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A NEW APPROACH TO THE SOLUTION OF SENSITIVITY MINIMIZATION IN LINEAR STATE FEEDBACK CONTROL

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Abstract. In this paper, it is shown that by exploiting the explicit parametric state feedback solution, it is feasible to obtain the ultimate solution to minimum sensitivity problem. A numerical algorithm for construction of a robust state feedback in eigenvalue assignment problem for a controllable linear system is presented. By using a generalized parametric vector companion form, the problem of eigenvalue assignment with minimum sensitivity is re-formulated as an unconstrained minimization problem. The derived explicit expressions of the solutions allow minimization of the sensitivity problem by using a powerful search technique.

Keywords: State Feedback, Control Theory, Matrix Algebra, Sensitivity Minimization, Robustness.

2000 Mathematics subject classification: X90

1. Introduction

Sensitivity minimization of the state feedback controllers has received considerable attention in recent years by many authors, for example, see $[1-5]$ and $[9-12]$. It is well established that the condition number of the closed-loop eigenvector matrix should be kept as small as possible [3], i.e. minimum, in order to minimize the sensitivity of the closed-loop system to unwanted perturbations of the system parameters which may arise from aging. At the same time, it is desirable to keep the transient response of the closed-loop system satisfactory. However, these two objectives are conflicting and each require different numerical treatment. Indeed, a parameterized state feedback controller is needed for a constructive investigation into these aspects of the work.

Different methods of parametric eigenvalue assignment for multi-input systems have been established. Most recently implementation of vector companion forms $[6-8]$ have been proposed. An important advantage of this latter method is the explicit parametric solution which makes it attractive for mathematical operations. In addition, the methods of [7] and [8], does not involve eigenvectors in obtaining the parametric state feedback matrix, nor it does require a prior knowledge of the open-loop eigenvalues and any restriction on the nature and multiplicity of the desired eigenvalues. But most of the other methods produce implicit parametric controllers with non-linear parameters.

In this respect, a new efficient computational algorithm for minimizing the sensitivity of the closed-loop eigenvector matrix in arbitrary eigenvalue assignment is presented. This algorithm is an extension of the algorithm proposed in [8] and is based on elementary similarity operations which lead to parametric vector companion forms $[6-7]$. The problem of minimizing the condition number of the closed-loop eigenvector matrix and other measures of robustness using the explicit parametric state feedback matrix are dealt with in detail.

2. Problem statement

Consider a controllable linear time-invariant system defined by the state equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

or its discrete-time version

$$x(t+1) = Ax(t) + Bu(t), \tag{2.2}$$

where $x(t) \in I\!R^n$, $u(t) \in I\!R^m$ and the matrices A and B are real constant matrices of dimensions $n \times n$ and $n \times m$ respectively, with $\mathrm{rank}(B) = m$. The aim of eigenvalue assignment in view of minimum sensitivity to small perturbations in the parameters is to design a state feedback controller, K, producing a closed-loop system with a satisfactory response by shifting controllable poles from undesirable to desirable locations in such a manner that a small change in system parameters will produce a negligible change in the eigenvalue spectrum. Karbassi and Bell $[6-7]$ have introduced an algorithm for obtaining an explicit parametric controller matrix K by performing similarity operations on the controllable pair (B, A). In fact K is chosen such that the eigenvalues of the closed-loop system

$$\Gamma = A + BK, \tag{2.3}$$

lie in the self conjugate eigenvalue spectrum $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. Recently, Karbassi and Tehrani [8] extended the previous results as to obtain an explicit formula involving nonlinear parameters in the control law. Now a similar approach is presented in this paper to obtain a controller gain matrix K such that the sensitivity of the closed-loop system to small perturbations in the system variables is minimized. In the next section, the state feedback controller proposed by Karbassi and Tehrani [8] is briefly reviewed and then the minimization of sensitivity is enhanced by using a powerful search technique.

3. Synthesis

Consider the state transformation

$$x(t) = T\tilde{x(t)}, \tag{3.1}$$

where T can be obtained by elementary similarity operations as described in [7]. In this way, $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$ are in a compact canonical form known as vector companion form:

$$\tilde{A} = \begin{bmatrix} G_0 \\ I_{n-m}, 0_{n-m,m} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_0 \\ 0_{n-m,m} \end{bmatrix}, \tag{3.2}$$

where G_0 is an $m \times n$ matrix and B_0 is an $m \times m$ upper triangular matrix. Note that if the Kronecker invariants of the pair (B, A) are regular, then $\tilde{A}$ and $\tilde{B}$ are always in the above form. In the case of irregular Kronecker invariants, some rows of I_{n-m} in $\tilde{A}$ are displaced. We may also conclude that if the vector companion form of $\tilde{A}$ obtained from similarity operations has the above structure, then the Kronecker invariants associated with the pair (B, A) are regular [8].

The state feedback matrix which assigns all the eigenvalues to zero, for the transformed pair $(\tilde{B}, \tilde{A})$, is then chosen as

$$u = -B_0^{-1}G_0\tilde{x} = \tilde{F}\tilde{x}, \tag{3.3}$$

which results in the primary state feedback matrix for the pair (B, A) defined as

$$F_p = \tilde{F}T^{-1}. \tag{3.4}$$

The transformed closed-loop matrix $\tilde{\Gamma_0} = \tilde{A} + \tilde{B}\tilde{F}$ assumes a compact Jordan form with zero eigenvalues

$$\tilde{\Gamma_0} = \begin{bmatrix} 0_{m,n} \\ I_{n-m}, 0_{n-m,m} \end{bmatrix}. \tag{3.5}$$

If $\tilde{A_\lambda}$ is any matrix in vector companion form, i.e.

$$\tilde{A_\lambda} = \begin{bmatrix} G_\lambda \\ I_{n-m}, 0_{n-m,m} \end{bmatrix}, \tag{3.6}$$

with the eigenvalue spectrum $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$, containing a set of self conjugate eigenvalues, then as shown in [8]

$$\tilde{K} = B_0^{-1}(-G_0 + G_\lambda) \tag{3.7}$$

is the feedback matrix which assigns the eigenvalue spectrum to the closed-loop matrix $\tilde{\Gamma} = \tilde{A} + \tilde{B}\tilde{K}$, and K may then be obtained by $K = \tilde{K}T^{-1}$. Note that G_λ is an $m \times n$ parametric matrix in the form:

$$G_\lambda = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{bmatrix}. \tag{3.8}$$

To obtain the non-linear system of equations relating the parameters of G_λ, the characteristic polynomial of $\tilde{A_\lambda}$ must be obtained. Thus, let

$$det(\tilde{A_\lambda} - \lambda I) = P_n(\lambda) = 0, \tag{3.9}$$

where

$$P_n(\lambda) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n) \tag{3.10}$$

is the characteristic polynomial of the closed-loop system. Since it is required that the zeros of this polynomial lie in the set $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$, it is clear that

$$P_n(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n). \tag{3.11}$$

Now by direct computation of $det(\tilde{A_\lambda} - \lambda I)$ in parametric form and equating the coefficients of the characteristic polynomial with (3-11), the following non-linear system of equations is obtained:

$$\begin{gathered} f_1(g_{11}, g_{12}, ..., g_{1n}, g_{21}, g_{22}, ..., g_{2n}, ..., g_{m1}, g_{m2}, ..., g_{mn}) = a_1 \\ f_2(g_{11}, g_{12}, ..., g_{1n}, g_{21}, g_{22}, ..., g_{2n}, ..., g_{m1}, g_{m2}, ..., g_{mn}) = a_2 \\ \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \\ f_n(g_{11}, g_{12}, ..., g_{1n}, g_{21}, g_{22}, ..., g_{2n}, ..., g_{m1}, g_{m2}, ..., g_{mn}) = a_n \end{gathered} \tag{3.12}$$

where $g_{ij}, (i = 1, ..., m, j = 1, ..., n)$, are the elements of G_λ.
In this way, a non-linear system of n equations with $n \times m$ unknowns is obtained. By choosing $N = n(m-1)$ unknowns arbitrarily it is then possible to solve the system. Thus different selections can be made to obtain different solutions. Indeed, the MAPLE software can be used to obtain all the possible combinations of the solutions to the system of nonlinear equations thus obtained.

Now to discuss the problem of robust eigenvalue assignment, recall that for a given controllable pair (B, A) and D we need to find a real matrix K such that

$$(A + BK)X = XD, \tag{3.13}$$

for some nonsingular X, where

$$D = diag(\lambda_1, \lambda_2, ...\lambda_n). \tag{3.14}$$

From equation $(3-11)$ it can be seen that the columns $x_j, j = 1, 2, ..., n$ of the matrix X are the right eigenvectors of the matrix $\Gamma = A + BK$ corresponding to the assigned eigenvalue λ_j. Similarly, the rows $(y_j)^t$, $j = 1, 2, ..., n$ of the matrix $Y^t = X^{-1}$ are the corresponding left eigenvectors. It has been shown by Wilkinson [13] that the sensitivity of the eigenvalues λ_j to perturbations in the components of A, B and K depends upon the magnitude of the condition number $c_j = 1/s_j$, where

$$s_j = \frac{|(y_j)^t x_j|}{\|y_j\|_2 \|x_j\|_2} \leq 1. \tag{3.15}$$

In the case of multiple eigenvalues, a particular choice of eigenvector is assumed. (For λ_j the sensitivity s_j, is just the cosine of the angle between the right and left eigenvectors corresponding to λ_j). It can also be observed that a bound on the sensitivity of the eigenvalues is given by Wilkinson [13], and

$$\max c_j \leq \kappa_2(X) \equiv \|X\|_2 \|X^{-1}\|_2, \tag{3.16}$$

where $\kappa_2(X)$ is the condition number of the matrix $X = [x_1, x_2, ..., x_n]$.

A common measure of robustness is taken as $\nu = \kappa_2(X)$, the condition number of the matrix X. Of course, other measures of robustness may also be used, (such as defining a cost function $J = \|X\|_2 + \|X^{-1}\|_2$). However, to minimize sensitivity it dose not make any difference which definition is used for measuring robustness [12]. It has been shown that, high condition number lead to increased sensitivity of the eigenvalues of the closed-loop system [10], therefore, an important aspect of eigenvalue assignment is to achieve a small condition number for the eigenvector matrices of the closed-loop system. In this paper a new method is developed to minimize sensitivity (i.e. to obtain a small condition number for the eigenvector matrices of the closed-loop system) by employing the nonlinear system of equations (3.12) which include all the possible combinations of the solutions. By using MAPLE software all possible solutions of the nonlinear system of equations can be obtained for a specified set of eigenvalue spectrum. Then, a search algorithm is employed in such a way as to calculate the condition number of the closed-loop eigenvector matrices in order to achieve a given bound for the condition number. In this manner, for each choice of solution a local minima is obtained. By solving the problem repeatedly with different choice of solutions, the state feedback matrix which produces the lowest value of the condition number for the closed-loop system can be selected. This method leads to a series of local sensitivity minima. The following algorithm is devised for obtaining the state feedback matrix with minimum sensitivity.

The algorithm for obtaining K with minimum sensitivity:
OBJECT: To obtain parameters (g_{ij}), in order to calculate the state feedback matrix K, for which the condition number of the eigenvector matrix X of the closed loop system Γ is minimized, i.e. to obtain K with minimum sensitivity.
INPUT: The controllable pair (B, A) and the eigenvalue spectrum $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$.
OUTPUT: K with minimum sensitivity and the corresponding condition number $\kappa_2(X)$ of the eigenvector matrix X of the closed-loop system.
STEP 1: Employ the algorithm given by [6] to obtain ${B_0}^{-1}$, G_0 and T^{-1}.
STEP 2: Obtain the coefficients of the characteristic polynomial whose roots are the same as the desired eigenvalue spectrum $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$.
STEP 3: Obtain the characteristic polynomial of $\tilde{A_\lambda}$ as defined in (9).
STEP 4: Obtain the nonlinear system of equations relating parameters g_{ij}, by equating the coefficients of the characteristic polynomials obtained in steps 2 and 3.
STEP 5: Employ MAPLE software to obtain all the solutions to the nonlinear system of equations obtained in step 4.
STEP 6: Specify a relatively high condition number as an upper bound. Then choose one of the parametric solutions, designate random values to free parameters of this parametric solution and obtain the feedback matrix K by $K = B_0^{-1}(-G_0 + G_\lambda)T^{-1}$ and the corresponding condition number for the eigenvector matrix X of the closed-loop system.

STEP 7: If the condition number obtained in step 6 is less than the specified bound, then halve the bound and repeat step 6 until no further decrease is obtained.
STEP 8: Store this condition number and repeat step 6 for all the solutions.
STEP 9: Choose the minimum condition number obtained in step 8 and obtain K, the corresponding feedback matrix.

In the next section of the paper some examples which are commonly found in the literature are intentionally presented in order to compare the numerical results obtained by our method with the previously reported results.

4. Illustrative examples

EXAMPLE 1. Consider the system [5]

$$B = \begin{bmatrix} 0.0755 & 0 & 0.0246 \\ 4.4800 & 5.2200 & -0.7420 \\ -5.0300 & 0.0998 & 0.9840 \\ 0.0755 & 0 & 0.0246 \end{bmatrix}, A = \begin{bmatrix} -0.3400 & 0.0517 & 0.0010 & -0.9970 \\ 0 & 0 & 1 & 0 \\ -2.6900 & 0 & -1.1500 & 0.7380 \\ 5.9100 & 0 & 0.1380 & -0.5060 \end{bmatrix}$$

. The transformation matrix which transforms the controllable (B, A) into vector companion form $(\tilde{B}, \tilde{A})$ is:

$$T^{-1} = \begin{bmatrix} -15.1780 & 0.1256 & -0.3327 & -1.1924 \\ -0.3108 & 0.1885 & 0.1583 & -0.3329 \\ 12.4427 & -0.0072 & 0.3752 & 12.9820 \\ 2.4285 & 0 & 0 & -2.4285 \end{bmatrix}.$$

It is desired to assign the eigenvalue spectrum $\Lambda = \{-1, -2, -3, -0.5\}$ to the closed-loop system. It is easy to verify that the nonlinear system of equations governing this eigenvalue assignment are:

$$-(g_{11} + g_{22} + g_{33}) = 6.5 \tag{4.1}$$

$$\begin{aligned} &g_{11}g_{22} - g_{31}g_{13} - g_{12}g_{21} - g_{23}g_{32} + g_{11}g_{33} + g_{22}g_{33} - g_{14} = 14 \\ &-g_{12}g_{24} - g_{13}g_{34} + g_{22}g_{14} + g_{14}g_{33} - g_{31}g_{12}g_{23} + g_{31}g_{13}g_{22} \end{aligned} \tag{4.2}$$

$$\begin{aligned} &-g_{21}g_{32}g_{13} + g_{11}g_{32}g_{23} - g_{11}g_{22}g_{33} + g_{21}g_{12}g_{33} = 11.5 \\ &g_{22}g_{13}g_{34} + g_{12}g_{24}g_{33} - g_{12}g_{23}g_{34} - g_{32}g_{13}g_{24} + g_{32}g_{14}g_{23} \end{aligned} \tag{4.3}$$

$$-g_{22}g_{14}g_{33} = 3 \tag{4.4}$$

which consists of four equations and twelve unknowns. MAPLE software can be used to obtain all the possible combinations of linear and nonlinear parametric solutions. In this way, from the nonlinear system of equations (4.1) – (4.4),42 parametric relationships and for each of them a minimum value for condition number can then be obtained. All the condition numbers which produce minimum sensitivity for each parametric solution are listed in table 1 in asending order.

Table 1:Condition numbers in ascending order from left to right

3.9788	4.3113	4.3525	4.4736	5.6759	6.9557
7.1892	7.3779	7.5989	7.8693	8.6556	8.9975
9.7512	10.6917	10.9931	11.3875	11.7894	12.7890
12.7931	18.8639	21.9911	22.9875	27.9263	28.6343
29.5793	29.8052	30.2976	31.9714	32.3132	36.7690
37.6907	42.5762	50.7288	50.9137	50.9884	54.3886
54.9894	81.8706	83.7711	99.7260	152.7981	156.2221

The parametric solution which yields the lowest value for condition number, after a thorough search, was found to be given by :

$$g_{23} = 0, \; g_{32} = 0,$$

and by defining

$$g_{12} = a, \; g_{13} = b, \; g_{21} = c, \; g_{22} = d, \; g_{31} = e, \; g_{33} = f.$$

The other parametric relationships were then found to be:

$$\begin{aligned}
g_{11} &= -(d + f + 6.5), \\
g_{14} &= -(f^2 + d^2 + ac + be + df + 6.5(f + d) + 14, \\
g_{24} &= (2d^4 + 13d^3 + 28d^2 + 2d^2ac + 23d - 2acdf + 6)/2a(f - d), \\
g_{34} &= (2f^4 + 13f^3 + 28f^2 + 2f^2be + 23f - 2bdef + 6)/2b(d - f).
\end{aligned}$$

Applying our search program yields

$$G_\lambda = \begin{bmatrix} -3.5869 & 0.4370 & 0.2777 & -1.7953 \\ 0.0610 & -1.9848 & 0 & 0.1699 \\ 0.2165 & 0 & -0.9283 & 0.4334 \end{bmatrix}, \tag{4.5}$$

which produces the state feedback matrix

$$K = \begin{bmatrix} -17.2870 & 0.1312 & -0.2482 & 1.2598 \\ 2.3913 & -0.3504 & -0.2948 & -0.4196 \\ -85.2690 & -0.6092 & -1.7857 & 5.3354 \end{bmatrix}. \tag{4.6}$$

The results are compared in table 2:

Table 2

Method	Condition number
Purposed	3.9788
Ibbini	119.7421

In fact 3.9788 is the smallest value for the condition number of the above eigenvalue assignment problem yet achieved.

EXAMPLE 2. Consider the system [9]

$$B = \begin{bmatrix} 0 & 0 \\ 5.6790 & 0 \\ 1.1360 & -3.1460 \\ 1.1360 & 0 \end{bmatrix} \quad , \quad A = \begin{bmatrix} 1.3800 & -0.2077 & 6.7150 & -5.6760 \\ -0.5814 & -4.2900 & 0 & 0.6750 \\ 1.0670 & 4.2730 & -6.6540 & 5.8930 \\ 0.0480 & 4.2730 & 1.3430 & -2.1040 \end{bmatrix}$$

. The transformation matrix which transforms the controllable (B, A) into vector companion form $(\tilde{B}, \tilde{A})$ is given by:

$$T^{-1} = \begin{bmatrix} -0.0040 & 0.1839 & 0 & -0.0393 \\ -0.0653 & 0.0098 & -0.3179 & 0.2687 \\ -0.0071 & -0.0071 & 0 & 0.0356 \\ -0.0473 & 0 & 0 & 0 \end{bmatrix}.$$

It is desired to assign the eigenvalue spectrum $\Lambda = \{-0.2, -0.5, -5.0566, -8.6659\}$ to the closed-loop system. The nonlinear system of equations governing this eigenvalue assignment is (as found in [8]):

$$-(g_{11} + g_{22}) = 14.4225 \tag{4.7}$$

$$g_{11}g_{22} - g_{12}g_{21} - g_{13} - g_{24} = 53.5257 \tag{4.8}$$

$$g_{22}g_{13} - g_{12}g_{23} + g_{11}g_{24} - g_{14}g_{21} = 32.0462 \tag{4.9}$$

$$g_{24}g_{13} - g_{14}g_{23} = 4.3820 \tag{4.10}$$

which consists of four equations and eight unknowns. Once again, with using MAPLE software, 11 combinations of linear and nonlinear parametric solutions and for each solution a minimum value for condition number is obtained by a suitably written program incorporating a special search algorithm. The increasing sequence of condition numbers which produce minimum sensitivity for each parametric solution is shown in table 3:

Table 3:Condition numbers in ascending order from left to right

3.1781	3.4189	3.4965	3.5268	3.6613	3.6789
4.3929	4.3945	4.8757	52.9566	56.9268	

The parametric solution which yielded the optimal value for condition number was found from the nonlinear system of equations (4.7) – (4.10) defined by:

$$g_{21} = a,\ g_{22} = b,\ g_{23} = c,\ g_{24} = d$$

and

$$e = a^2 d - abc - c^2$$

The other parametric relationships were then found to be:

$$\begin{aligned}
g_{11} &= -(b + 14.4225),\\
g_{12} &= (-ad^2 - ab^2 d - 14.4225abd - 53.5257ad - 4.3820a + 2bcd\\
&\quad +b^3 c + 53.5257bc + 14.4225b^2 c + 14.4225cd + 32.0462c)/e,\\
g_{13} &= (c^2 d - abcd + 4.3820a^2 - 14.4225acd - 32.0462ac + b^2 c^2\\
&\quad +14.4225bc^2 + 53.5257c^2)/e,\\
g_{14} &= (-abd^2 + 4.3820ab + 4.3820c + 53.5257cd + cd^2 - 32.0462ad\\
&\quad -14.4225ad^2 + 14.4225bcd + b^2 cd)/e.
\end{aligned}$$

After using the search program it was found that

$$G_\lambda = \begin{bmatrix} -5.7182 & -0.3922 & -1.9530 & 0.2800 \\ -1.6602 & -8.7043 & 1.4452 & -2.4509 \end{bmatrix}, \tag{4.11}$$

which produces the state feedback matrix

$$K = \begin{bmatrix} 0.1632 & -0.0857 & 0.2041 & -0.1797 \\ 1.1024 & -0.1622 & 0.7296 & -0.1606 \end{bmatrix}. \tag{4.12}$$

The results are copared in table 4:

Table 4

	Condition number	Frobenius norm
Purposed method	3.1781	1.3811
Kautsky's method	3.32	1.40

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EXAMPLE 3. Consider the system [11]

$$B = \begin{bmatrix} 0 & 0 \\ 0.0638 & 0 \\ 0.0838 & -0.1396 \\ 0.1004 & -0.2060 \\ 0.0063 & -0.0128 \end{bmatrix}, A = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.306 & -2.132 & 0.9807 & 0 & 0 \\ 0 & 1.595 & -3.149 & 1.547 & 0 \\ 0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0 & 0.00227 & 0 & 0.1636 & -0.1625 \end{bmatrix}.$$

The transformation matrix which transforms the controllable (B, A) into vector companion form $(\tilde{B}, \tilde{A})$ is:

$$T^{-1} = \begin{bmatrix} -2.1074 & 14.9824 & -37.3835 & 32.2226 & -9.2525 \\ -10.2764 & 8.6959 & -4.2921 & -2.1823 & 3.8072 \\ -1.9828 & -1.7797 & 7.2011 & -4.9247 & 0.7208 \\ 89.8109 & -0.3454 & 1.4777 & 0.2659 & -20.3942 \\ 22.5843 & 0.3736 & -1.5290 & 0.7674 & 4.3250 \end{bmatrix}.$$

The desired eigenvalue spectrum, $\Lambda = \{-0.2, -0.5, -1, -1 \pm i\}$ is to be assigned to the closed-loop system in this case. It is easy to verify that the nonlinear system of equations governing this eigenvalue assignment are :

$$-(g_{11} + g_{22}) = 3.7 \tag{4.13}$$

$$g_{21}g_{12} - g_{11}g_{22} + g_{13} + g_{24} = -6.2 \tag{4.14}$$

$$g_{15} - g_{22}g_{13} + g_{12}g_{23} - g_{11}g_{24} + g_{14}g_{21} = -5.1 \tag{4.15}$$

$$g_{23}g_{14} - g_{13}g_{24} + g_{12}g_{25} - g_{22}g_{15} = -1.8 \tag{4.16}$$

$$g_{14}g_{25} - g_{15}g_{24} = -0.2 \tag{4.17}$$

which consists of five equations and ten unknowns.To obtain all the possible combinations of linear and nonlinear parametric solutions, MAPLE software was again used. For each solution a minimum value for condition number was obtained as before. The parametric solution which yields the lowest value for condition number from the nonlinear system of equations (4.13) – (4.17) was found to be:

$$g_{14} = a,\ g_{21} = b,\ g_{23} = c,\ g_{24} = d$$

with the other parametric relationships as follows:

$$\begin{aligned} g_{11} &= -(5d + 3.5), \\ g_{12} &= 5a, \\ g_{13} &= -(25d^2 + 17.5d + 5ba + 5.5), \\ g_{15} &= -(125d^3 + 87.5d^2 + 27.5d + 25abd + 5ac + 4), \\ g_{22} &= 5d - 0.2, \\ g_{25} &= -(125d^4 + 87.5d^3 + 27.5d^2 + 4d + 25d^2ba + 5acd + 0.2)/a. \end{aligned}$$

Applying our search program yields

$$G_\lambda = \begin{bmatrix} -4.3675 & 1.3470 & -5.6479 & 0.2694 & -0.2316 \\ -2.7810 & 0.6875 & -6.5189 & 0.1775 & -0.8950 \end{bmatrix}, \tag{4.18}$$

which produces the state feedback matrix

$$K = \begin{bmatrix} -0.7686 & 48.5339 & -47.6569 & -1.9305 & 0.9691 \\ -5.0299 & 1.6182 & 39.3997 & -62.9201 & 20.8260 \end{bmatrix}. \tag{4.19}$$

For this example, the results are compared in table 5:

Table 5

	Condition number	Frobenius norm
Purposed method	218.7343	102.9801
Tam's method	1.6337×10^3	103.2225

5. Conclusions

The minimum sensitivity of the closed-loop eigenvalues was achieved by exploitation of the parameterized state feedback controller matrix proposed by Karbassi and Tehrani [8], assigning the closed-loop system eigenvalues to the prescribed locations. The derived explicit expressions allow the use of standard search based minimization techniques. The main advantage of this methodology is that not only all the possible combination of parametric state feedback controllers can be generated, but also that for each specific parametric form a minima can be obtained. Several numerical examples that were tested, showed that the minimum sensitivity occurs with the parametric state feedback matrix which contains the highest number of free parameters, as expected. The proposed robust eigenvalue assignment approach produced better results for both minimum sensitivity and minimum norm than recent sensitivity analysis results [5], [9] and [11]. With the available softwares, the algorithm presented in this paper is much simpler to implement than the existing numerical methods [5], [9] and [11].

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