polar representation, the modulus of the complex signal is identified as a local quantitative measure of a signal, called local amplitude, and the argument of the complex signal is identified as a local measure for the qualitative information of a signal, called local phase. First step for generalizing such representation system was quaternion-valued representation, on which a signal can be expressed by four parameters as its local quantitative measures.

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On the other hand wavelets are a very useful and wide applied tools for practical applications in signal and image processing, multi-satellite measurements of electromagnetic wave fields, analysis of climate-related time-series and analysis space weather effects and so on. One usual way to construct wavelets pass through multiresolution analysis (MRA), which is a procedure for constructing wavelets from a scaling function. Now if the scaling function is a matrix of functions, we deal with matrix-valued MRAs. In this paper we show that any real or complex Clifford algebra can be identified with a suitable matrix algebra, then via this representation, Clifford-valued scaling functions, Clifford-valued MRAs and Clifford wavelets are given.

Notations. For an algebra $K$, we denote its product with ".". $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are algebra of real numbers, complex numbers and quaternions, respectively. $K[n]$ is the algebra of $n \times n$ matrices over field $K$. $\otimes_K$ denotes tensor product over field $K$.

This paper is organized as follow: in second section we introduce the n-dimensional Clifford algebra (on brief) and some useful theorems on it, then we discuss the $\text{Cl}(\mathbb{R}^4)$ and $\text{Cl}(\mathbb{C}^4)$ (real and complex forms of Clifford algebra on $\mathbb{R}^4$, resp.) and their matrix representations. Section 3 consists of multiresolution analysis (MRA) and Clifford wavelet structures. In section 4, we compute Clifford wavelets matrices on $\mathbb{R}^4$.

2. Clifford Algebra

In this section we mention some definitions and basic facts about Clifford algebras.

Definition 2.1. let $V$ be a finite dimensional vector space on the field $\mathbb{F}$. A quadratic form (q-form) on $V$ is a function $h: V \times V \rightarrow \mathbb{F}$, such that

$h(\alpha x_1 + x_2, y) = \alpha h(x_1, y) + h(x_2, y)$
$h(x, \alpha y_1 + y_2) = \alpha h(x, y_1) + h(x, y_2)$.

Furthermore if $h(x, y) = h(y, x)$ then $h$ is called symmetric. For any q-form $h$, there exists a matrix representation $A = (A_{ij})$ such that $A_{ij} = h(e_i, e_j)$ where $\{e_1, e_2, \cdots, e_n\}$ is a basis for $V$. The q-form $h$ is called nondegenerate, if $\text{det}(h(e_i, e_j)) \neq 0$.

Let $V$ be an $n$-dimensional vector space on the field $\mathbb{F}$, and $h$ be a nondegenerate symmetric q-form on $V$, then there exists an ordered basis $B = \{e_1, e_2, \cdots, e_n\}$ for $V$ such that $A = (A_{ij})$ is diagonal. In particular for $\mathbb{F} = \mathbb{R}$

$h(e_i, e_j) = \begin{cases} 
\pm 1 & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}$

If the matrix $A$ have $p$-times $1$ and $q$-times $-1$ on its diameter such that $p + q = n$, then $h$ will be shown with $h(p, q)$. For $h$, a nondegenerate q-form on
real vector space $V$, the pair $(V,h)$ is called a quadratic space (q-space). For describing the Clifford algebra on vector space $V$, consider the commutative tensor algebra $T(V) = \bigoplus_{r=0}^{\infty} V^r$ on real q-space $(V,h)$ with unit $1$. Let $I_h(V) = (V \otimes V + h(V,V))$ then $I_h$ is a two-sided ideal in $T(V)$. The quotient space $T(V)/I_h(V)$ is called the **Clifford algebra** on $V$ and is denoted by $Cl(V,h)$. The induced product, from tensor product on $T(V)$, is called **Clifford product** and will be shown with “.”, $(Cl(V,h),\cdot)$ is again a commutative algebra with unit. If $h$ is $h(p,q)$ then $Cl(V,h)$ will be shown by $Cl(p,q)$.

By considering the canonical projection map $\pi_h: T(V) \longrightarrow Cl(V,h)$, one can find that the map $\theta_V: V \longrightarrow Cl(V,h)$ is one-to-one. This fact says that $Cl(V,h)$ is generated by vector space $V \subset Cl(V,h)$ and identity $1$, and its product satisfies the following relations:

1) $v \cdot v = -h(v,v)1$ for any $v \in Cl(V,h)$
2) $v \cdot w + w \cdot v = -2h(v,w)$.

In view of previous equations we can obtain the universal map for Clifford algebras as follow:

**Proposition 2.1.** Let $\mathcal{A}$ be a commutative $\mathbb{K}$-Algebra with unit 1, and $f: V \longrightarrow \mathcal{A}$ be a linear map such that: $f(v) \cdot f(v) = -h(v,v)1$ for any $v \in V$, then $f$ can be uniquely extended to the algebraic homomorphism $\tilde{f}: Cl(V,h) \longrightarrow \mathcal{A}$. Furthermore, $Cl(V,h)$ is the unique associated $\mathbb{K}$-Algebra with this property.

In other word if $(V,h)$ is a q-space, then there exists a Clifford algebra associated to it and is unique up to an isomorphism. This is easy to show that if $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal basis for real vector space $V$, then the set $\{1, e_1, e_2, e_{ij} = e_1 e_2 \cdots e_{ij}, \cdots : i + 1 = j, j + 1 = k\}$ is a basis for $Cl(V,h)$.

Note that $Cl(V,h) = \frac{T(V)}{I_h} = \frac{R \oplus V \oplus V \otimes V \otimes V \oplus \cdots}{(V \otimes V + h(V,V))}$, and

$T(V) = a_0 + \sum_{i=1}^{n} a_i e_i + \sum a_{ij} e_i e_j + \sum a_{ijk} e_i e_j e_k + \cdots + a_{i_1 \cdots i_n} e_1 e_2 \cdots e_n$.

Also $V \otimes V + h(V,V)1 = 0$ implies that $V \otimes V = -h(V)V$.

**Example 2.2.** Let $V = \mathbb{R}^2$, and $h$ be the quadratic form obtained by the matrix

$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e $V = \mathbb{R}^2 = \langle e_1, e_2 \rangle$. Dim$V = 2$, so Dim$Cl(V) = 4$ and

$Cl(V) = Cl(\mathbb{R}^2) = \langle 1, e_1, e_2, e_1 e_2 \rangle$

$= \{a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_1 e_2 : e_1^2 = e_2^2 = -1, e_1 \cdot e_2 = e_2 \cdot e_1 \}$

where $(e_1 \cdot e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = (-1)(-1)(-1) = -1$.

So if we define $\psi: Cl(\mathbb{R}^2) \longrightarrow \mathbb{H}$ by

$\psi(1) = 1, \psi(e_1) = i, \psi(e_2) = j, \psi(e_1 e_2) = \psi(e_3) = k$

then, since $\psi$ is an algebraic homomorphism, $Cl(\mathbb{R}^2) \cong \mathbb{H}$. 
There are useful algebraic isomorphisms for $Cl(p, q)$ such as
\begin{equation}
Cl(n, 0) \otimes Cl(0, 2) \cong Cl(0, n + 2)
\end{equation}
\begin{equation}
Cl(0, n) \otimes Cl(2, 0) \cong Cl(n + 2, 0)
\end{equation}
\begin{equation}
Cl(p, q) \otimes Cl(1, 1) \cong Cl(p + 1, q + 1),
\end{equation}
where $n, p, q \geq 0$ such that $n = p + q$.

Now we introduce a useful tool. Complexification is one of the important tools in linear algebra which make it more flexible. Let $(V, h)$ be a real q-space. The complexification of $V$ is the vector space $W = V \otimes_{R} C$ such that for $w \in W : w = v \otimes \lambda = v \otimes (a + ib) = v \otimes a + v \otimes ib = 1 \otimes av + i(1 \otimes bv)$. This means that any element of $W$ can be written as $x + iy$ where $x, y \in V$. Now let $g$ be a nondegenerate q-form on $V$. Then $g_{W} : W \times W \longrightarrow C$ is a nondegenerate q-form on $W = V \otimes_{C} C$ defined by $g_{W}(x \otimes \lambda, y \otimes \gamma) = \lambda \gamma g(x, y)$. From this point of view the complexification of $Cl(V)$ is $Cl(V) \otimes C$ and if $W = V \otimes_{C} C$ then $Cl(W) = Cl(V) \otimes_{R} C$.

**Lemma 2.3.** Let $V$ be a real $n$-dimensional vector space, then
\begin{equation}
Cl(V \oplus \mathbb{R}^{2}) \otimes C \cong (Cl(V) \otimes_{R} C) \otimes_{C} (Cl(\mathbb{R}^{2}) \otimes C).
\end{equation}

**Proof.** Let $\{\nu_{1}, \ldots, \nu_{n}\}$ be an orthonormal basis for $V$ and $\{e_{1}, e_{2}\}$ be the standard basis for $\mathbb{R}^{2}$. Consider the real map $\theta : V \oplus \mathbb{R}^{2} \longrightarrow (Cl(V) \otimes_{R} C) \otimes_{C} (Cl(\mathbb{R}^{2}) \otimes C)$ defined by
\begin{equation}
(\nu_{j}, 0) \longmapsto i\nu_{j} \otimes e_{1}e_{2}, \quad 1 \leq j \leq n, \quad (0, e_{r}) \longmapsto 1 \otimes e_{r} \quad r = 1, 2.
\end{equation}
so $\theta$ extends to algebra homomorphism $Cl(V \oplus \mathbb{R}^{2}) \otimes C \cong (Cl(V) \otimes_{R} C) \otimes_{C} (Cl(\mathbb{R}^{2}) \otimes C)$. On the other hand domain and range of $\theta$ have the same dimension and it is onto, so $\theta$ is isometry.

\[ \Box \]

the following lemma is the key tool for describing the complex Clifford algebras.

**Lemma 2.4.** Let $V$ be a real vector space such that $\text{dim}V = 2n$, then $Cl(V) \otimes_{R} C$ is isomorphic to the matrix algebra $C[2^{n}]$. If $\text{dim}V = 2n + 1$ then $Cl(V) \otimes_{R} C$ is isomorphic to $C[2^{n}] \oplus C[2^{n}]$.

**Proof.** We refer interested reader to [2], for an extended proof. \[ \Box \]

2.1. **Construction of Clifford Algebra on $\mathbb{R}^{4}$.** Now we are going to show that for $V = \mathbb{R}^{4}$, $Cl(V)$ is $\mathbb{H}[2] \cong C[4]$. We know that, via the algebraic isomorphism
\begin{equation}
a + bi + cj + dk \longmapsto \begin{pmatrix} a + id & b + ic \\ -b + ic & a - id \end{pmatrix},
\end{equation}
$\mathbb{H}$ is isomorphic to $\mathbb{C}[2]$. Now if $V = \mathbb{R}^4 = \langle e_0, e_1, e_2, e_3 \rangle$ with Riemannian form

$$h = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$
on it, then

$$Cl(\mathbb{R}^4) = \{ a_0 + \sum_{i=1}^{4} a_i e_i + \sum_{i<j} a_{ij} e_i e_j + \sum_{i<j<k} a_{ijk} e_i e_j e_k + a_{1234} e_1 e_2 e_3 e_4 : e_i e_j = -e_j e_i, e_i^2 = -1, a_i \in \mathbb{R} \}$$

this means that $Cl(\mathbb{R}^4)$ is spanned by $2^4 = 16$ vectors:

$$1, E_1, E_2, E_3, E_4, E_1 E_2, E_1 E_3, E_1 E_4, E_2 E_3, E_2 E_4, E_3 E_4, E_1 E_2 E_3, E_1 E_2 E_4, E_1 E_3 E_4, E_2 E_3 E_4, E_1 E_2 E_3 E_4,$$

as a basis. On the other hand

$$Cl(0, 2) = Cl(\mathbb{R}^2, \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}) = \langle e_0, e_1, e_2, e_3 = e_1 e_2 \rangle$$

where $e_0 = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & -1 \\
1 & 0 \end{pmatrix}$

such that $e_0^2 = e_1^2 = e_2^2 = 1, (e_1 e_2)^2 = -1$ and

$$Cl(2, 0) = Cl(\mathbb{R}^2, \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}) \cong \mathbb{H} = \langle e_0', e_1', e_2', e_3' = e_1' e_2' \rangle$$

where $e_0' = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}$, $e_1' = \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}$, $e_2' = \begin{pmatrix} 1 & i \\
0 & 0 \end{pmatrix}$, $e_3' = \begin{pmatrix} i & 0 \\
0 & -i \end{pmatrix}$.

Now if in (2.1) we set $n = 2$ then

$$Cl(0, 2) \otimes Cl(2, 0) \cong Cl(4, 0).$$

Through the relation $A \otimes B = (A_{ij} B)$ between matrices we can find the matrix representation for $Cl(4, 0)$’s bases:

$$E_0 = e_0 \otimes e_0' = I, E_1 = e_0 \otimes e_3', E_2 = e_2 \otimes e_1', E_3 = e_1 \otimes e_1', E_4 = e_0 \otimes e_2', E_1 E_2 = e_2 \otimes e_2', E_1 E_3 = e_1 \otimes e_2', E_1 E_4 = -(e_0 \otimes e_1'), E_2 E_3 = e_3 \otimes e_0', E_2 E_4 = e_1 \otimes e_3', E_3 E_4 = e_2 \otimes e_3', E_1 E_2 E_3 = e_3 \otimes e_3', E_1 E_2 E_4 = -(e_2 \otimes e_0'), E_2 E_3 E_4 = e_3 \otimes e_2', E_1 E_3 E_4 = -(e_1 \otimes e_0'), E_1 E_2 E_3 E_4 = -(e_3 \otimes e_1').$$

This means that for any $\rho \in Cl(\mathbb{R}^4)$ we have

$$\rho = a_0 + a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4 + a_{12} E_1 E_2 + a_{13} E_1 E_3 + a_{14} E_1 E_4 + a_{23} E_2 E_3 + a_{24} E_2 E_4 + a_{34} E_3 E_4 + a_{234} E_2 E_3 E_4 + a_{1234} E_1 E_2 E_3 E_4 + a_{1234} E_1 E_2 E_3 E_4 + a_{1234} E_1 E_2 E_3 E_4$$

By the above matrix representation for $E_i$’s, associated matrix to $\rho$ is:
3.1. General construction and mask functions. Let \( L^2(\mathbb{R}, \mathbb{C}[r]) = \{ F(t) = (F_{m,n}(t)) : t \in \mathbb{R}, F_{m,n} \in L^2(\mathbb{R}), 1 \leq m, n \leq r \} \) be the space of matrix-valued functions defined on \( \mathbb{R} \) with values in \( \mathbb{C}[r] \). The norm on \( L^2(\mathbb{R}, \mathbb{C}[r]) \) is the Frobenious norm: \( \| F(t) \| = \sqrt{\sum_{m,n} \int_{\mathbb{R}} |F_{m,n}(t)|^2 dt} \). and for \( F, G \in L^2(\mathbb{R}, \mathbb{C}[r]) \), the "inner product" is defined by \( (F, G)_{L^2(\mathbb{R}, \mathbb{C}[r])} := \int_{\mathbb{R}} F(t) G^\dagger(t) dt \).

Now if we set
\[
\begin{align*}
A_1 &= a_0 + ia_1, \\
B_1 &= -a_{124} + ia_{34}, \\
A_2 &= a_2 + ia_4, \\
B_2 &= a_{14} + ia_{12}, \\
A_3 &= a_{23} + ia_{24}, \\
B_3 &= -a_{134} + ia_{123}, \\
A_4 &= a_3 + ia_{13}, \\
B_4 &= a_{1234} + ia_{234},
\end{align*}
\]
and then set \( A = A_1 + B_1, \ B = A_1 - B_1, \ C = A_2 - B_2, \ D = A_2 + B_2, \ E = A_3 + B_3, \ F = -A_3 + B_3, \ G = A_4 + B_4, \ H = A_4 - B_4, \rho \) can be shown as
\[
(2.2) \quad \rho \cong \begin{pmatrix} A & -C & F & -G \\ C & A & G & F \\ E & -H & B & -D \\ H & E & D & B \end{pmatrix} := M_Q,
\]
A simpler representation for \( \rho \) is \( \rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \lambda \end{pmatrix} \), which is a \( 2 \times 2 \)-matrix in \( \mathbb{H} \), with \( \alpha = A - jC, \ \beta = F - jG, \ \gamma = E - jH, \ \lambda = B - jD \).

Till now we've found the matrix representations for \( Cl(\mathbb{R}^4) \) such that \( \mathbb{H}[2] \cong \mathbb{C}[4] \). By considering the complexification of \( Cl(\mathbb{R}^4) \) we will work with \( \mathbb{C}[4] \), which is a more general and flexible case.

Let \( M_Q \) be the set of all \( 4 \times 4 \)-matrices in \( \mathbb{C}[4] \) which are like above then \( M_Q \) excepting the zero matrix is a subgroup of \( GL(2, \mathbb{C}) \) in the sense of matrix multiplication.

In next step we generalize these concepts to an MRA.

3. \( Cl(\mathbb{R}^4) \)-valued MRA

### 3.1. General construction and mask functions.

Let \( L^2(\mathbb{R}, \mathbb{C}[r]) = \{ F(t) = (F_{m,n}(t)) : t \in \mathbb{R}, F_{m,n} \in L^2(\mathbb{R}), 1 \leq m, n \leq r \} \) be the space of matrix-valued functions defined on \( \mathbb{R} \) with values in \( \mathbb{C}[r] \). The norm on \( L^2(\mathbb{R}, \mathbb{C}[r]) \) is the Frobenious norm: \( \| F(t) \| = \sqrt{\sum_{m,n} \int_{\mathbb{R}} |F_{m,n}(t)|^2 dt} \). and for \( F, G \in L^2(\mathbb{R}, \mathbb{C}[r]) \), the "inner product" is defined by \( (F, G)_{L^2(\mathbb{R}, \mathbb{C}[r])} := \int_{\mathbb{R}} F(t) G^\dagger(t) dt \).

1. \( \langle F_1, aF_2 + bF_3 \rangle = a^\dagger \langle F_1, F_2 \rangle + b^\dagger \langle F_1, F_3 \rangle \)
2. \( \langle F_1, F_2 \rangle = \langle F_2, F_1 \rangle^\dagger. \)
Here the orthogonality of $\mathbf{F}_j$ and $\mathbf{F}_k$ is identified with $\langle \mathbf{F}_j, \mathbf{F}_k \rangle = I_r \delta_{jk}$ where $I_r$ is identity matrix and $\delta_{jk}$ the Kronecker delta. Now let $\mathbf{X}(t)$ be a $\mathbb{C}l(\mathbb{R}^4)$-valued function. Then $\mathbf{X}(t)$ via its components has a representation like $M_Q$, as shown in (2.2) and matrix representation of $\mathbf{X}(t)$ is shown with $M_Q(\mathbf{X})$.

Define $L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4]) = \{M_Q(\mathbf{X}) : x_{ij} \in L^2(\mathbb{R}), 1 \leq i, j \leq 4 \} \subseteq L^2(\mathbb{R}, \mathbb{C}[4])$,

and

\[ L^2(\mathbb{R}, \mathbb{C}[4]) = \{\mathbf{X}(t) = x_0(t) + x_1(t)E_1 + \ldots + x_{1324}(t)E_{1324} : x_i \in L^2(\mathbb{R})\}, \]

then we can identify $L^2(\mathbb{R}, \mathbb{C}[4])$ with $L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])$ by $T : L^2(\mathbb{R}, \mathbb{C}[4]) \rightarrow L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4])$ such that

\[ \mathbf{X}(t) \rightarrow \left( \begin{array}{cccc} x_A & -x_G & x_F & -x_G \\ x_C & x_A & x_G & x_F \\ x_E & -x_H & x_B & -x_D \\ x_H & x_E & x_D & x_B \end{array} \right) = M_Q(\mathbf{X}), \]

where $x_A = x_0(t) + ix_1(t) + ix_{34}(t) - x_{124}(t)$ and all other entries are similar to $M_Q$’s entries.

Immediately we realize that $\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2(\mathbb{R}, \mathbb{C}[4])} = \int_{\mathbb{R}} \mathbf{X}^\dagger \mathbf{Y} \, dt$.

Now by considering $\mathbb{C}l(\mathbb{R}^4) \cong \mathbb{C}[4]$, we will investigate some results in matrix-valued MRAs.

**Definition 3.1.** The matrix-valued function $\Phi(t) = (\varphi_{m,n}(t))_{m,n} \in L^2(\mathbb{R}, \mathbb{C}[r])$ generates a matrix-valued multiresolution analysis for $L^2(\mathbb{R}, \mathbb{C}[r])$ if the subspaces $\mathbf{V}_j = \text{span}\{2^{j/2} \Phi(2^j t - k) : k \in \mathbb{Z}\}$ are nested: $\ldots \subseteq \mathbf{V}_{-1} \subseteq \mathbf{V}_0 \subseteq \mathbf{V}_1 \subseteq \mathbf{V}_2 \ldots$, and the following conditions hold:

1) $\bigcup_{j \in \mathbb{Z}} \mathbf{V}_j = L^2(\mathbb{R}, \mathbb{C}[r])$,
2) $\bigcap_{j \in \mathbb{Z}} \mathbf{V}_j = 0_r$, in which $0_r$ is the $r \times r$-zero matrix.
3) $\mathbf{X}(t) \in \mathbf{V}_0 \iff \mathbf{X}(2^j t) \in \mathbf{V}_j, \quad j \in \mathbb{Z}$,
4) $\mathbf{X}(t) \in \mathbf{V}_0 \iff \mathbf{X}(t - k) \in \mathbf{V}_0, \quad k \in \mathbb{Z}$,
5) $\{\Phi(t - k) : k \in \mathbb{Z}\}$ form an orthonormal basis for $\mathbf{V}_0$.

**Remark 3.1.** A sequence $\{\Phi_k\}_{k \in \mathbb{Z}}$ in $L^2(\mathbb{R}, \mathbb{C}[r])$ is called an orthonormal basis if it is an orthonormal set, $\langle \Phi_j, \Phi_k \rangle = I_r \delta_{jk}$, and for any $\mathbf{X}(t) \in L^2(\mathbb{R}, \mathbb{C}[r])$ there exists constant matrix-sequence $\{\mathbf{A}_k\}_{k \in \mathbb{Z}}$ such that $\mathbf{X}(t) = \sum_{k \in \mathbb{Z}} \mathbf{A}_k \Phi_k(t)$.

Condition (5) means that $X(t) = \sum_{k \in \mathbb{Z}} \mathbf{A}_k \Phi_k(t - k)$, which Ferobenious norm will guarantee the convergence of infinite sum, and $\mathbf{A}_k = \langle \mathbf{X}, \Phi_k(t - k) \rangle$ by orthonormality. Also since $\Phi(t) \in \mathbf{V}_0 \subset \mathbf{V}_1$, then the two-scale matrix dilation equation is

\[
\Phi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} \mathbf{G}_k \Phi(t - k)
\]
which combined with orthonormality of \( \Phi \)'s means

\[
(3.2) \quad \sum_{k \in \mathbb{Z}} G_k G_{2l+k}^\dagger = I_r \delta_{l0}, \quad l \in \mathbb{Z}.
\]

Let \( \hat{G}(f) = \sum_{k \in \mathbb{Z}} G_k e^{-2\pi i kf} \) be the matrix mask function, then (3.2) implies that

\[
(3.3) \quad \hat{G}(f) \hat{G}^\dagger(f) + \hat{G}(f + \frac{1}{2}) \hat{G}^\dagger(f + \frac{1}{2}) = 2I_r,
\]

Define matrix Fourier transform for \( \Phi(t) \) by \( \hat{\Phi}(f) := \int_{\mathbb{R}} \Phi(t)e^{-2\pi i f t} \) dt. Then (3.1) gives \( \hat{\Phi}(f) = \frac{1}{\sqrt{2}} \hat{G}(\frac{f}{2}) \hat{\phi}(\frac{f}{2}) \), where by setting \( f = 0 \) we get \( \hat{G}(0) = \sum G_k = \sqrt{2} I_r \), \( \hat{G}(\frac{1}{2}) = 0 \). Define the function matrix \( \Psi(t) = (\psi_{m,n}(t))_{r \times r} \in L^2(\mathbb{R}, \mathbb{C}[r]) \) and corresponding subspace \( W_j = \text{span}\{2^j \Psi(2^j t - k) : k \in \mathbb{Z}\} \). \( W_j \) is orthogonal complement of \( V_j \) in \( V_{j+1} \) i.e. \( V_{j+1} = V_j \oplus W_j \), \( V_j \perp W_j \) and \( \bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{R}, \mathbb{C}[r]) \). Since \( \Psi(t) \in W_0 \subseteq V_1 \), then \( \Psi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} H_k \Phi(2t - k) \). Combining this formula with (3.1) gives us

\[
(3.4) \quad \sum_{k \in \mathbb{Z}} G_k H_{2l+k}^\dagger = 0, \quad l \in \mathbb{Z}.
\]

Now if \( \hat{H}(f) = \sum_{k \in \mathbb{Z}} H_k e^{-2\pi i kf} \) then

\[
(3.5) \quad \hat{H}(f) \hat{G}^\dagger(f) + \hat{H}(f + \frac{1}{2}) \hat{G}^\dagger(f + \frac{1}{2}) = 0, \quad r \in \mathbb{Z},
\]

and \( \hat{\Psi}(f) = \frac{1}{\sqrt{2}} \hat{H}(\frac{f}{2}) \hat{\phi}(\frac{f}{2}) \). If \( \{\Psi(t - k) : k \in \mathbb{Z}\} \) is an orthonormal basis for \( W_0 \) then

\[
\langle \Psi, \Psi(t - k) \rangle = \int_{\mathbb{R}} \Psi(t) \Psi(t - k) dt = I_r \delta_{k0}, \quad k \in \mathbb{Z},
\]

which implies the following relation for the matrix of wavelet mask function:

\[
(3.6) \quad \sum_{k \in \mathbb{Z}} H_k H_{2l+k} = I_r \delta_{l0}, \quad l \in \mathbb{Z}.
\]

This is equivalent to

\[
(3.7) \quad \hat{H}(f) \hat{H}^\dagger(f) + \hat{H}(f + \frac{1}{2}) \hat{H}^\dagger(f + \frac{1}{2}) = 2I_r.
\]

Define \( \hat{M}(f) = \begin{pmatrix} \hat{G}(f) & \hat{G}(f + \frac{1}{2}) \\ \hat{H}(f) & \hat{H}(f + \frac{1}{2}) \end{pmatrix} \) then equations (3.3),(3.5),(3.7) all together are equivalent to

\[
(3.8) \quad \hat{M}(f) \hat{M}^\dagger(f) = 2I_{2r},
\]

which means \( \hat{M}(f) \) is a paraunitary matrix.
3.2. Construction of filters. After constructing the mask function representation, now we are ready to describe and build filters. Suppose that \( \hat{\Phi}(f) \) is a finite polynomial matrix in \( e^{-2\pi i f} \), i.e. can be written in the form \( \hat{G}(f) = \sum_{l=0}^{L'-1} G_l e^{-2\pi i f l} \) with \( \hat{G}(0) = \sqrt{2} \mathbf{I} \), and satisfies (3.1). Then from [8] if
\[
(3.9) \quad \inf_{|f| \leq \frac{1}{2}} |\lambda_i(\hat{G}(f))| > 0
\]
for any eigenfunction \( \lambda_i[\hat{G}(f)] \) of polynomial matrix \( \hat{G}(f) \), the solution \( \Phi(t) \) of the two-scale dilation equation is a matrix-valued scaling function for a matrix-valued MRA, and \( \{ \Psi_{j,k}(t) = 2^{j \hat{f}} \Psi(2^j t - k) : j, k \in \mathbb{Z} \} \) forms an orthonormal basis for matrix-valued space \( L^2(\mathbb{R}, \mathbb{C}[r]) \). For designing the matrix filters with transforms \( \hat{G}(f) \) and \( \hat{H}(f) \) that satisfies (3.2) and for that \( \hat{M}(f) \) is paraunitary, we consider
\[
(3.10) \quad \hat{G}(f) = \frac{\epsilon 2\pi i f \gamma}{\sqrt{2}}(\mathbf{I} + e^{2\pi i f} \hat{P}(2f)), \quad \epsilon \in \{-1, 1\}
\]
where \( \gamma \) is a finite integer and \( \hat{P}(2f) \) is a (normalized ) paraunitary matrix, i.e. \( \hat{P}(f)\hat{P}^\dagger(f) = \mathbf{I} \), which satisfies \( \hat{P}(f+1) = \hat{P}(f) \), and such that \( \hat{P}(0) = \mathbf{I} \). The matrix \( \hat{G}(f) \) satisfies conditions (3.1) and (3.2). Notice that the eigenvalues of the polynomial matrix \( \hat{G}(f) \) are related to the eigenvalues of \( \hat{P}(2f) \) via \( \lambda_i[\hat{G}(f)] = \frac{2\pi i f \gamma}{\sqrt{2}} \{ 1 + e^{2\pi i f} \lambda_i[\hat{P}(2f)] \} \). Since \( \hat{M}(f) \) is paraunitary, \( \hat{H}(f) \) may be chosen as
\[
(3.11) \quad \hat{H}(f) = e^{-2\pi i f(L' - 1 + \delta)} \hat{G}^\dagger(f + \frac{1}{2})
\]
where \( L' \) is the design length of the filter \( G_L \), and \( \delta \in \{0, 1\} \) is chosen so that \( L' - 1 + \delta \) is odd, because by 3.5
\[
\hat{H}(f) \hat{G}^\dagger(f) + \hat{H}(f + \frac{1}{2}) \hat{G}^\dagger(f + \frac{1}{2})
\]
\[
= e^{-2\pi i f(L' - 1 + \delta)} \left[ \hat{G}^\dagger(f + \frac{1}{2}) \hat{G}^\dagger(f) + e^{-\pi i(L' - 1 + \delta)} \hat{G}^\dagger(f) \hat{G}^\dagger(f + \frac{1}{2}) \right]
\]
\[
= e^{-2\pi i f(L' - 1 + \delta)} \left[ \hat{G}^\dagger(f + \frac{1}{2}) \hat{G}^\dagger(f + \frac{1}{2}) - \hat{G}^\dagger(f) \hat{G}^\dagger(f + \frac{1}{2}) \right] = 0_r,
\]
which provide \( \hat{G}(f) \) is commutative in the sense that \( \hat{G}(f) \hat{G}(f + \frac{1}{2}) = \hat{G}(f + \frac{1}{2}) \hat{G}(f) \), and indeed this condition holds when \( \hat{G}(f) \) is defined as in (3.10).
The matrix \( \hat{H} \) given by (3.11) is a polynomial which can be written in the form
\[
\hat{H} = \sum_{m=\delta}^{L'-1+\delta} (-1)^{L'-1+\delta-m} G_{L'-1+\delta-m} e^{-2\pi i f m}.
\]
If \( L' \) is even (and \( \delta = 0 \)), then comparison with \( \hat{H} = \sum_{l=0}^{L'-1} H_l e^{-2\pi i f l} \) we obtain \( H_l = (-1)^{l+1} G_{L'-1-l} \) for \( l = 0, 1, \ldots, L' - 1 \) and we set \( L = L' \). If \( L' \) is odd (\( \delta = 1 \)) we can increase the filter length to an even length \( L' + 1 \) by setting \( G_{L'} = 0_r. \) Then we have \( H_l = (-1)^{l+1} G_{(L'+1)-l-1} \) for \( l = 0, \ldots, L' - 1 \).
0, . . . , L′, with $H_0 = 0_r$. In this case we set $L = L′ + 1$. For constructing the matrix $\hat{P}(f)$ we first consider the class of paraunitary matrices, defined by $\hat{P}(f) = \hat{U}(f)\hat{D}(f)U^\dagger(f)$, where $\hat{U}(f)$ is an arbitrary (normalized) paraunitary polynomial matrix with $\hat{U}(0) = I_r$, and $\hat{D}(f)$ is a diagonal matrix with diagonal elements $\hat{D}_{l,l} = e^{-2\pi ifk_l}, k_l \in \{0,1\}$. Using the general lattice structure, the $r \times r$-matrix $\hat{U}(f)$ may be constructed by $\hat{U}(f) = \hat{U}_q(f), \ldots , \hat{U}_1(f)F$, where $q$ is a positive integer, $F$ is an $r \times r$ constant unitary matrix, i.e. $F^\dagger F = FF^\dagger$, and $\hat{U}_l(f) = I_r + (e^{2\pi if} - 1)z_lz_l^\dagger$ for $l = 0, \ldots , q$ with $z_l^\dagger z_l = 1$, unit-norm constant $r \times 1$-vectors. The advantage of this construction is that the matrices $\hat{D}(f)$ and $\hat{P}(f)$ are similar and hence have the same eigenvalues, and those of $\hat{D}(f)$ are known. It is thus possible to compute the eigenvalues of $\hat{G}(f)$ to check that the sufficient condition (3.9) is satisfied.

4. Main results for $Cl(\mathbb{R}^4)$-MRA

Case I:
Let $r = 4$, by the previous section $\hat{D}_{l,l} = e^{-2\pi ikf}, k \in \{0,1\}, l = 1, 2, 3, 4$. So we have

$$\hat{P}(f) = \hat{U}(f)\hat{D}(f)U^\dagger(f)$$

If $\hat{U}(f) = I_4$, $\hat{U}$ is a paraunitary polynomial matrix which $\hat{U}(0) = I_4$, so $\hat{P}(f) = e^{-2\pi ikf}I_4$, this gives the diagonal matrix $\hat{G}(f) = \frac{e^{2\pi if\gamma}}{\sqrt{2}}(1 + e^{(r-2k)2\pi if})I_4$. $\hat{G}(f)$ has only one eigenvalue which is repeated and is $\lambda[\hat{G}(f)] = \frac{e^{2\pi if\gamma}}{\sqrt{2}}(1 + e^{(r-2k)2\pi if})$. Now if we set $\epsilon = 1$ we obtain

$$\lambda[\hat{G}(f)] = \frac{e^{2\pi if\gamma}}{\sqrt{2}}(1 + e^{2\pi if}), (k = 0)$$

$$\lambda[\hat{G}(f)] = \frac{e^{2\pi if\gamma}}{\sqrt{2}}(1 + e^{-2\pi if}), (k = 1)$$

which in both case the condition $|\lambda[\hat{G}(f)]| = \sqrt{1 + \cos 2\pi f} > 0$, for $|f| < \frac{1}{4}$, is full-faith. Hence the sufficient condition (3.9) is satisfied.

If we set $\gamma = 0, \epsilon = 1, k = 1$, then

$$\hat{G}(f) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + e^{-2\pi if} & 0 & 0 & 0 \\ 0 & 1 + e^{-2\pi if} & 0 & 0 \\ 0 & 0 & 1 + e^{-2\pi if} & 0 \\ 0 & 0 & 0 & 1 + e^{-2\pi if} \end{pmatrix}.$$
Let \( f = 0 \), then \( \tilde{G}(0) = \sqrt{2}I_4 \), \( \tilde{G}(\frac{1}{2}) = 0_4 \) and in comparison with \( \tilde{G}(f) = \sum_{l=0}^{L'-1} G_l e^{-2\pi if} \) we have

\[
\tilde{G}(f) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} e^{-2\pi if}.
\]

This means that \( G_0 = G_1 = \frac{1}{\sqrt{2}}I_4 \) so, \( H_l = (-1)^{l+1}G_{L-l-1}^\dagger \) for \( l = 0, 1 \).

**Case II:**

From now on we consider \( \tilde{G}(f) = \frac{e^{2\pi if}}{\sqrt{2}}(I_4 + e^{2\pi if}\tilde{P}(2f)) \), we can make \( \tilde{P}(f) \) as

\[
\tilde{P}(f) = \tilde{U}(f)\tilde{D}(f)U^\dagger(f)
\]

(for \( L^2_{M_Q}(\mathbb{R}, \mathbb{C}[4]) \) we set \( \tilde{U}(f) \in M_{Q}\cap U(4) \)).

Set \( q = 1 \) and \( F = 4 \times 4 \)-rotation matrix

\[
F = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & -\sin \theta \\
0 & 0 & \sin \theta & \cos \theta
\end{bmatrix},
\]

(note that \( F \in M_{Q} \)). Then \( \tilde{U}(f) = \tilde{U}_1(f)F \) such that \( \tilde{U}_1(f) = I_4 + (e^{2\pi if} - 1)z_1z_1^\dagger \).

Now let \( z_1 = \frac{e^{i\theta}}{\alpha}(a,b,c,d)^T \) so \( z_1^\dagger = \frac{e^{-i\theta}}{\alpha}(a,b,c,d) \) such that \( \alpha = a^2+b^2+c^2+d^2 \).

For instance if \((a,b,c,d) = (0,0,0,\alpha), \alpha \in \mathbb{R}, \) then \( z_1 z_1^\dagger \) is a \( 4 \times 4 \)-matrix with all entries zero except \( e_{4,4} = \frac{e^{2\pi if}}{\sqrt{2}} \) and by choosing \( D \) such that \( D_{1,1} = 1, D_{2,2} = D_{3,3} = D_{4,4} = e^{-2\pi if} \) finally we have:

\[
\tilde{G}(f) = \frac{1}{\sqrt{2}}\begin{pmatrix}
\cos^2 \theta + e^{-2\pi if} + e^{-4\pi if} \sin^2 \theta & \sin \theta \cos \theta - e^{-4\pi if} \cos \theta & 0 & 0 \\
\sin \theta \cos \theta - e^{-4\pi if} \sin \theta \cos \theta & \cos^2 \theta + e^{-4\pi if} \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

This means that

\[
G_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}
\cos^2 \theta & \sin \theta \cos \theta & 0 & 0 \\
\sin \theta \cos \theta & \cos^2 \theta & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad G_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
\]

\[
G_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}
\sin^2 \theta & \sin \theta \cos \theta & 0 & 0 \\
\sin \theta \cos \theta & \cos^2 \theta & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{and since } L'-1 = 3 \text{ then } L' = 4 \text{ so } \delta = 0.
\]

Then we set \( L = L' = 4 \).

Now by \( H_l = (-1)^{l+1}G_{L-l-1}^\dagger \) \((l = 0, 1, 2, 3) \) we have

\[
H_0 = -G_3^\dagger = 0_4, \quad H_1 = G_2^\dagger, \quad H_2 = -G_1^\dagger, \quad H_3 = G_0^\dagger.
\]

So from (3.1) and (3.2) we obtain the desired wavelets.
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