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Some Properties of Ideal Extensions in Ternary Semigroups

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Abstract. A concept of ideal extensions in ternary semigroups is introduced and throughly investigated. The connection between an ideal extensions and semilattice congruences in ternary semigroups is considered.

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1. Introduction and Prerequisites

The literature of ternary algebraic system was introduced by Lehmer [4] in 1932. He investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. The notion of ternary semigroups was known to S. Banach. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup. We can see that any semigroup can be reduced to a ternary semigroup. In 1965, Sioson [7] studied ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. In 1995, Dixit and Dewan [1] introduced and studied the properties of ideals in ternary semigroups. In 2007, Kar and Maity [3] introduced the notion of congruences on ternary semigroups and studied some interesting properties. They also introduced the notions of cancellative congruences, group congruences and Rees

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congruences and characterized these congruences in ternary semigroups. Congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures.

The concept of ideals is an interesting and important idea in many algebraic structures. Several researches have characterized the many type of ideals on the algebraic structures such as: In 2009, Iampan [2] studied the concept and gave some characterizations of (0-)minimal and maximal ordered bi-ideals in ordered Γ-semigroups. In 2010, Nezhad [5] gave several characterizations of strongly prime ideals of commutative integral domains. In this year, Shabir, Jun and Bano [6] introduced and studied the prime, strongly prime, semiprime and irreducible fuzzy bi-ideals of semigroups. They characterized those semigroups for which each fuzzy bi-ideal is semiprime and also characterized those semigroups for which each fuzzy bi-ideal is strongly prime.

Our purpose in this paper is threefold.

(1) To give the definition of ideal extensions in ternary semigroups.
(2) To characterize the properties of ideal extensions in ternary semigroups.
(3) To characterize the connection between ideal extensions and some congruences in ternary semigroups.

We first recall the definition of a ternary semigroup which is important in this paper.

A nonempty set T is called a *ternary semigroup* [4] if there exists a ternary operation $[\ ] : T \times T \times T \to T$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]].$$

For nonempty subsets A, B and C of a ternary semigroup T, let

$$[ABC] := \{[abc] \mid a \in A, b \in B \text{ and } c \in C\}.$$

If $A = \{a\}$, then we also write $[\{a\}BC]$ as $[aBC]$, and similarly if $B = \{b\}$ or $C = \{c\}$ or $A = \{a\}$ and $B = \{b\}$ or $A = \{a\}$ and $C = \{c\}$ or $B = \{b\}$ and $C = \{c\}$. A nonempty subset S of a ternary semigroup T is called a *ternary subsemigroup* [1] of T if $[SSS] \subseteq S$. A ternary semigroup T is said to be *commutative* if for any $x_1, x_2, x_3 \in T$ and permutation σ of $\{1, 2, 3\}$,

$$[x_1x_2x_3] = [x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}].$$

For any positive integers m and n with $m \leq n$ and any elements $x_1, x_2, ..., x_{2n}$ and x_{2n+1} of a ternary semigroup T [7], we can write

$$\begin{aligned} [x_1x_2...x_{2n+1}] &= [x_1...x_mx_{m+1}x_{m+2}...x_{2n+1}] \\ &= [x_1...[[x_mx_{m+1}x_{m+2}]x_{m+3}x_{m+4}]...x_{2n+1}]. \end{aligned}$$

Example 1.1. [1] *Let* $T = \{-i, 0, i\}$. *Then* T *is a ternary semigroup under the multiplication over complex number while* T *is not a semigroup under complex number multiplication.*

Example 1.2. [1] *Let* $O = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], A_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], A_2 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], A_3 = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ *and* $A_4 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. *Then* $T = \{O, I, A_1, A_2, A_3, A_4\}$ *is a ternary semigroup under matrix multiplication.*

The following definitions are introduced analogously to some definitions in semigroups.

A nonempty subset I of a ternary semigroup T is called an *ideal* [7] of T if $[TTI] \subseteq I, [TIT] \subseteq I$ and $[ITT] \subseteq I$. An ideal I of a ternary semigroup T is called

a *prime ideal* of T if $[aTb] \subseteq I$ implies $a \in I$ or $b \in I$,

a *strongly prime ideal* of T if $[atb] \in I$ implies $a \in I$ or $b \in I$,

a *semiprime ideal* of T if $[aTa] \subseteq I$ implies $a \in I$,

a *strongly semiprime ideal* of T if $[ata] \in I$ implies $a \in I$

for all $a, b, t \in T$. Then we have that I is a prime ideal of T if for any $A, B \subseteq T, [ATB] \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. Analogous results holds if I is a strongly prime ideal, a semiprime ideal and a strongly semiprime ideal of T. Hence the following implications hold in a ternary semigroup.

$$\begin{array}{ccc} \text{strongly prime ideal} & \Rightarrow & \text{prime ideal} \\ \Downarrow & & \Downarrow \\ \text{strongly semiprime ideal} & \Rightarrow & \text{semiprime ideal} \end{array}$$

The intersection of all ideals of a ternary semigroup T containing a nonempty subset A of T is called the *ideal of T generated by A* and denoted by $I(A)$. We also write $I(x)$ for $I(\{x\})$. It is clear that for any subset A of a commutative ternary semigroup $T, I(A) = [TTA] \cup A$. An equivalence relation ρ on a ternary semigroup T is called a *congruence* [3] on T if for any $a, b \in T, (a, b) \in \rho$ implies $([axy], [bxy]), ([xay], [xby]), ([xya], [xyb]) \in \rho$ for all $x, y \in T$. If ρ is a congruence on a ternary semigroup T, then $T/\rho = \{(x)_\rho \mid x \in T\}$ is a ternary semigroup with $[(a)_\rho (b)_\rho (c)_\rho] = [abc]_\rho$ for all $a, b, c \in T$. A congruence ρ on a ternary semigroup T is called a *semilattice congruence* on T if $([xxx], x) \in \rho$ for all $x \in T$ and $([x_1 x_2 x_3], [x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}]) \in \rho$ for all $x_1, x_2, x_3 \in T$ and permutation σ of $\{1, 2, 3\}$.

If I is an ideal of a ternary semigroup T and $A \subseteq T$, then

$$< A, I > := \{t \in T \mid [ATt] \subseteq I\}$$

contains I and it is called the *extension* of I by A. Also, let $< a, I >$ stand for $< \{a\}, I >$. Notice that $[AT < A, I >] \subseteq I$ and for $B \subseteq T, [ATB] \subseteq I$ implies $B \subseteq < A, I >$. In fact, $< A, I >$ is an ideal of T containing I if T is commutative. Since

$$\begin{aligned} [AT[< A, I > TT]] &= [[AT < A, I >]TT] \\ &\subseteq [ITT] \\ &\subseteq I, \end{aligned}$$

it follows that $[< A, I > TT] \subseteq < A, I >$.

For an ideal I of a ternary semigroup T, define the equivalence relation ϕ_I on T as follows:

$$\phi_I := \{(x, y) \mid < x, I >=< y, I >\}.$$

In the remainder, let T be a ternary semigroup. Our purpose is to provide various properties of extensions of ideals in T. The equivalence relation ϕ_I of T is considered. It will be shown that if T is commutative and I is a strongly semiprime ideal of T, then ϕ_I is a semilattice congruence. In addition, if I is a prime ideal of T, then

$$\phi_I = (I \times I) \cup (T \setminus I \times T \setminus I).$$

2. Main Results

We start to prove the characterizations of ideal extensions in ternary semigroups. Our main result is as follows.

Lemma 2.1. *Let I be an ideal of T and $A, B \subseteq T$. Then the following statements hold.*

(a) *If $A \subseteq B$, then $< B, I > \subseteq < A, I >$.*
(b) *If $A \subseteq I$, then $< A, I >= T$.*
(c) *$< A, I > \subseteq < A \setminus I, I >$.*

Proof. (a) Suppose $A \subseteq B$. Then

$$\begin{aligned} x \in < B, I > &\Rightarrow [BTx] \subseteq I \\ &\Rightarrow [ATx] \subseteq I \text{ (because } [ATx] \subseteq [BTx]) \\ &\Rightarrow x \in < A, I > . \end{aligned}$$

Hence $< B, I > \subseteq < A, I >$.

(b) Suppose $A \subseteq I$. Clearly, $< A, I > \subseteq T$. If $x \in T$, then

$$[ATx] \subseteq [ITx] \subseteq I.$$

Thus $x \in < A, I >$, so $T \subseteq < A, I >$. Hence $< A, I >= T$.

(c) Since $A \setminus I \subseteq A$, it follows by statement (a) that $< A, I > \subseteq < A \setminus I, I >$. $\square$

Proposition 2.2. *If I is an ideal of T and $A \subseteq T$, then for any subsets X and Y of T,*

$$< A, I > \subseteq < [XTA], I > \subseteq < [XYA], I >.$$

Proof. Since

$$\begin{aligned} [[XTA]T < A, I >] &= [XT[AT < A, I >]] \\ &\subseteq [XTI] \\ &\subseteq I \end{aligned}$$

and

$$\begin{aligned} [[XYA]T < [XTA], I >] &\subseteq [[XTA]T < [XTA], I >] \\ &\subseteq I. \end{aligned}$$

It follows that $< A, I > \subseteq < [XTA], I >$ and $< [XTA], I > \subseteq < [XYA], I >$, respectively. □

Proposition 2.3. *Let I and I_i be ideals of T and $A, A_i \subseteq T$ for all $i \in \Lambda$. Then we have*

(a) $< A, \bigcap_{i\in\Lambda} I_i > = \bigcap_{i\in\Lambda} < A, I_i >$ *and*

(b) $< \bigcup_{i\in\Lambda} A_i, I > = \bigcap_{i\in\Lambda} < A_i, I >$.

Proof. (a) For $x \in T$,

$$\begin{aligned} x \in < A, \bigcap_{i\in\Lambda} I_i > &\Leftrightarrow [ATx] \subseteq \bigcap_{i\in\Lambda} I_i \\ &\Leftrightarrow [ATx] \subseteq I_i \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in < A, I_i > \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in \bigcap_{i\in\Lambda} < A, I_i > . \end{aligned}$$

Hence $< A, \bigcap_{i\in\Lambda} I_i > = \bigcap_{i\in\Lambda} < A, I_i >$.

(b) For $x \in T$,

$$\begin{aligned} x \in < \bigcup_{i\in\Lambda} A_i, I > &\Leftrightarrow [(\bigcup_{i\in\Lambda} A_i)Tx] \subseteq I \\ &\Leftrightarrow [A_iTx] \subseteq I \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in < A_i, I > \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in \bigcap_{i\in\Lambda} < A_i, I > . \end{aligned}$$

Hence $< \bigcup_{i\in\Lambda} A_i, I > = \bigcap_{i\in\Lambda} < A_i, I >$. □

Proposition 2.4. *Let I be an ideal of T. Then I is a prime ideal of T if and only if $< A, I >= I$ for all $A \subseteq T$ with $A \nsubseteq I$.*

Proof. Assume that I is a prime ideal of T and let $A \nsubseteq I$. Since $[AT < A, I >] \subseteq I, A \nsubseteq I$ and I is a prime ideal of T, it follows that $< A, I > \subseteq I$ which implies that $< A, I >= I$.

Conversely, assume that $< A, I >= I$ for all $A \nsubseteq I$. To show that I is a prime ideal of T, let $A, B \subseteq T$ be such that $[ATB] \subseteq I$ and $A \nsubseteq I$. Then $B \subseteq < A, I >= I$. Hence I is a prime ideal of T. □

Recall that for an ideal I of T and $A \subseteq T, < A, I >$ is an ideal of T if T is commutative.

Corollary 2.5. *Assume that T is commutative. If I is a prime ideal of T and $A \subseteq T$, then $< A, I >$ is a prime ideal of T.*

Proof. This follows directly from Lemma 2.1(b) and Proposition 2.4. □

It is obviously seen that a nonempty intersection of prime ideals of T is a semiprime ideal of T.

Corollary 2.6. *Assume that T is commutative and $A \subseteq T$. If $\{I_i \mid i \in \Lambda\}$ is a collection of prime ideals of T such that $\bigcap_{i\in\Lambda} I_i \neq \emptyset$, then $< A, \bigcap_{i\in\Lambda} I_i >$ is a semiprime ideal of T.*

Proof. By Corollary 2.5, $< A, I_i >$ is a prime ideal of T for all $i \in \Lambda$. But $< A, \bigcap_{i\in\Lambda} I_i >= \bigcap_{i\in\Lambda} < A, I_i >$ by Proposition 2.3(a). It follows that $< A, \bigcap_{i\in\Lambda} I_i >$ is a semiprime ideal of T. □

Proposition 2.7. *For $A, B \subseteq T$, if T is commutative and $I(A) \subseteq I(B)$, then $< B, I > \subseteq < A, I >$ for every ideal I of T.*

Proof. Assume that $I(A) \subseteq I(B)$ and let I be an ideal of T. If $x \in < B, I >$, then $[BTx] \subseteq I$. Since $A \subseteq I(B)$, it follows from Lemma 2.1(a) that $< I(B), I > \subseteq < A, I >$. By Lemma 2.1($a$) and Propositions 2.2 and 2.3(b), we have

$$\begin{aligned} < B, I > &\subseteq < B, I > \cap < [TTB], I > \\ &= < B \cup [TTB], I > \\ &= < I(B), I > \\ &\subseteq < A, I > . \end{aligned}$$

□

Lemma 2.8. *If T is commutative and I is an ideal of T, then ϕ_I is a congruence on T.*

Proof. Let $(x, y) \in \phi_I$ and $a, b \in T$. Then $< x, I >=< y, I >$. For $t \in T$,

$$\begin{aligned} t \in< [xab], I > &\Leftrightarrow [[xab]Tt] \subseteq I \\ &\Leftrightarrow [xT[abt]] \subseteq I \\ &\Leftrightarrow [abt] \in< x, I > \\ &\Leftrightarrow [abt] \in< y, I > \\ &\Leftrightarrow [yT[abt]] \subseteq I \\ &\Leftrightarrow [[yab]Tt] \subseteq I \\ &\Leftrightarrow t \in< [yab], I > . \end{aligned}$$

Thus $([xab], [yab]) \in \phi_I$. Since T is commutative, we have $([axb], [ayb]) \in \phi_I$ and $([abx], [aby]) \in \phi_I$. Hence ϕ_I is a congruence on T. □

Proposition 2.9. *If T is commutative and I is a strongly semiprime ideal of T, then ϕ_I is a semilattice congruence on T.*

Proof. By Proposition 2.8, ϕ_I is a congruence on T. Since T is commutative, we have $([x_1x_2x_3], [x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}]) \in \phi_I$ for all $x_1, x_2, x_3 \in T$ and permutation σ of $\{1, 2, 3\}$. Let $x \in T$. Then for $t \in T$,

$$\begin{aligned} t \in< [xxx], I > &\Rightarrow [[xxx]Tt] \subseteq I \\ &\Rightarrow [[[xxx]Tt]Tt] \subseteq [ITT] \subseteq I \\ &\Rightarrow [[xTt]x[xTt]] \subseteq I \\ &\Rightarrow [xTt] \subseteq I \\ &\Rightarrow t \in< x, I > . \end{aligned}$$

Thus $< [xxx], I >\subseteq< x, I >$. By Proposition 2.2, $< x, I >\subseteq< [xxx], I >$. Therefore, we have $< [xxx], I >=< x, I >$, so $([xxx], x) \in \phi_I$. Hence ϕ_I is a semilattice congruence on T. □

Proposition 2.10. *If I is a prime ideal of T, then*

$$\phi_I = (I \times I) \cup (T \setminus I \times T \setminus I).$$

Proof. This follows directly from Lemma 2.1(b) and Proposition 2.4. □

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