SPECTRUM OF THE FOURIER-STIELTJES ALGEBRA OF A SEMIGROUP

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Abstract. For a unital foundation topological $*$-semigroup $S$ whose representations separate points of $S$, we show that the spectrum of the Fourier-Stieltjes algebra $B(S)$ is a compact semitopological semigroup.

We also calculate $B(S)$ for several examples of $S$.

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1. INTRODUCTION

In [3] Lau studied the subalgebra $F(S)$ of $WAP(S)$ of a topological semigroup $S$ with involution. If $G$ is an abelian topological group, then $F(G) ≃ M(\hat{G})$ where $\hat{G}$ is the dual group of $G$. If $S$ is a topological $*$-semigroup with an identity, then $F(S)$ is the linear span of positive definite functions on $S$. The authors introduced and studied Fourier and Fourier-Stieltjes algebras $A(S)$ and $B(S)$ of a foundation topological $*$-semigroup $S$ in [1]. When $S$ is unital, $B(S) = F(S)$.

Let $S$ be a locally compact topological semigroup and $M(S)$ be the Banach algebra of all bounded regular Borel measures on $S$. We consider the mappings
Let \( L_\mu \) and \( R_\mu \) of \( S \) into \( M(S) \) defined by
\[
L_\mu(x) = \mu * \delta_x, \quad R_\mu(x) = \delta_x * \mu \quad (x \in S, \mu \in M(S)),
\]
where \( \delta_x \) is the point mass at \( x \). Then the semigroup algebra \( L(S) \) consists of those \( \mu \in M(S) \) for which \( L_\mu \) and \( R_\mu \) are continuous with respect to the weak topology of \( M(S) \), and \( L(S) \) is a Banach subalgebra of \( M(S) \). The semigroup \( S \) is called foundation if \( \cup \{ \text{supp}(\mu) : \mu \in L(S) \} \) is dense in \( S \) [6].

A representation of \( S \) is a pair \( \{ \pi, H_\pi \} \) of a Hilbert space \( H_\pi \) and a semigroup homomorphism \( \pi : S \rightarrow B(H_\pi) \) such that \( \pi \) is (weakly) continuous, i.e. the mappings \( x \mapsto \langle \pi(x)\xi, \eta \rangle \) are continuous on \( S \), for all \( \xi, \eta \in H_\pi \), and that \( \pi \) is bounded if \( \|\pi\| = \sup_{x \in S} \|\pi(x)\| < \infty \). Also \( \pi \) is called a *-representation if moreover \( \pi(x^*) = \pi(x)^* \) \( (x \in S) \), where the right hand side is the adjoint operator. A *-representation \( \{ \sigma, H \} \) of \( L(S) \) is called non-vanishing if for every \( 0 \neq \xi \in H \), there exists \( \mu \in L(S) \) with \( \sigma(\mu)\xi \neq 0 \). Let \( \Sigma(L(S)) \) be the family of all *-representations of \( L(S) \) on a Hilbert space which are non-vanishing, and \( \Sigma(S) \) be the family of all continuous *-representations \( \pi \) of \( S \) with \( \|\pi\| \leq 1 \), then one has a bijective correspondence between \( \Sigma(S) \) and \( \Sigma(L(S)) \) via
\[
\langle \tilde{\pi}(\mu)\xi, \eta \rangle = \int_S \langle \pi(x)\xi, \eta \rangle d\mu(x) \quad (\mu \in L(S), \xi, \eta \in H_\pi = H_\pi).
\]

Given \( \rho \subseteq \Sigma = \Sigma(S) \) and \( \mu \in L(S) \), define \( \|\mu\|_\rho = \sup\{||\tilde{\pi}(\mu)|| : \pi \in \rho\} \) and \( I_\rho = \{ \mu \in L(S) : \|\mu\|_\rho = 0 \} \). Then \( I_\rho \) is clearly a closed two-sided ideal of \( L(S) \) and \( \|\mu + I_\rho\| = \|\mu\|_\rho \) defines a \( C^* \)-norm on \( L(S)/I_\rho \). The completion of this quotient space in this norm is a \( C^* \)-algebra which is denoted by \( C^*_\rho(S) \).

When \( \rho = \Sigma \), then the \( C^* \)-algebra \( C^*_\rho(S) = C^*_\Sigma(S) \) is called the (full) semigroup \( C^* \)-algebra of \( S \). If \( S \) is foundation and \( \Sigma \) separates the points of \( S \), then \( L(S) \) is *-semisimple and so \( I_\Sigma = \{0\} \). In this case \( L(S) \) is a norm dense subalgebra of \( C^*_\Sigma(S) \) (see [1] for more details).

A complex valued function \( u : S \rightarrow \mathbb{C} \) is called positive definite if for all positive integers \( n \) and all \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \), and \( x_1, x_2, \ldots, x_n \in S \), we have
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j u(x_i x_j^*) \geq 0.
\]

Let \( P(S) \) denotes the set of all continuous positive definite functions on \( S \). We denote the linear span of \( P(S) \) by \( B(S) \) and call it the Fourier-Stieltjes algebra of \( S \). Let \( S \) be a topological *-semigroup and \( C_\rho(S) \) be the algebra of all continuous functions on \( S \) with compact support. Then the closed subalgebra \( (B(S) \cap C_\rho(S)) \subseteq B(S) \) is denoted by \( A(S) \) and is called the Fourier algebra of \( S \).

### 2. Fourier-Stieltjes Algebra

It is well known that for an abelian topological group \( G \), the Fourier and Fourier-Stieltjes algebras \( A(G) \) and \( B(G) \) are isometrically isomorphic to the group and measure algebras \( L^1(G) \) and \( M(G) \) of the dual group \( \hat{G} \). For a class of commutative foundation topological *-semigroup with identity we show that
B(S) is isometrically isomorphic to M(\hat{S}). Here \hat{S} is the set of continuous semi-characters on S which is a locally compact topological semigroup [3].

**Theorem 2.1.** Let S be a commutative foundation topological *-semigroup with identity. For \lambda \in L(\hat{S}), define \lambda : S \to \mathbb{C} by

\[
\hat{\lambda}(x) = \int_S \chi(x) d\lambda(\chi) \quad (x \in S).
\]

Then the map \lambda \mapsto \hat{\lambda} is a continuous monomorphism from L(\hat{S}) into B(S).

**Proof.** \hat{S} is a locally compact topological semigroup [3]. Also for each \lambda \in L(\hat{S}) there is a probability measure \gamma on \hat{S} and \phi \in L^1(\hat{S}, \gamma) such that d\lambda = \phi d\gamma.

We can decompose \phi as

\[
\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4),
\]

where \phi_i \geq 0, for i = 1, \ldots, 4. Put d\lambda_i = \phi_i d\gamma. Then for each n \geq 1, c_1, \ldots, c_n \in \mathbb{C}, and x_1, \ldots, x_n \in S,

\[
\sum_{i,j=1}^n c_i c_j \hat{\lambda}_k(x_i x_j^*) = \int_{\hat{S}} \sum_{i,j=1}^n c_i c_j \hat{\chi}(x_i x_j^*) d\lambda_k(\chi) \geq 0,
\]

for k = 1, \ldots, 4. Next we show that \hat{\lambda}_k is also continuous. Given \varepsilon > 0, there is a measurable subset K \subseteq \hat{S} such that

\[
\int_{\hat{S} \setminus K} \phi_k(\chi) d\gamma(\chi) < \varepsilon.
\]

By Ascoli’s Theorem, K is equicontinuous. Now given \chi \in \hat{S}, there is a neighborhood U of x_0 in S such that

\[
|\chi(x) - \chi(x_0)| < \varepsilon \quad (\chi \in K, x \in U).
\]

For each x \in U,

\[
|\hat{\lambda}_k(x) - \hat{\lambda}_k(x_0)| \leq \int_{\hat{S}} |\chi(x) - \chi(x_0)| d\lambda_k(\chi)
\leq \int_K |\chi(x) - \chi(x_0)| d\lambda_k(\chi) + \int_{\hat{S} \setminus K} |\chi(x) - \chi(x_0)| d\lambda_k(\chi)
\leq \varepsilon \lambda_k(K) + 2\varepsilon \leq (2 + \lambda_k(\hat{S}))\varepsilon.
\]

This shows that \hat{\lambda}_k \in P(S), for k = 1, \ldots, 4, and so \hat{\lambda} \in B(S). Next we have

\[
\|\hat{\lambda}\|_{B(S)} = \sup \int_S |\hat{\lambda}(x)| d\mu(x) = \sup \int_S \int_S |\chi(x) d\lambda(\chi) d\mu(x)|
\leq \int_S \int_S |\chi(x)| d\mu(x) \|d\lambda\| \leq \|\lambda\|,
\]

where the supremum is taken over all \mu \in L(S) with \|\mu\|_{\Sigma} \leq 1 (see [1]). Also the last inequality follows from the fact that each semi-character \chi \in \hat{S} could be regarded as a representation of S.
When $\lambda$ is positive, we also have
\[
\|\lambda\| = \int_{\hat{S}} \chi(e)d\lambda(\chi) = \hat{\lambda}(e) \leq \|\hat{\lambda}\|_{B(S)},
\]
since $\chi(e) = 1$, for each $\chi \in \hat{S}$, where $e$ is the identity of $S$. In general, $\lambda = (\lambda_1 - \lambda_2) + i(\lambda_3 - (\lambda_4)$, with $\lambda_k$’s positive, and we have
\[
\|\lambda\| \leq \sum_{i=1}^{4} \|\lambda_i\| \leq \sum_{i=1}^{4} \|\hat{\lambda_i}\|.
\]
In particular the map $\lambda \mapsto \hat{\lambda}$ is injective.

Finally, for $\lambda, \mu \in L(S)$ and $x \in S$ we have
\[
(\lambda \ast \mu)(x) = \int_{S} \chi(x)d(\lambda \ast \mu)(\chi) = \int_{S} \int_{S} \chi(x)\zeta(x)d\lambda(\chi)d\mu(\zeta) = \hat{\lambda}(x)\hat{\mu}(x),
\]
and we are done. $\Box$

Remark 2.2. In the group case, the range of the above map is $A(S)$. We don’t know if this is the case for foundation semigroups.

Following [4] we say that $S$ is of type $U$ if it has a dense subsemigroup $U$ which is a union of groups. Then to each $x \in U$ there corresponds an element $x' \in U$ (the inverse of $x$ in the group to which $x$ belongs) such that $xx'$ and $x'x$ are idempotents and
\[
x'x = x, \quad x'x' = x'.
\]

In [5] a concept of positive definite functions is defined for semigroups of type $U$. We denote the set of positive definite functions on $U$ by $P(U)$. When $U$ is an increasing union or a disjoint union of groups, this element $x'$ is unique for each $x \in U$. When the latter holds and the map $x \mapsto x'$ is continuous we say that $S$ is of type $\bar{U}$. In this case the map $x \mapsto x'$ on $U$ extends to a continuous map $x \mapsto x'$ on $S$ and $S$ becomes a topological $*$-semigroup. In this case we can talk about positive definite functions on $S$ in the sense of section 1. If $U$ is an increasing union or a disjoint union of groups, each open in $S$, then $S$ is of type $\bar{U}$. If $S$ is of type $\bar{U}$, then it is easy to see that for each $f \in C_b(S)$, $f \in P(S)$ if and only if $f|_{U} \in P(U)$. In particular for a unital commutative semigroup $S$ of type $\bar{U}$ we have $B(S) = R(S)$ [5, 7.2.5]. Now the following result follows from [5] immediately.

Proposition 2.3. If $S$ is a commutative foundation $*$-semigroup of type $\bar{U}$ with identity, then the map $\lambda \mapsto \hat{\lambda}$ is a linear isometry of $M(\hat{S})$ onto $B(S)$.

Note that if we consider the semigroup of integers $\mathbb{Z}$ with trivial involution $n^* = n$, then we have $B(\mathbb{Z}) \neq R(\mathbb{Z})$ [7].
3. Spectrum of the Fourier algebra

In this section we show that for a unital foundation topological $*$-semigroup $S$, the spectrum of $B(S)$ is a compact unital semitopological semigroup. Let $S$ be a unital foundation topological $*$-semigroup with identity $e$ and $\Omega = \Omega(S)$ be the family of all continuous $*$-representations $\omega$ of $S$ in a $W^*$-algebra $M_\omega$ with $\|\omega\| \leq 1$. Let $\omega_\Omega$ be the universal representation of $S$ in the $\ell^\infty$ direct sum $M_\Omega = \sum_{\omega \in \Omega} \oplus M_\omega$. Then the predual $(M_\Omega)_*$ is the $\ell^1$ direct sum $\sum_{\psi \in \Omega} (M_\omega)_*$ and for each $\psi \in (M_\Omega)_*$ we have $u = \psi \circ \omega_\Omega \in B(S)$ and $\|u\| \leq \|\psi\| [1, 3.1, 3.4], [7].

For $u \in B(S)$ and $x, y \in S$ let $u_x(y) = u(yx)$ then $u_x \in B(S)$ with $\|u_x\| \leq \|u\| [1, 3.4]. This means that the right translation operators $\tau_x : B(S) \to B(S)$ defined by

$$\tau_x(u) = u_x \quad (x \in S, u \in B(S)),$$

are bounded with $\|\tau_x\| \leq 1$.

**Definition 3.1.** For $u \in B(S)$ and $f \in B(S)^* = W^*_\Omega(S)$ define $E_f(u) : S \to \mathbb{C}$ by

$$E_f(u)(x) = \{f, u_x\} \quad (x \in S).$$

**Lemma 3.2.** For $f \in W^*_\Omega(S)$, $E_f : B(S) \to B(S)$ is a bounded linear operator which commutes with right translation operators and $\|E_f\| = \|f\|$.

**Proof.** Let $u \in B(S)$ and choose $\psi \in (M_\Omega)_*$ with $u = \psi \circ \omega_\Omega$ and $\|u\| = \|\psi\|$, then $u(x) = \langle \omega_\Omega(x), \psi \rangle$, for $x \in S$. Given $\zeta \in (M_\Omega)_*$ and $m \in M_\Omega$ define $\zeta \cdot m \in (M_\Omega)_*$ by $\langle n, \zeta \cdot m \rangle = \langle mn, \zeta \rangle$ for $n \in M_\Omega$. Also $m, \zeta$ is defined similarly. For each $x, y \in S$,

$$u_x(y) = u(yx) = \langle \omega_\Omega(yx), \psi \rangle = \langle \omega_\Omega(y), \psi \rangle = \langle \zeta \cdot \omega_\Omega(x), \psi \rangle,$$

hence $u_x = (\zeta \cdot \omega_\Omega(x)) \circ \omega_\Omega$. To each $f \in W^*_\Omega(S)$ there corresponds $f^\circ \in M_\Omega$ defined by $(f^\circ, \zeta) = (f, \zeta \circ \omega_\Omega)$, for $\zeta \in (M_\Omega)_*$. Then

$$E_f(u)(x) = \{f, u_x\} = \{f, (\zeta \cdot \omega_\Omega(x)) \circ \omega_\Omega\} = \{f^\circ, \psi \circ \omega_\Omega(x)\} = \langle \omega_\Omega(x), f^\circ \cdot \psi \rangle,$$

so $E_f(u) = (f^\circ \cdot \psi) \circ \omega_\Omega \in B(S)$ with $\|E_f(u)\| \leq \|f^\circ \cdot \psi\| \leq \|u\| \|f\|$, that is $\|E_f\| \leq \|f\|$. On the other hand $\|f\| = \|E_f(u)(e)\| \leq \|E_f(u)\| \leq \|E_f\| \|u\|$, hence $\|E_f\| = \|f\|$. Finally, for $x, y \in S$,

$$(E_f(u))_{xy} = E_f(u)(yx) = \{f, u_{xy}\} = \{f, (u_x)_y\} = E_f(u_x)(y),$$

and so $E_f$ commutes with right translation operators. \qed

Let $L(B(S))$ be the space of all bounded linear operators on $B(S)$ and $L_0(B(S))$ be the closed subspace of $L(B(S))$ consisting of those operators which commute with all right translation operators $\tau_x$ on $B(S)$.

**Theorem 3.3.** Let $S$ be a unital foundation topological $*$-semigroup with identity $e$, then $B(S)^*$ is isometrically isomorphic to $L_0(B(S))$ and $B(S)^*$ is homeomorphic to the space $\text{End}(L_0(B(S)))$ consisting of non-zero endomorphisms of $L_0(B(S))$. In particular $B(S)^*$ is a compact unital semitopological semigroup.
Proof. By above lemma, the map \( f \mapsto E_f \) is an isometric isomorphism from \( B(S)^* \) into \( L_0(B(S)) \). Given \( E \in L_0(B(S)) \) define \( f \in B(S)^* \) by \( \langle f, u \rangle = E(u)(e) \), for \( u \in B(S) \). Then

\[
E_f(u)(x) = (f, u_x) = E(u_x)(e) = E(u)_x(e) = E(u)(x),
\]

for \( x \in S \) and \( u \in B(S) \). Therefore \( E_f = E \). Now it is easy to check that \( f \) is multiplicative if and only if \( E_f \) is an endomorphism. Next \( B(S)^* \) is isomorphic with the \( w^* \)-closed linear span of \( \{ \omega_\Omega(x) : x \in S \} \) in \( M_\Omega \) [1, 2.1]. Now for each net \( \{ f_\alpha \} \subseteq B(S)^* \), \( E_{f_\alpha} \to E_f \) in \( WOT \) if and only if \( E_{f_\alpha}(u) \to E_f(u) \) weakly, for each \( u \in B(S) \), that is \( \langle m, E_{f_\alpha}(u) \rangle \to \langle m, E_f(u) \rangle \), for \( m \in B(S)^* \), which in turn is equivalent to \( \langle f_\alpha, \psi, m \rangle = \langle m, f_\alpha, \psi \rangle \to \langle f, \psi, m \rangle = \langle m, f, \psi \rangle \), for \( m \in B(S)^* \) and \( \psi \in (M_\Omega)_* \). But \( B(S) \) is unital and so \( (M_\Omega)_*, B(S) = (M_\Omega)_* \), hence the latter is equivalent to \( \langle f_\alpha, \psi \rangle \to \langle f, \psi \rangle \), for \( \psi \in (M_\Omega)_* \), that is \( f_\alpha \to f \) in \( w^* \)-topology. \( \square \)

4. Examples

In this section we calculate the algebras \( A(S) \) and \( B(S) \) in various examples. One class of examples are semigroups of type \( \mathcal{U} \) [4].

The following example shows that the existence of an identity is needed in Proposition 2.3.

Example 4.1. Let \( S = \mathbb{N} \cup \{ 0 \} \) with discrete topology and multiplication \( n.m = \delta_{nm}n \), for \( n, m \in S \). Then each singleton \( \{ n \} \) is the trivial group and \( S \) is of type \( \mathcal{U} \). In this case \( R(S) = \ell^1(\mathbb{N}) \cup \mathbb{C} \) [5, 3.1.6], whereas \( B(S) = \text{span}\{ f \in c_0(S) : f(n) \geq f(0) \geq 0 \} \).

Example 4.2. Let \( S \) be the unit ball of \( L^\infty(\Omega, \mu) \) with pointwise multiplication and \( w^* \)-topology. We assume that \( \mu \) is a finite measure on \( \Omega \). Put

\[
U = \{ f \in S : |f| = 1 \text{ or } 0 \}.
\]

In this case \( f' = f \) if \( f \neq 0 \) and \( 0' = 0 \). We claim that the map \( f \mapsto f' = \check{f} \) is continuous on \( U \). Let \( f_\alpha \to f \) in \( w^* \)-topology, i.e.

\[
\int_\Omega |g| f_\alpha d\mu \to \int_\Omega |g| f d\mu \quad (g \in L^1(\Omega, \mu)).
\]

Then we have

\[
\int_\Omega g(\check{f}_\alpha - \check{f}) d\mu = (\int_\Omega \check{g}(f_\alpha - f) d\mu) \to 0,
\]

for each \( g \in L^1(\Omega, \mu) \). This shows that \( S \) is of type \( \mathcal{U} \). In particular \( B(S) = R(S) \).

Example 4.3. Let \( S = G \cup \{ \infty \} \) be a one-point compactification of a locally compact group \( G \). If \( \{ g_\alpha \} \) is a net in \( G \) and \( g_\alpha \to \infty \) in \( S \), then \( g_\alpha^{-1} \to \infty \) in \( S \). If \( g_\alpha \to g \) in \( G \) then \( g_\alpha^{-1} \to g^{-1} \) in \( G \). Hence \( S \) is of type \( \mathcal{U} \). Also \( S \) is unital with identity \( \infty \). If \( G \) is abelian, then \( B(S) = R(S) = M_0(G) \oplus \mathbb{C} \), where \( M_0(G) = \{ \mu \in M(G) : \check{\mu} \in C_0(G) \} \) [5, 5.1.3].
Example 4.4. Let $S = ([0, 1], \text{max})$ with involution $x^* = x$. Then $S$ is a compact abelian unital semigroup and $\hat{S}$ is an idempotent semigroup. Indeed

$\hat{S} = \{x_{[0, x]} : x \in S\}.$

In particular $\hat{S}$ separates the points of $S$ (and so does $\Sigma(S)$.) Also

$L^1(S) = \{f : S \to \mathbb{C} : f \text{ measurable and } \int_0^1 |f(x)| dx < \infty\}$

is a Banach algebra with convolution

$$f * g(x) = f(x) \int_0^x g(t) dt + g(x) \int_0^x f(t) dt.$$ 

$L^1(S)$ has a bounded approximate identity. Let $f : S \to \mathbb{C}$ be positive definite, then

$$n \sum_{i,j=1}^n c_i c_j f(x_i x_j^*) \geq 0,$$

for each $n \geq 1, c_1, \ldots, c_n \in \mathbb{C},$ and $x_1, \ldots, x_n \in S.$ Once put $n = 1, c_1 = 1,$ and $x_1 = x,$ and then put $n = 2, c_1 = c_2 = \sqrt{-1},$ and $x_1 = x, x_2 = y$ to get

$$f(x) \geq 0, \quad f(x) - 2f(xy) + f(y) \geq 0,$$

for each $x, y \in S$. This shows that $f$ is non-negative and non-increasing. Conversely all such functions are positive definite, and so $A(S) = B(S) = BV[0, 1]$. In particular $A(S)$ is regular and natural [4, 4.4.35]. Also $B(S)$ is not a dual space [5]. Note that in this case $S$ is not foundation [7] (compare with [1].) The convolution product of two elements in $L^2(S)$ is defined as above.

In particular for $g(x) = 1$ and

$$f(x) = \begin{cases} x \sin \left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases},$$

we have $f, g \in L^2(S)$, but

$$f * g(x) = x^2 \sin \left(\frac{1}{x}\right) + \int_0^x t \sin \left(\frac{1}{t}\right) dt,$$

for $x \neq 0$ and $f * g(0) = 0$. It is easy to see that $f * g \notin BV[0, 1]$. In particular $A(S) \neq L^2(S) \ast L^2(S)$. 

Example 4.5. Let $S = (\mathbb{R}^+, +)$ with involution $x^* = x$. Then $S$ is a locally compact commutative unital $*$-semigroup. If $f : S \to \mathbb{C}$ is continuous and positive definite, then in the corresponding inequality, once put $n = 1, c_1 = 1,$ and $x_1 = \frac{x}{2}$, and then put $n = 2, c_1 = 1, c_2 = -1,$ and $x_1 = \frac{x}{2}, x_2 = \frac{y}{2}$ to get

$$f(x) \geq 0, \quad f(x) - 2f\left(\frac{x}{2} + \frac{y}{2}\right) + f(y) \geq 0,$$
for each \(x, y \in S\). This shows that \(f\) is non-negative and convex. Conversely we know that \(\mathbb{R}^+ \simeq \mathbb{R}^+\) \([4]\) and we have the Laplace transform
\[
\hat{\mu}(x) = \int_0^\infty e^{-xt}d\mu(t),
\]
for \(\mu \in M(\mathbb{R}^+)\), and these are exactly the elements of \(B(\mathbb{R}^+)\) \([2]\).

**Example 4.6.** Let \(S = (\mathbb{N} \cup \{0\}, +)\) with involution \(x^* = x\). Then \(S\) is a discrete abelian unital semigroup. If \(f : S \to \mathbb{C}\) is positive definite, then in the corresponding inequality, once put \(n = 1, c_1 = 1, c_2 = -1\), and \(x_1 = m, x_2 = n\) to get
\[
f(2n) \geq 0, \ f(0) - 2f(n) + f(2n) \geq 0, \ f(2m) - 2f(m + n) + f(2n) \geq 0,
\]
for each \(m, n \in S\). It follows from the first and second inequality that \(f\) is real valued. In this case \(\hat{S} \simeq [-1, 1]\) with multiplication. Hence \(B(S) \simeq M[-1, 1]\).

**References**


