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Second Hankel Determinant for a Certain Subclass of λ-Pseudo-Starlike Bi-Univalent Functions

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Abstract. In this paper, we discuss the upper bounds for the second Hankel determinant $H_2(2)$ of a new subclass of λ-pseudo-starlike bi-univalent functions defined in the open unit disk U.

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1. Introduction

Let $\mathcal{A}$ stand for the family of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. Let S indicate the class of all functions in $\mathcal{A}$ which are univalent in U.

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A function $f \in S$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ in U if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in U)$$

and convex of order $\alpha (0 \leq \alpha < 1)$ in U if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in U)$$

As usual, we denote these subclasses of S by $S^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively.

The Koebe one-quarter theorem [12] ensures that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus, every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ be the class of bi-univalent functions in U given by (1.1).

For a brief history and interesting examples of functions that are in (or are not in) the class Σ, together with various other properties of the bi-univalent functions class Σ, one can refer the work of Srivastava et al. [22] and the references stated therein. Recently, many authors introduced various subclasses of the bi-univalent function class Σ and investigated non sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) (see [2, 4, 5, 8, 14, 20, 23, 24]). The problem of finding the coefficient estimates on $|a_n|\,(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

One of the significant tools in the theory of univalent functions is Hankel determinant which are utility, for example, in showing that a function of bounded characteristic in U, i.e., a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [9]. Also the Hankel determinant plays an important role in the study of singularities. Noonan and Thomas [19] defined the q^{th} Hankel determinant of $f \in \mathcal{A}$ for $n \geq 0$ and $q \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

The Hankel determinants

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

are well known as Fekete-Szegö and second Hankel determinant functionals, respectively. Fekete and Szegö [13] consider the further generalized functional $a_3 - \mu a_2^2$, where μ is real number. Recently, several authors established upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions (see [1, 3, 10, 16, 17, 18]). On the other hand, Zaprawa [25, 26] extended the study of the Fekete-Szegö problem for some classes of bi-univalent functions. Very recently, the upper bounds of $H_2(2)$ for some classes were discussed by Deniz et al. [11] (see also [6]).

Babalola [7] defined the class $\mathcal{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as follows:

Definition 1.1. Let $f \in \mathcal{A}$. Suppose that $0 \leq \beta < 1$ and $\lambda \geq 1$. Then $f \in \mathcal{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β in U if and only if

$$Re\left\{\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right\} > \beta, \quad (z \in U).$$

In particular, Babalola [7] showed all λ-pseudo-starlike functions are Bazilevic of type $1 - \frac{1}{\lambda}$ and order $\beta^{\frac{1}{\lambda}}$ and are univalent in U.

In this work, we introduce a new subclass of λ-pseudo-starlike bi-univalent functions and seek an upper bound to the functional $H_2(2)$ for functions in this subclass.

To derive the desired bounds in our study, we shall require the following lemmas.

Lemma 1.2. [21] *If the function $p \in \mathcal{P}$ is given by the series $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$, then the sharp estimate $|p_k| \leq 2$ $(k = 1, 2, 3, \cdots)$ holds.*

Lemma 1.3. [15] *If the function $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z,$$

fore some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2. Main Results

We begin this section by defining the function class $\mathcal{N}_\Sigma(\delta, \lambda, \gamma)$ as follows:

Definition 2.1. A function $f \in \Sigma$ is said to be in the class $\mathcal{N}_\Sigma(\delta, \lambda, \gamma)$ $(\delta \in C \setminus \{0\}, \lambda \geq 1, 0 \leq \gamma < 1)$ if it satisfies the conditions:

$$Re\left\{1 + \frac{1}{\delta}\left[\frac{z\left(f'(z)\right)^\lambda}{f(z)} - 1\right]\right\} > \gamma \quad (z \in U)$$

and

$$Re\left\{1+\frac{1}{\delta}\left[\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right]\right\}>\gamma \quad (w\in U),$$

where $g=f^{-1}$ is given by (1.2).

Theorem 2.2. *Let f of the form (1.1) be in the class $\mathcal{N}_{\Sigma}(\delta,\lambda,\gamma)$. Then*

$$\left|a_2a_4-a_3^2\right|\leq\begin{cases}\frac{4\delta^2(1-\gamma)^2}{3(4\lambda-1)(2\lambda-1)^3}\left(4\delta^2\lambda(2\lambda+1)\left(1-\gamma\right)^2+3\left(2\lambda-1\right)^2\right),\\ \gamma\in\left[0,1-\frac{3(2\lambda-1)\left((4\lambda-1)+\sqrt{(4\lambda-1)^2-\frac{16}{3}\lambda(2\lambda+1)\left[(4\lambda-1)(2\lambda-1)-2(3\lambda-1)^2\right]}\right)}{8\delta\lambda(2\lambda+1)(3\lambda-1)}\right]\\ \frac{\delta^2(1-\gamma)^2}{(4\lambda-1)(2\lambda-1)}\,\frac{\psi_1+\psi_2}{\psi_3+\psi_4},\\ \gamma\in\left(1-\frac{(2\lambda-1)\left(3(4\lambda-1)+\sqrt{9(4\lambda-1)^2+96\lambda(2\lambda+1)(3\lambda-1)^2}\right)}{16\delta\lambda(2\lambda+1)(3\lambda-1)},1\right),\end{cases}$$

where

$$\psi_1=\delta^2(4\lambda-1)\left(1-\gamma\right)^2\left[16\lambda(2\lambda+1)(2\lambda-1)-3(4\lambda-1)\right],$$

$$\psi_2=3\left(2\lambda-1\right)^2\left[4(4\lambda-1)(2\lambda-1)-9\left(3\lambda-1\right)^2\right] - 18\delta(4\lambda-1)(3\lambda-1)(2\lambda-1)(1-\gamma),$$

$$\psi_3=4\delta^2\lambda(2\lambda+1)\left(3\lambda-1\right)^2\left(1-\gamma\right)^2-3\delta(4\lambda-1)(3\lambda-1)(2\lambda-1)(1-\gamma),$$

and

$$\psi_4=3(4\lambda-1)\left(2\lambda-1\right)^3-6\left(3\lambda-1\right)^2\left(2\lambda-1\right)^2.$$

Proof. Let $f\in\mathcal{N}_{\Sigma}(\delta,\lambda,\gamma)$. Then there exists $p,q\in\mathcal{P}$ such that

$$1+\frac{1}{\delta}\left[\frac{z\left(f'(z)\right)^{\lambda}}{f(z)}-1\right]=\gamma+(1-\gamma)p(z) \tag{2.1}$$

and

$$1+\frac{1}{\delta}\left[\frac{w\left(f'(w)\right)^{\lambda}}{f(w)}-1\right]=\gamma+(1-\gamma)q(w), \tag{2.2}$$

where $g=f^{-1}$ and p, q have the following series representations:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots.$$

By equating the coefficients in (2.1) and (2.2), we have

$$\frac{2\lambda-1}{\delta}a_2=(1-\gamma)p_1, \tag{2.3}$$

$$\frac{1}{\delta}\left[(3\lambda-1)a_3+\left(2\lambda(\lambda-2)+1\right)a_2^2\right]=(1-\gamma)p_2, \tag{2.4}$$

$$\frac{1}{\delta}\left[(4\lambda-1)a_4+(6\lambda^2-11\lambda+2)a_2a_3+\left(\frac{2}{3}\lambda(\lambda-2)(2\lambda-5)-1\right)a_2^3\right]$$
$$=(1-\gamma)p_3, \tag{2.5}$$

$$-\frac{2\lambda-1}{\delta}a_2=(1-\gamma)q_1, \tag{2.6}$$

$$\frac{1}{\delta}\left[(3\lambda-1)\left(2a_2^2-a_3\right)+\left(2\lambda(\lambda-2)+1\right)a_2^2\right]=(1-\gamma)q_2 \tag{2.7}$$

and

$$-\frac{1}{\delta}\left[(4\lambda-1)(5a_2^3-5a_2a_3+a_4)+(6\lambda^2-11\lambda+2)(2a_2^2-a_3)a_2\right.$$
$$\left.+\left(\frac{2}{3}\lambda(\lambda-2)(2\lambda-5)-1\right)a_2^3\right]=(1-\gamma)q_3, \tag{2.8}$$

In view of (2.3) and (2.6), it easy to see that

$$p_1=-q_1 \tag{2.9}$$

and

$$a_2=\frac{\delta(1-\gamma)}{2\lambda-1}p_1. \tag{2.10}$$

By subtracting (2.4) from (2.7) and using (2.10), we get

$$a_3=\frac{\delta^2\left(1-\gamma\right)^2}{\left(2\lambda-1\right)^2}p_1^2+\frac{\delta(1-\gamma)}{2(3\lambda-1)}(p_2-q_2). \tag{2.11}$$

Also, subtracting (2.5) from (2.8), further computations using (2.10) and (2.11) lead to

$$a_4=\frac{\left(-\frac{8}{3}\lambda^3+\frac{26}{3}\lambda-2\right)\delta^3\left(1-\gamma\right)^3}{2(4\lambda-1)\left(2\lambda-1\right)^3}p_1^3+\frac{5\delta^2\left(1-\gamma\right)^2}{4(3\lambda-1)(2\lambda-1)}p_1(p_2-q_2)$$
$$+\frac{\delta(1-\gamma)}{2(4\lambda-1)}(p_3-q_3). \tag{2.12}$$

Thus, using (2.10), (2.11) and (2.12), we deduce that

$$\left|a_2a_4-a_3^2\right|=\left|\frac{-\lambda(2\lambda+1)\delta^4\left(1-\gamma\right)^4}{3(4\lambda-1)\left(2\lambda-1\right)^3}p_1^4+\frac{\delta^3\left(1-\gamma\right)^3}{4(3\lambda-1)\left(2\lambda-1\right)^2}p_1^2(p_2-q_2)\right.$$
$$\left.+\frac{\delta^2\left(1-\gamma\right)^2}{2(4\lambda-1)(2\lambda-1)}p_1(p_3-q_3)-\frac{\delta^2\left(1-\gamma\right)^2}{4\left(3\lambda-1\right)^2}\left(p_2-q_2\right)^2\right|. \tag{2.13}$$

According to Lemma 1.3 and (2.9), we write

$$p_2-q_2=\frac{\left(4-p_1^2\right)(x-y)}{2} \tag{2.14}$$

and

$$p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1\left(4-p_1^2\right)(x+y)}{2} - \frac{p_1\left(4-p_1^2\right)\left(x^2+y^2\right)}{4} + \frac{\left(4-p_1^2\right)\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right]}{2}, \tag{2.15}$$

fore some x, y, z and w with $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$ and $|w| \leq 1$.

Substituting the calculated values from (2.14) and (2.15) in the right hand side of (2.13), it follows that

$$\begin{aligned}
\left|a_2a_4 - a_3^2\right| = &\left|\frac{-\lambda(2\lambda+1)\delta^4(1-\gamma)^4 p_1^4}{3(4\lambda-1)(2\lambda-1)^3} + \frac{\delta^3(1-\gamma)^3 p_1^2\left(4-p_1^2\right)(x-y)}{8(3\lambda-1)(2\lambda-1)^2}\right.\\
&+\frac{\delta^2(1-\gamma)^2 p_1^4}{4(4\lambda-1)(2\lambda-1)} + \frac{\delta^2(1-\gamma)^2 p_1^2\left(4-p_1^2\right)(x+y)}{4(4\lambda-1)(2\lambda-1)}\\
&-\frac{\delta^2(1-\gamma)^2 p_1^2\left(4-p_1^2\right)\left(x^2+y^2\right)}{8(4\lambda-1)(2\lambda-1)}\\
&+\frac{\delta^2(1-\gamma)^2 p_1\left(4-p_1^2\right)\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right]}{4(4\lambda-1)(2\lambda-1)}\\
&\left.-\frac{\delta^2(1-\gamma)^2\left(4-p_1^2\right)^2(x-y)^2}{16(3\lambda-1)^2}\right|\\
\leq &\frac{\lambda(2\lambda+1)\delta^4(1-\gamma)^4 p_1^4}{3(4\lambda-1)(2\lambda-1)^3} + \frac{\delta^2(1-\gamma)^2 p_1^4}{4(4\lambda-1)(2\lambda-1)}\\
&+\frac{\delta^2(1-\gamma)^2 p_1\left(4-p_1^2\right)}{2(4\lambda-1)(2\lambda-1)}\\
&+\left[\frac{\delta^3(1-\gamma)^3 p_1^2\left(4-p_1^2\right)}{8(3\lambda-1)(2\lambda-1)^2} + \frac{\delta^2(1-\gamma)^2 p_1^2\left(4-p_1^2\right)}{4(4\lambda-1)(2\lambda-1)}\right](|x|+|y|)\\
&+\left[\frac{\delta^2(1-\gamma)^2 p_1^2\left(4-p_1^2\right)}{8(4\lambda-1)(2\lambda-1)} - \frac{\delta^2(1-\gamma)^2 p_1\left(4-p_1^2\right)}{4(4\lambda-1)(2\lambda-1)}\right]\left(|x|^2+|y|^2\right)\\
&+\frac{\delta^2(1-\gamma)^2\left(4-p_1^2\right)^2}{16(3\lambda-1)^2}(|x|+|y|)^2.
\end{aligned}$$

Since the function p is in the class $\mathcal{P}$, so $|p_1| \leq 2$. Choosing $p_1 = p$, we can assume without loss of generality that $p \in [0,2]$. Then, for $\eta_1 = |x| \leq 1$ and $\eta_2 = |y| \leq 1$, we have

$$\left|a_2a_4 - a_3^2\right| \leq L_1 + L_2(\eta_1+\eta_2) + L_3(\eta_1^2+\eta_2^2) + L_4(\eta_1+\eta_2)^2 = M(\eta_1,\eta_2),$$

where

$$L_1 = L_1(p) = \frac{\delta^2 (1-\gamma)^2}{4(4\lambda-1)(2\lambda-1)} \left[\left(\frac{4\lambda(2\lambda+1)\delta^2 (1-\gamma)^2}{3(2\lambda-1)^2} + 1 \right) p^4 - 2p^3 + 8p \right] \geq 0,$$

$$L_2 = L_2(p) = \frac{\delta^2 (1-\gamma)^2 p^2 (4-p^2)}{8(2\lambda-1)} \left(\frac{\delta(1-\gamma)}{(3\lambda-1)(2\lambda-1)} + \frac{2}{4\lambda-1} \right) \geq 0,$$

$$L_3 = L_3(p) = \frac{\delta^2 (1-\gamma)^2 p (4-p^2)(p-2)}{8(4\lambda-1)(2\lambda-1)} \leq 0,$$

$$L_4 = L_4(p) = \frac{\delta^2 (1-\gamma)^2 (4-p^2)^2}{16(3\lambda-1)^2} \geq 0.$$

We next maximize the function $M(\eta_1, \eta_2)$ on the closed square $[0,1] \times [0,1]$. We must investigate the maximum of $M(\eta_1, \eta_2)$ according to $p \in (0,2)$, $p = 0$ and $p = 2$ taking into account the sign of $M_{\eta_1\eta_1} \cdot M_{\eta_2\eta_2} - (M_{\eta_1\eta_2})^2$.
Since $L_3 < 0$ and $L_3 + 2L_4 > 0$ for $p \in (0,2)$, we conclude that

$$M_{\eta_1\eta_1} \cdot M_{\eta_2\eta_2} - (M_{\eta_1\eta_2})^2 < 0.$$

Therefore the function M cannot have a local maximum in the interior of the closed square $[0,1] \times [0,1]$.
Now, we investigate the maximum M on the boundary of the closed square $[0,1] \times [0,1]$.

When $\eta_1 = 0$ and $0 \leq \eta_2 \leq 1$ (similarly $\eta_2 = 0$ and $0 \leq \eta_1 \leq 1$), we have

$$M(0, \eta_2) = E(\eta_2) = L_1 + L_2\eta_2 + (L_3 + L_4)\eta_2^2.$$

(1) The case $L_3 + L_4 \geq 0$:
In this case for $0 < \eta_2 < 1$ and any fixed p with $0 \leq p \leq 2$, it is easily observed that $E'(\eta_2) = L_2 + 2(L_3 + L_4)\eta_2 > 0$. Therefore $E(\eta_2)$ is increasing function and hence, for fixed $p \in [0,2)$, the maximum of $E(\eta_2)$ occurs at $\eta_2 = 1$ and

$$\max E(\eta_2) = E(1) = L_1 + L_2 + L_3 + L_4.$$

(2) The case $L_3 + L_4 < 0$:
Since $L_2 + 2(L_3 + L_4) \geq 0$ for $0 < \eta_2 < 1$ and any fixed p with $0 \leq p \leq 2$, it is easily observed that $L_2 + 2(L_3 + L_4) < L_2 + 2(L_3 + L_4)\eta_2 < L_2$. Therefore $E'(\eta_2) > 0$ and hence, for fixed $p \in [0,2)$, the maximum of $E(\eta_2)$ occurs at $\eta_2 = 1$.

Also, for $p = 2$, we find

$$M(\eta_1, \eta_2) = \frac{4\delta^2 (1-\gamma)^2}{(4\lambda-1)(2\lambda-1)} \left(\frac{4\lambda(2\lambda+1)\delta^2 (1-\gamma)^2}{3(2\lambda-1)^2} + 1 \right). \tag{2.16}$$

Taking into account the value (2.16) and the cases 1 and 2, for $0 \leq \eta_2 \leq 1$ and any fixed p with $0 \leq p \leq 2$,

$$\max E(\eta_2) = E(1) = L_1 + L_2 + L_3 + L_4.$$

When $\eta_1 = 1$ and $0 \leq \eta_2 \leq 1$ (similarly $\eta_2 = 1$ and $0 \leq \eta_1 \leq 1$), we have

$$M(1, \eta_2) = K(\eta_2) = L_1 + L_2 + L_3 + L_4 + (L_2 + 2L_4)\eta_2 + (L_3 + L_4)\eta_2^2.$$

Similarly to the above cases of $L_3 + L_4$, we find that

$$\max K(\eta_2) = K(1) = L_1 + 2L_2 + 2L_3 + 4L_4.$$

Since $E(1) \leq K(1)$ for $p \in [0, 2]$, $\max M(\eta_1, \eta_2) = M(1, 1)$ on the boundary of the closed square $[0, 1] \times [0, 1]$. Hence, the maximum of M occurs at $\eta_1 = 1$ and $\eta_2 = 1$ in the closed square $[0, 1] \times [0, 1]$.
Assume that $T : [0, 2] \longrightarrow \mathbb{R}$ be defined by

$$T(p) = \max M(\eta_1, \eta_2) = M(1, 1) = L_1 + 2L_2 + 2L_3 + 4L_4. \tag{2.17}$$

Now, substituting the values of L_1, L_2, L_3 and L_4 in (2.17), we conclude that

$$\begin{aligned} T(p) = {} & \frac{\delta^2 (1-\gamma)^2}{4(4\lambda - 1)(3\lambda - 1)^2 (2\lambda - 1)^3} \left\{ \left[\frac{4}{3}\delta^2 \lambda(2\lambda + 1)(3\lambda - 1)^2 (1-\gamma)^2 \right.\right. \\ & - \delta(4\lambda - 1)(3\lambda - 1)(2\lambda - 1)(1-\gamma) - 2(3\lambda - 1)^2 (2\lambda - 1)^2 \\ & \left. +(4\lambda - 1)(2\lambda - 1)^3 \right] p^4 + 4(2\lambda - 1)\left[\delta(4\lambda - 1)(3\lambda - 1)(1-\gamma) \right. \\ & \left.\left. +3(3\lambda - 1)^2 (2\lambda - 1) - 2(4\lambda - 1)(2\lambda - 1)^2 \right] p^2 + 16(4\lambda - 1)(2\lambda - 1)^3 \right\}. \end{aligned}$$

Suppose that $T(p)$ has a maximum value in an interior of $p \in [0, 2]$, then

$$\begin{aligned} T'(p) = {} & \frac{\delta^2 (1-\gamma)^2}{(4\lambda - 1)(3\lambda - 1)^2 (2\lambda - 1)^3} \left\{ \left[\frac{4}{3}\delta^2 \lambda(2\lambda + 1)(3\lambda - 1)^2 (1-\gamma)^2 \right.\right. \\ & - \delta(4\lambda - 1)(3\lambda - 1)(2\lambda - 1)(1-\gamma) - 2(3\lambda - 1)^2 (2\lambda - 1)^2 \\ & \left. +(4\lambda - 1)(2\lambda - 1)^3 \right] p^3 + 2(2\lambda - 1)\left[\delta(4\lambda - 1)(3\lambda - 1)(1-\gamma) \right. \\ & \left.\left. +3(3\lambda - 1)^2 (2\lambda - 1) - 2(4\lambda - 1)(2\lambda - 1)^2 \right] p \right\}. \end{aligned}$$

After some calculations, we consider the following cases:
Case1: Assume that

$$\begin{aligned} & \frac{4}{3}\delta^2 \lambda(2\lambda + 1)(3\lambda - 1)^2 (1-\gamma)^2 - \delta(4\lambda - 1)(3\lambda - 1)(2\lambda - 1)(1-\gamma) \\ & + (2\lambda - 1)^2 \left[(4\lambda - 1)(2\lambda - 1) - 2(3\lambda - 1)^2 \right] \geq 0. \end{aligned}$$

Thus

$$\gamma \in \left[0, 1 - \frac{3(2\lambda - 1)\left((4\lambda - 1) + \sqrt{(4\lambda - 1)^2 - \frac{16}{3}\lambda(2\lambda + 1)\left[(4\lambda - 1)(2\lambda - 1) - 2(3\lambda - 1)^2 \right]} \right)}{8\delta\lambda(2\lambda + 1)(3\lambda - 1)} \right]$$

and so $T'(p) > 0$ for $p \in (0, 2)$. Since T is an increasing function in the interval $(0, 2)$, hence the maximum point of T must be on the boundary of $p \in [0, 2]$.

Then, we have

$$\max_{0\le p\le 2} T(p)=T(2)=\frac{4\delta^2\left(1-\gamma\right)^2}{3(4\lambda-1)\left(2\lambda-1\right)^3}\left(4\delta^2\lambda(2\lambda+1)\left(1-\gamma\right)^2+3\left(2\lambda-1\right)^2\right).$$

Case2: Assume that

$$\frac{4}{3}\delta^2\lambda(2\lambda+1)\left(3\lambda-1\right)^2\left(1-\gamma\right)^2-\delta(4\lambda-1)(3\lambda-1)(2\lambda-1)(1-\gamma)$$
$$+\left(2\lambda-1\right)^2\left[(4\lambda-1)(2\lambda-1)-2\left(3\lambda-1\right)^2\right]<0,$$

that is,

$$\gamma\in\left(1-\frac{3(2\lambda-1)\left((4\lambda-1)+\sqrt{\left(4\lambda-1\right)^2-\frac{16}{3}\lambda(2\lambda+1)\left[(4\lambda-1)(2\lambda-1)-2\left(3\lambda-1\right)^2\right]}\right)}{8\delta\lambda(2\lambda+1)(3\lambda-1)},1\right).$$

Therefore $T'(p)=0$ implies the real critical point $p_{0_1}=0$ or

$$p_{0_2}=\sqrt{\frac{-6(2\lambda-1)\left[\delta(4\lambda-1)(3\lambda-1)(1-\gamma)+3\left(3\lambda-1\right)^2(2\lambda-1)-2(4\lambda-1)\left(2\lambda-1\right)^2\right]}{4\delta^2\lambda(2\lambda+1)\left(3\lambda-1\right)^2\left(1-\gamma\right)^2-3\delta(4\lambda-1)(3\lambda-1)(2\lambda-1)(1-\gamma)-6\left(3\lambda-1\right)^2\left(2\lambda-1\right)^2+3(4\lambda-1)\left(2\lambda-1\right)^3}}.$$

When

$$\gamma\in\left(1-\frac{3(2\lambda-1)\left((4\lambda-1)+\sqrt{\left(4\lambda-1\right)^2-\frac{16}{3}\lambda(2\lambda+1)\left[(4\lambda-1)(2\lambda-1)-2\left(3\lambda-1\right)^2\right]}\right)}{8\delta\lambda(2\lambda+1)(3\lambda-1)},\right.$$
$$\left.1-\frac{(2\lambda-1)\left(3(4\lambda-1)+\sqrt{9\left(4\lambda-1\right)^2+96\lambda(2\lambda+1)\left(3\lambda-1\right)^2}\right)}{16\delta\lambda(2\lambda+1)(3\lambda-1)}\right],$$

we observe that $p_{0_2}\ge 2$, that is, p_{0_2} is out of the interval $(0,2)$. Hence the maximum value of $T(p)$ occurs at $p_{0_1}=0$ or p_{0_2} which contradicts our assumption of having the maximum value at the interior point of $p\in[0,2]$. Since T is an increasing function in the interval $(0,2)$, so the maximum point of T must be on the boundary of $p\in[0,2]$, that is, $p=2$. Therefore, we obtain

$$\max_{0\le p\le 2} T(p)=T(2)=\frac{4\delta^2\left(1-\gamma\right)^2}{3(4\lambda-1)\left(2\lambda-1\right)^3}\left(4\delta^2\lambda(2\lambda+1)\left(1-\gamma\right)^2+3\left(2\lambda-1\right)^2\right).$$

When

$$\gamma\in\left(1-\frac{(2\lambda-1)\left(3(4\lambda-1)+\sqrt{9\left(4\lambda-1\right)^2+96\lambda(2\lambda+1)\left(3\lambda-1\right)^2}\right)}{16\delta\lambda(2\lambda+1)(3\lambda-1)},1\right),$$

we observe that that $p_{0_2} < 2$, that is, p_{0_2} is an interior of the interval $[0, 2]$. Since $T''(p_{0_2}) < 0$, so the maximum value of $T(p)$ occurs at $p = p_{0_2}$. Therefore, we obtain

$$\max_{0 \le p \le 2} T(p) = T(p_{0_2}) = \frac{\delta^2 (1-\gamma)^2}{(4\lambda-1)(2\lambda-1)} \frac{\psi_1 + \psi_2}{\psi_3 + \psi_4}.$$

This completes the proof of our Theorem. □

Remark 2.3. Taking $\delta = \lambda = 1$ in Theorem 2.2, we obtain the second Hankel determinant for the well-known class $S_\Sigma^*(\gamma)$ as in [11].

Corollary 2.4. [11] *Let f given by (1.1) be in the class $S_\Sigma^*(\gamma)$, $0 \le \gamma < 1$. Then*

$$\left|a_2 a_4 - a_3^2\right| \le \begin{cases} \frac{4}{3}(1-\gamma)^2 \left(4\gamma^2 - 8\gamma + 5\right), & \gamma \in \left[0, \frac{29-\sqrt{137}}{32}\right] \\ (1-\gamma)^2 \left(\frac{13\gamma^2 - 14\gamma - 7}{16\gamma^2 - 26\gamma + 5}\right), & \gamma \in \left(\frac{29-\sqrt{137}}{32}, 1\right) \end{cases}.$$

Remark 2.5. For $\gamma = 0$ and $\delta = \lambda = 1$, Theorem 2.2 readily yields the following coefficient estimates for bi-starlike functions.

Corollary 2.6. [11] *Let f given by (1.1) be in the class S_Σ^*. Then*

$$\left|a_2 a_4 - a_3^2\right| \le \frac{20}{3}.$$

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