On Diameter of Line Graphs

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Abstract. The diameter of a connected graph $G$, denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices of $G$. Let $L(G)$ be the line graph of $G$. We establish necessary and sufficient conditions under which for a given integer $k \geq 2$, $\text{diam}(L(G)) \leq k$.

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1. Introduction

Let $G$ be a simple connected graph on $n$ vertices. Let the vertices of $G$ be labeled as $v_1, v_2, \ldots, v_n$. The distance between the vertices $v_i$ and $v_j$ in $G$ is equal to the length of a shortest path joining $v_i$ and $v_j$, and is denoted by $d_G(v_i, v_j)$. The diameter of $G$, denoted by $\text{diam}(G)$ is the maximum distance between any pair of vertices of $G$.

The above distance provides the simplest and most natural metric in graph theory, and is one of the popular areas of research in discrete mathematics. Details on distance in graph theory can be found in the books [3, 5, 8] and the papers [1, 6, 7, 15, 16] published in this journal.

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As usual, by $K_n$, $P_n$, and $K_{1,n-1}$ we denote, respectively, the complete graph, the path, and the star on $n$ vertices.

The line graph $L(G)$ of $G$ is the graph whose vertices correspond to the edges of $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent. The second line graph of $G$ is $L^2(G) = L(L(G))$.

Metric properties of line graphs have been much studied in the mathematical literature [2,4,9,12,14,17–20], and found remarkable applications in chemistry [10,11,13,14].

We first recall some known established properties of line graphs, needed for the considerations that follow.

**Lemma 1.1.** [17] If $G_1$ is an induced subgraph of $G$ then $L(G_1)$ is an induced subgraph of $L(G)$.

**Theorem 1.2.** [19] If $\text{diam}(G) \leq 2$ and if none of the three graphs $F_1$, $F_2$, and $F_3$ depicted in Fig. 1 are induced subgraphs of $G$, then $\text{diam}(L(G)) \leq 2$.

![Fig. 1. The graphs mentioned in Theorem 1.2](image)

In this paper we establish structural conditions for the graph $G$, under which for a given integer $k$, $k \geq 2$, the diameter of $L(G)$ does not exceed $k$. We also establish conditions under which for a given integer $k$, $k \geq 3$, the diameter of $L(G)$ is not less than $k$.

2. **Main results**

Let $F_1^k$ be the path on $(k+3)$ vertices, $k \geq 2$. The vertices of $F_1^k$ are $v_1, v_2, \ldots, v_{k+3}$, labeled so that $v_i$ is adjacent to $v_{i+1}$, $i = 1, 2, \ldots, k + 2$.

Let $F_2^k$ be the graph obtained from $F_1^k$ by adding to it an edge between the vertices $v_1$ and $v_3$. Let $F_3^k$ be the graph obtained from $F_1^k$ by adding to it edges between $v_1$ and $v_3$ and between $v_{k+1}$ and $v_{k+3}$ (see Fig. 2).

**Theorem 2.1.** Let $k \geq 2$. For a connected graph $G$, $\text{diam}(L(G)) \leq k$, if and only if none of the three graphs $F_1^k$, $F_2^k$, and $F_3^k$, depicted in Fig. 2, are an induced subgraph of $G$. 
Proof. The result can be easily verified for graphs of order \( n \leq 4 \). We thus assume that \( n > 4 \).

Let \( k \geq 2 \) and let \( \text{diam}(L(G)) \leq k \). Suppose that \( F_k \) is an induced subgraph of \( G \). By Lemma 1.1, \( L(F_k) \) is an induced subgraph of \( L(G) \). It is straightforward to check that \( \text{diam}(L(F_k)) = \text{diam}(P_{k+1}) = k + 1 > k \). Hence \( \text{diam}(L(G)) > k \), a contradiction. Therefore \( F_k \) is not an induced subgraph of \( G \).

![Graphs mentioned in Theorems 2.1 and 2.3](image)

Fig. 2 The graphs mentioned in Theorems 2.1 and 2.3

Similarly we can show that \( F_2 \) and \( F_3 \) are also not induced subgraphs of \( G \).

Conversely, suppose that \( k \geq 2 \) and that \( \text{diam}(L(G)) > k \). Then \( G \) must possess two independent edges, say \( e_i = (uv) \) and \( e_j = (xy) \), such that neither \( u \) nor \( v \) are adjacent to either \( x \) or \( y \). If so, then because the diameter of \( L(G) \) is greater than \( k \), there must exist \( k - 1 \) vertices, say \( u_1, u_2, \ldots, u_{k-1} \) such that \( u \) is adjacent to \( u_1 \), \( u_i \) is adjacent to \( u_{i+1} \), \( i = 1, 2, \ldots, k - 2 \), and \( u_{k-1} \) is adjacent to \( x \). If \( u_i \), \( i = 1, 2, \ldots, k - 1 \) are not adjacent to either \( v \) or \( y \), then \( G \) has \( F_1 \) as an induced subgraph (spanned by the vertices \( v, u, u_1, u_2, \ldots, u_{k-1}, x, y \)). If \( u_1 \) is adjacent to \( v \) (or \( u_{k-1} \) is adjacent to \( y \)), then \( G \) has \( F_2 \) as an induced subgraph. If \( u_1 \) is adjacent to \( v \) and \( u_{k-1} \) is adjacent to \( y \), then \( G \) has \( F_3 \) as an induced subgraph, a contradiction. Hence \( \text{diam}(L(G)) \leq k \). \( \square \)

Theorem 1.2 is a special case of Theorem 2.1, for \( k = 2 \). From Theorem 2.1, we observe that the condition \( \text{diam}(G) \leq 2 \), in Theorem 1.2 was not necessary.
Theorem 2.2. Let $G$ be a connected graph with $n \geq 3$ vertices. Then $\text{diam}(L(G)) = 1$ if and only if $G \cong K_3$ or $G \cong K_{1,n-1}$.

Proof. If $G \cong K_3$, then $L(K_3) = K_3$ and $\text{diam}(L(K_3)) = \text{diam}(K_3) = 1$. If $G \cong K_{1,n-1}$, then all the edges of $K_{1,n-1}$ are incident to a common vertex. Therefore all vertices are adjacent to each other in $L(K_{1,n-1})$ and thus $L(K_{1,n-1}) \cong K_{n-1}$. Hence $\text{diam}(L(K_{1,n-1})) = 1$.

Conversely, let $\text{diam}(L(G)) = 1$. Suppose that $G \not\cong K_3$, $K_{1,n-1}$. Then in $G$ there exists at least two independent edges, say $e_i = (uv)$ and $e_j = (xy)$. Therefore $d_{L(G)}(e_i,e_j) > 1$. Thus $\text{diam}(L(G)) > 1$, a contradiction. Hence it must be $G \cong K_3$ or $G \cong K_{1,n-1}$.

\(\square\)

Evidently, the diameter of $L(G)$ is zero if and only if $G \cong K_1$ or $G \cong K_2$.

A statement equivalent to Theorem 2.1 is:

Theorem 2.3. Let $G$ be a connected graph with $n \geq 3$ vertices. Let $k \geq 2$. Then $\text{diam}(L(G)) > k$, if and only if either $F^k_1$ or $F^k_2$ or $F^k_3$, depicted in Fig. 2, is an induced subgraph $G$.

3. A RESULT FOR SECOND LINE GRAPH

Let $P_{k-1}$ be the path with vertices $u_1, u_2, \ldots, u_{k-1}$, where $u_i$ is adjacent to $u_{i+1}$, $i = 1, 2, \ldots, k-2$, $k \geq 3$. Let $F^k_4$ be the graph obtained from $P_{k-1}$ by joining two vertices to $u_1$ and another two vertices to $u_{k-1}$ (see Fig. 3). $F^k_4$ has $k + 3$ vertices and $k + 2$ edges.

\[F^k_4\]

Fig. 3. The graph mentioned in Theorem 3.1

Theorem 3.1. Let $k \geq 3$. If $F^k_4$ is an induced subgraph of $G$, then $\text{diam}(L^2(G)) \geq k - 1$.

Proof. Let $k \geq 3$. Let $F^k_4$ be the induced subgraph of $G$. Then $L(F^k_4)$ is isomorphic to $F^{k-1}_3$, and by Lemma 1.1, $L(F^k_4)$ is an induced subgraph of $L(G)$. Therefore $F^{k-1}_3$ is an induced subgraph of $L(G)$. Hence by Theorem 2.3, $diam(L(L(G))) = diam(L^2(G)) > k - 1$.

\(\square\)

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