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# Horadam Polynomials Estimates for $\lambda$-Pseudo-Starlike Bi-Univalent Functions

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Abstract. In the present investigation, we use the Horadam Polynomials to establish upper bounds for the second and third coefficients of functions belongs to a new subclass of analytic and $\lambda$-pseudo-starlike bi-univalent functions defined in the open unit disk $U$. Also, we discuss Fekete-Szegö problem for functions belongs to this subclass.

**Keywords:** Bi-univalent functions, Coefficient bounds, Horadam polynomials, $\lambda$-Pseudo-starlike functions, Fekete-Szegö problem, Subordination.

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## 1. Introduction

Let $\mathcal{A}$ stand for the family of functions $f$ which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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Further, let $S$ indicate the subclass of $\mathcal{A}$ consisting of the form (1.1) which are univalent in $U$. It is well known (see [3]) that every function $f \in S$ has an inverse $f^{-1}$ defined by $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ stands for the class of bi-univalent functions in $U$ given by (1.1). In fact, Srivastava et al. [13] has apparently revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [4], Goyal and Goswami [5], Caglar et al. [2] and others (see, for example [9, 10, 11, 12, 14]).

A function $f \in S$ is said to be $\lambda$-pseudo-starlike function of order $\beta$ $0 \leq \beta < 1$ in $U$, if it satisfies (see [1]): This class of functions was denoted by $\mathcal{L}_\lambda(\beta)$. It is observed that for $\lambda = 1$, we have the class of starlike functions.

With a view to recalling the principal of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $U$. We say that the function $f$ is said to be subordinate to $g$, if there exists a Schwarz function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known that, if the function $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [6]):

$$h_n(r) = pr h_{n-1}(r) + q h_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N}),$$

with

$$h_1(r) = a \quad \textit{and} \quad h_2(r) = br, \tag{1.3}$$

for some real constant $a, b, p$ and $q$.

The generating function of the Horadam polynomials $h_n(r)$ (see [7]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \tag{1.4}$$

## 2. Main Results

We begin this section by defining the subclass $\mathcal{L}_\Sigma(\delta, \lambda, r)$ as follows:

**Definition 2.1.** For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$ if it satisfies the subordinations:

$$1 + \frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^\lambda}{f(z)} - 1\right) \prec \Pi(r, z) + 1 - a$$

and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)\prec \Pi(r,w)+1-a,$$

where $a$ is real constant and the function $g=f^{-1}$ is given by (1.2).

**Theorem 2.2.** *For $\delta\in\mathbb{C}\setminus\{0\}$, $\lambda\geq 1$ and $r\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the class $\mathcal{L}_{\Sigma}(\delta,\lambda,r)$. Then*

$$|a_2|\leq \frac{\sqrt{2}\delta\,|br|\sqrt{|br|}}{\sqrt{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p\left(\lambda-1\right)^2\right]br^2-2qa\left(\lambda-1\right)^2\right|}}$$

*and*

$$|a_3|\leq \frac{\delta\,|br|}{3\lambda-1}+\frac{\delta^2b^2r^2}{\left(\lambda-1\right)^2}.$$

*Proof.* Let $f\in\mathcal{L}_{\Sigma}(\delta,\lambda,r)$. Then there are two analytic functions $u,v:U\longrightarrow U$ given by

$$u(z)=u_1z+u_2z^2+u_3z^3+\cdots\quad(z\in U)\tag{2.1}$$

and

$$v(w)=v_1w+v_2w^2+v_3w^3+\cdots\quad(w\in U),\tag{2.2}$$

with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, $z,w\in U$ such that

$$1+\frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right)=\Pi(r,u(z))+1-a$$

and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)=\Pi(r,v(w))+1-a.$$

Or, equivalently

$$1+\frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right)=1+h_1(r)+h_2(r)u(z)+h_3(r)u^2(z)+\cdots\tag{2.3}$$

and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)=1+h_1(r)+h_2(r)v(w)+h_3(r)v^2(w)+\cdots.\tag{2.4}$$

Combining (2.1), (2.2), (2.3) and (2.4) yields

$$1+\frac{1}{\delta}\left(\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right)=1+h_2(r)u_1z+\left[h_2(r)u_2+h_3(r)u_1^2\right]z^2+\cdots\tag{2.5}$$

and

$$1+\frac{1}{\delta}\left(\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right)=1+h_2(r)v_1w+\left[h_2(r)v_2+h_3(r)v_1^2\right]w^2+\cdots.\tag{2.6}$$

It is quite well-known that if $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in U$, then

$$|u_i| \leq 1 \quad and \quad |v_i| \leq 1 \; forall \; i \in \mathbb{N}. \tag{2.7}$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$\frac{2\lambda - 1}{\delta} a_2 = h_2(r) u_1, \tag{2.8}$$

$$\frac{3\lambda - 1}{\delta} a_3 + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r) u_2 + h_3(r) u_1^2, \tag{2.9}$$

$$-\frac{2\lambda - 1}{\delta} a_2 = h_2(r) v_1 \tag{2.10}$$

and

$$\frac{3\lambda - 1}{\delta} (2a_2^2 - a_3) + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r) v_2 + h_3(r) v_1^2. \tag{2.11}$$

It follows from (2.8) and (2.10) that

$$u_1 = -v_1 \tag{2.12}$$

and

$$\frac{2\left(\lambda - 1\right)^2}{\delta^2} a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \tag{2.13}$$

If we add (2.9) to (2.11), we find that

$$\frac{4\lambda^2 - 5\lambda + 1}{\delta} a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \tag{2.14}$$

Substituting the value of $u_1^2 + v_1^2$ from (2.13) in the right hand side of (2.14), we deduce that

$$a_2^2 = \frac{\delta^2 h_2^3(r)(u_2 + v_2)}{\delta\left(4\lambda^2 - 5\lambda + 1\right) h_2^2(r) - 2h_3(r)\left(\lambda - 1\right)^2}. \tag{2.15}$$

Further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2| \leq \frac{\sqrt{2}\delta\,|br|\sqrt{|br|}}{\sqrt{\left|\left[\delta\left(4\lambda^2 - 5\lambda + 1\right) b - 2p\left(\lambda - 1\right)^2\right] br^2 - 2qa\left(\lambda - 1\right)^2\right|}}.$$

Next, if we subtract (2.11) from (2.9), we can easily see that

$$\frac{2(3\lambda - 1)}{\delta}(a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2). \tag{2.16}$$

In view of (2.12) and (2.13), we get from (2.16)

$$a_3 = \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + \frac{\delta^2 h_2^2(r)(u_1^2 + v_1^2)}{2\left(\lambda - 1\right)^2}.$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{\delta\,|br|}{3\lambda - 1} + \frac{\delta^2 b^2 r^2}{\left(\lambda - 1\right)^2}.$$

This completes the proof of Theorem 2.2 □

In the next theorem, we discuss the Fekete-Szegö problem for the subclass $\mathcal{L}_\Sigma(\delta, \lambda, r)$.

**Theorem 2.3.** *For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r, \mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\delta|br|}{3\lambda-1}; \\ \qquad for \ |\mu - 1| \leq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}, \\ \\ \frac{2\delta^2|br|^3|\mu-1|}{\left|\left[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}; \\ \\ \qquad for \ |\mu - 1| \geq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}. \end{cases}$$

*Proof.* It follows from (2.15) and (2.16) that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + (1 - \mu)a_2^2 \\ &= \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + \frac{\delta^2 h_2^3(r)(u_2 + v_2)(1 - \mu)}{\delta\left(4\lambda^2 - 5\lambda + 1\right) h_2^2(r) - 2h_3(r)\left(\lambda - 1\right)^2} \\ &= h_2(r)\left[\left(\psi(\mu, r) + \frac{\delta}{2(3\lambda - 1)}\right) u_2 + \left(\psi(\mu, r) - \frac{\delta}{2(3\lambda - 1)}\right) v_2\right], \end{aligned}$$

where

$$\psi(\mu, r) = \frac{\delta^2 h_2^2(r)(1 - \mu)}{\delta\left(4\lambda^2 - 5\lambda + 1\right) h_2^2(r) - 2h_3(r)\left(\lambda - 1\right)^2}.$$

According to (1.3), we find that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\delta|br|}{3\lambda-1}, & 0 \leq |\psi(\mu, r)| \leq \frac{\delta}{2(3\lambda-1)}, \\ \\ 2\left|br\right|\left|\psi(\mu, r)\right|, & |\psi(\mu, r)| \geq \frac{\delta}{2(3\lambda-1)}. \end{cases}$$

After some computations, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\delta|br|}{3\lambda-1}; \\ \qquad for \ |\mu - 1| \leq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}, \\ \\ \frac{2\delta^2|br|^3|\mu-1|}{\left|\left[\delta(4\lambda^2-5\lambda+1)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}; \\ \\ \qquad for \ |\mu - 1| \geq \frac{\left|\left[\delta\left(4\lambda^2-5\lambda+1\right)b-2p(\lambda-1)^2\right]br^2-2qa(\lambda-1)^2\right|}{2\delta b^2r^2(3\lambda-1)}. \end{cases}$$

□

Putting $\mu = 1$ in Theorem 2.3, we obtain the following result:

**Corollary 2.4.** *For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{\delta\,|br|}{3\lambda - 1}.$$

*Remark* 2.5. If we put $\lambda = 1$ in our Theorems, we have the result for well-known class $S^*_\Sigma(r)$ of bi-starlike functions which was considered recently by Srivastava et al. [8].

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## References

1. K. O. Babalola, On $\lambda$-Pseudo-Starlike Functions, *J. Class. Anal.*, **3**(2), (2013), 137-147.
2. M. Caglar, H. Orhan, N. Yagmur, Coefficient Bounds for New Subclasses of Bi-Univalent Functions, *Filomat*, **27**, (2013), 11651171.
3. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
4. B. A. Frasin, M. K. Aouf, New Subclasses of Bi-Univalent Functions, *Appl. Math. Lett.*, **24**, (2011), 1569-1573.
5. S. P. Goyal, P. Goswami, Estimate for Initial Maclaurin Coefficients of Bi-Univalent Functions for a Class Defined by Fractional Derivatives, *J. Egyptian Math. Soc.*, **20**, (2012), 179-182.
6. A. F. Horadam, J. M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, **23**, (1985), 7-20.
7. T. Hörcum, E. G. Kocer, On some Properties of Horadam polynomials, *Int. Math. Forum*, **4**, (2009), 1243-1252.
8. H. M. Srivastava, S. Altinkaya, S. Yalin, Certain Subclasses of Bi-Univalent Functions Associated with the Horadam Polynomials, *Iran. J. Sci. Technol. Trans. Sci.*, (2018), 1-7.
9. H. M. Srivastava, D. Bansal, Coefficient Estimates for a Subclass of Analytic and Bi-Univalent Functions, *J. Egyptian Math. Soc.*, **23**, (2015), 242-246.
10. H. M. Srivastava, S. S. Eker, R. M. Ali, Coefficient Bounds for a Certain Class of Analytic and Bi-Univalent Functions, *Filomat*, **29**, (2015), 1839-1845.
11. H. M. Srivastava, S. S. Eker, S. G. Hamidi, J. M. Jahangiri, Faber Polynomial Coefficient Estimates for Bi-Univalent Functions Defined by the Tremblay Fractional Derivative Operator, *Bull. Iranian Math. Soc.*, **44**(1), (2018), 149-157.
12. H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient Estimates for some General Subclasses of Analytic and Bi-Univalent Functions, *Africa Math.*, **28**, (2017), 693-706.
13. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain Subclasses of Analytic and Bi-Univalent Functions, *Appl. Math. Lett.*, **23**, (2010), 1188-1192.
14. A. K. Wanas, A. H. Majeed, Certain New Subclasses of Analytic and m-Fold Symmetric Bi-Univalent Functions, *Applied Mathematics E-Notes*, *18*, (2018), 178-188.