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A Study of the Total Graph

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Abstract. Let R be a commutative ring with $\mathrm{Z}(R)$ its set of zero-divisors. In this paper, we study the total graph of R, denoted by $\mathrm{T}(\Gamma(R))$. It is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in \mathrm{Z}(R)$. We study the chromatic number and edge connectivity of this graph.

Keywords: Zero-divisor graph, Total graph.

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1. Introduction

Let R be a commutative ring with non-zero unity. The concept of the graph of the zero divisors of R was first introduced by Beck [6], where he was mainly interested in colorings. In his work all elements of the ring were vertices of the graph. The investigation of colorings of a commutative ring was then continued by D. D. Anderson and Naseer in [5]. Let $\mathrm{Z}(R)$ be the set of zero-divisors of R. In [4], D. F. Anderson and Livingston associate a graph, $\Gamma(R)$, to R with vertices $\mathrm{Z}(R) \setminus \{0\}$, the set of non-zero zero-divisors of R, and for distinct $x, y \in \mathrm{Z}(R) \setminus \{0\}$, the vertices x and y are adjacent if and only if $xy = 0$.

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In [3], D. F. Anderson and Badawi introduced the total graph of R, denoted by $\mathrm{T}(\Gamma(R))$, as the graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in \mathrm{Z}(R)$. They studied some graphical parameters of this graph such as diameter and girth. In addition, they studied some special subgraphs of $\mathrm{T}(\Gamma(R))$, and study the total graph based on these subgraphs. In [1] Akbari and et. al., prove that the total graph is a Hamiltonian graph if this graph is connected. In [12], Maimani and et. al., gave the necessary and sufficient codition for the planarity of total graphs of commutative rings. They also characterized all rings with total graph of genus 1.
This paper organized as follows:
In Section 1, we recall some definitions and known results. In Section 2, we study the structure of total graphs when the set of zero divisor of R is an ideal. In Section 3, some graphical parameters of total graphs are studied. Finally, the edge chromatic number of total graph are obtained and independency number and edge connectivity of this graph are calculated.

2. Definitions and Preliminaries

For a graph G, let $V(G)$ denotes the set of vertices, and let $E(G)$ denotes the set of edges. For a graph G and vertex $x \in V(G)$, the *degree* of x, denoted by $\deg(x)$, is the number of edges of G incident with x. A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. For r a nonnegative integer, an *r-partite graph* is one whose vertex-set is partitioned into r disjoint parts in such a way that the two end vertices for each edge lie in distinct partitions. A *complete r-partite graph* is one in which each *vertex* is joined to every vertex that is not in the same partition. The *complete* 2-*partite graph* (also called the *complete bipartite graph*) with exactly two partitions of size m and n, is denoted by $K_{m,n}$. A *clique* of a graph G is a complete subgraph of G. A *coclique* in a graph G is a set of pairwise nonadjacent vertices. A subgraph H of a graph G is called a *spanning subgraph* if $V(H) = V(G)$. A 1-regular spanning subgraph H of G is called a *perfect matching* of G. For every nonnegative integer r, the graph G is called *r-regular* if the degree of each vertex of G is equal to r. Recall that the complement graph of a graph G is denoted by $\bar{G}$ with vertices $\mathrm{V}(\bar{G})$, and for distinct $x, y \in \mathrm{V}(\bar{G})$, the vertices x and y are adjacent if and only if $(x, y) \in \mathrm{E}(\bar{G})$.

Let R be a commutative ring and let $Z(R)$ be the set of zero-divisors of R. The *total graph* of R, denoted $T(\Gamma(R))$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x + y \in Z(R)$.

The study of $T(\Gamma(R))$ breaks naturally into two cases depending on whether or not $Z(R)$ is an ideal of R. In the next section we study the structure of total graphs of commutative rings, when the set of zero divisor of rings is an ideal.

3. DEGREE OF VERTICES OF TOTAL GRAPHS

In this section, we study the degree of vertices of total graphs. First we bring a useful lemma.

Lemma 3.1. [12]. *The graph* $\mathrm{T}(\Gamma(R))$ *is a* $|Z(R)|-1$*-regular graph if* $2 \in Z(R)$. *In addition, If* $2 \notin Z(R)$, *then any vertices of* $\mathrm{T}(\Gamma(R))$ *have degree* $(|Z(R)|-1)$ *or* $(|Z(R)|)$.

Proof. Let $2 \in \mathrm{Z}(R)$. Then $2x \in R$ for any $x \in R$. In the other hand x is adjacent to $a-x$ for any $a \in \mathrm{Z}(R)$ except for an unique element $z \in \mathrm{Z}(R)$, which $x = z - x$. Hence the degree of 2 is equal to $|\mathrm{Z}(R)| - 1$. If $2 \notin \mathrm{Z}(R)$, then by the same argument, $2x \in \mathrm{Z}(R)$ only for the elements $x \in \mathrm{Z}(R)$, and we have $\deg(x) = |\mathrm{Z}(R)$ for $x \in R \setminus \mathrm{Z}(R)$, and $\deg(x) = |\mathrm{Z}(R)| - 1$ for $x \in \mathrm{Z}(R)$. □

Corollary 3.2. *The graph* $\mathrm{T}(\Gamma(R))$ *has an isolated vertices, if and only if* R *is integral domain.*

Proof. If $\mathrm{T}(\Gamma(R))$ has an isolated vertices, then by Corollary 3.1, $|\mathrm{Z}(R)| = 1$ and so R is integral domain. If R is integral domain, then the element zero is an isolated vertices. □

Remark 3.3. *If* R *is an integral domain and* $S = \{x|2x = 0\}$, *then* $< S >$ *is an independent set of* $\mathrm{T}(\Gamma(R))$ *and so the induced subgraph* $< R \setminus S >$ *is a matching in the graph* $\mathrm{T}(\Gamma(R)) = |S|K_1 \cup |R \setminus S|K_2$.

By the definition of $\mathrm{T}(\Gamma(R))$, it is obvious that if $Z(R)$ is an ideal of R, then the induced subgraph $< Z(R) >$ is an complete connected component of $\mathrm{T}(\Gamma(R))$. In the following lemma, we show that the converse of this statement is also true.

Lemma 3.4. *The set of zero divisors* $Z(R)$ *is an ideal of* R *if and only if* $< Z(R) >$ *is a complete connected component of* $\mathrm{T}(\Gamma(R))$.

Proof. Suppose that $< Z(R) >$ is an complete connected component $T(\Gamma(R))$. If $x, y \in Z(R)$, then x, y are two vertices of induced subgraph $< Z(R) >$ and therefore they are adjacent. So $x + y \in Z(R)$. Hence $Z(R)$ is an ideal of R. □

Remark 3.5. *Suppose that* $Z(R)$ *is an ideal of* R *and* $\mid Z(R) \mid = \alpha$, *and* $\mid R/Z(R) \mid = \beta$. *If* $2 \in Z(R)$, *then the induced subgraph* $< (x + Z(R)) >$ *is a complete subgraph. On the other hand, if* y *is adjacent to* x, *then* $x+y \in Z(R)$. *Now since* $2x \in Z(R)$, *and* $Z(R)$ *is an ideal, we conclude that* $2x-(x+y) = x-$

$y \in Z(R)$. Hence $y \in x+Z(R)$. Therefore the induced subgraph $< (x+Z(R)) >$ is an connected component of $\mathrm{T}(\Gamma(R))$. Thus $\mathrm{T}(\Gamma(R)) = \beta K_\alpha$. If $2 \notin Z(R)$, then the induced subgraph $< (x + Z(R)) >$ is an empty subgraph. Now if $x \notin Z(R)$ is adjacent to y, then $y \notin x+Z(R)$, and all elements of coset $x+Z(R)$ are adjacent to all elements of coset $y+z(R)$. In this case the induced subgraph $< (x + Z(R)) \cup (y + Z(R)) >$ is a complete bipartite subgraph. On the other hand if x is adjacent to z, then $x + z \in Z(R)$. Now since $Z(R)$ is an ideal, we conclude that $x + y - (x + z) = y - z \in Z(R)$. Hence $z \in y + Z(R)$. Therefore the induced subgraph $< (x + Z(R)) \cup (y + Z(R)) >$ is a connected component of $\mathrm{T}(\Gamma(R))$. So in this case we have $\mathrm{T}\Gamma(R)) = K_\alpha \cup \frac{\beta-1}{2} K_{\alpha,\alpha}$.

4. GRAPHICAL PARAMETERS OF $T(\Gamma(R)$

In this section we study some graphical parameters of total graphs of a commutative ring. At first we study the independent number of this graph.
A clique of a graph is its maximal complete subgraph and the number of vertices in a largest clique of a graph G, denoted by $\omega(G)$, is called the clique number of G. The greatest integer r such that $\bar{K_r} \subseteq G$ is the independence number $\alpha(G) = \omega(\bar{G})$. In the following lemma we obtain the independent number of a total graph.

Lemma 4.1. *Let R be a finite local ring with maximal ideal $\mathfrak{m}$. Suppose that $|\mathfrak{m}| = \alpha$ and $|R/\mathfrak{m}| = \beta$. If $2 \in Z(R)$, then $\alpha(T(\Gamma(R))) = \beta$. If $2 \notin Z(R)$, then $\alpha(T(\Gamma(R))) = \alpha(\frac{\beta-1}{2}) + 1$*

Proof. Since R is a local ring, we conclude that $\mathfrak{m} = Z(R)$ is the maximal ideal of R. If $2 \in Z(R)$, then the total graph, $\mathrm{T}(\Gamma(R))$, is disjoint union of complete graph K_α by Remark 3.5. Hence every independent set of this graph, has at most one vertex from each connected components of $\mathrm{T}(\Gamma(R))$. Therefore $\alpha(T(\Gamma(R))) = \beta$. If $2 \notin Z(R)$, then $\mathrm{T}(\Gamma(R))$ is union of one complete graph (which the vertex set is equal to $\mathfrak{m}$) and $\frac{\beta-1}{2}$ complete bipartite $K_{\alpha,\alpha}$, by Remark 3.5. Hence if S is an independent set, then $|S \cap \mathfrak{m}| \leq 1$. In addition if S is a set, which its elements are a part of each $K_{\beta,\beta}$, and an element from $\mathfrak{m}$, then S is an independent set. this independent set is maximum and therefore $\alpha(R) = \beta(\alpha - 1)/2 + 1$.

□

Remark 4.2. *Suppose that R is a finite commutative ring which is not local. For computing independent number of total graph, we can compute the clique number of complement of the total graph. Note that the statement "x is not adjacent to y in a total graph" is equivalent with "$x+y$ is a unit element of R". The clique number of complement of a total graph were studied by Maimani and et. al. in* [13] *and reader may obtain the independent number of a total graph in* [13].

A *k-edge coloring* of a graph G is an assignment of k *labels*, also called *colors*, to the edges of G such that every pair of distinct edges meeting at a common vertex are assigned two different colors. If G has a k-edge coloring, then G is said to be *k-edge colorable*. The *chromatic index* of G, denoted $\chi'(G)$, is the smallest number k such that G is k-edge colorable. By Vizing's theorem, if G is a graph whose maximum vertex degree is Δ, then $\Delta \leq \chi'(G) \leq \Delta+1$. Vizing's theorem divides the graphs into two classes according to their chromatic index; graphs satisfying $\chi'(G) = \Delta$ are called *class* 1, those with $\chi'(G) = \Delta + 1$ are *class* 2

We now state the following result which shows that all total graphs are of class 1.

Theorem 4.3. *Let R be a finite ring. Then the unit graph $T(\Gamma(R))$ is of class 1.*

Proof. . We color the edge xy by the color $x+y$. By this coloring, clearly edges xy and xz have the different color. But if $2 \in Z(R)$, then for every edge xy we have $x + y \neq 0$. So $\chi(T(\Gamma(R))) \leq \mid Z(R) \mid -1$, and the above edge coloring implies that $\chi(T(\Gamma(R))) = \mid Z(R) \mid = \Delta$. □

For a graph G, a *k-coloring* of vertices of G is an assignment of k colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number k such that G admits a k-coloring. Obviously $\chi(G) \geq \text{clique}(G)$ for general graph G

Lemma 4.4. . *Let p be a prime number, and let $k \in N$. Then we have $\chi(T(\Gamma(Z_{p^k}))) = clique(T(\Gamma(Z_{p^k})))$.*

Proof. Here $Z(R)$ is an ideal of R of order p^{k-1}. Hence $T(\Gamma(Z_{p^k})) = K_{p^{k-1}} \cup (p - \frac{1}{2})K_{p^{k-1},p^{k-1}}$. If p is odd prime and $T(\Gamma(Z_{p^k})) = 2K_{2^{k-1}}$. In both cases $\chi(T(\Gamma(Z_{p^k}))) = clique(T(\Gamma(Z_{p^k})))$. □

Suppose that G is a graph with p vertex ($p > 1$), and $G \setminus F$ is connected for every set $F \subseteq E$ of fewer than l edge. Then G is called l- edge-connected. The greatest integer l, such that G is l-edge-connected is the edge-connectivity $\lambda(G)$ of G. It is clear that $\lambda(G) \leq \delta(G)$ where $\delta(G)$ is the minimum degree of vertices of G. It is proved that if $diam(G) = 2$, then $\lambda(G) = \delta(G)$.

Theorem 4.5. *Let R be a finite commutative ring which is not local. Thus $\lambda(T(\Gamma(R))) = \mid Z(R) \mid$, if $\lambda(T(\Gamma(R))) = 2 \notin Z(R)$ and $\mid Z(R) \mid -1$ if $2 \in Z(R)$*

Proof. Since R is finite and not local hence $R = R_1 \times R_2 \times \cdots \times R_t$ where $t \geq 2$. So $diam(T(\Gamma(R))) = 2$ (see [3]) and hence $\lambda(T(\Gamma(R))) = \delta(T(\Gamma(R)))$. Now the result follows from Lemma 3.1 □

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References

1. S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, The total graph and regular graph of a commutative ring, *J. Pure Appl. Algebra*, **213**(12), (2009), 2224-2228.
2. S. Akbari, H. R. Maimani and S. Yassemi, When a zero-divisor graph is planar or a complete r-partite graph, *J. Algebra*, **270**, (2003), 169-180.
3. D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra*, **320**, (2008), 2706-2719,.
4. D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217**, (1999), 434-447.
5. D. D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *J. Algebra*, **159**, (1993), 500-514.
6. I. Beck, Coloring of commutative rings, *J. Algebra*, **116**(1), (1988), 208-226.
7. M. Axtell, J. Coykendall and J. Stickles, Zero-divisor graphs of polynomial and power series over commutative rings, *Comm. Algebra*, **33**, (2005), 2043-2050.
8. M. Axtell and J. Stickles, Zero-divisor graphs of idealizations, *J. Pure Appl. Algebra*, **204**, (2006), 235-243.
9. B. Corbas, G.D. Williams, Ring of order p^5. II. Local rings, *J. Algebra*, **231**(2), (2000), 691-704.
10. F. DeMeyer and K. Schneider, Automorphisms and zero-divisor graphs of commutative rings, *Int. J. Commut. Rings*, **1**, (2002), 93-106.
11. T. G. Lucas, The diameter of a zero divisor graph, *J. Algebra*, **301**, (2006), 174-193.
12. H. R. Maimani, C. Wickham, S. Yassemi, Rings whose total graphs have genus at most one, *Rocky Mountain J. Math.*, to appear.
13. H.R. Maimani, M. R. Pournaki, S. Yassemi, Weakly perfect graphs arising from rings, *Glasgow Math. J.*, **52**, (2010), 417-425.
14. S. B. Mulay, Cycles and symmetries of zero-divisors, *Comm. Algebra*, **30**, (2002), 3533-3558.