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Iterative Process for an $\alpha-$ Nonexpansive Mapping and a Mapping Satisfying Condition(C) in a Convex Metric Space

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Abstract. We construct one−step iterative process for an $\alpha-$ nonexpansive mapping and a mapping satisfying condition (C) in the framework of a convex metric space. We study $\triangle-$convergence and strong convergence of the iterative process to the common fixed point of the mappings. Our results are new and valid in hyperbolic spaces, $CAT(0)$ spaces, Banach spaces and Hilbert spaces, simultaneously.

Keywords: Convex metric space, One−step iterative process, $\alpha-$Nonexpansive mapping, Condition(C), Common fixed point, Convergence.

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1. Introduction

Let K be a nonempty subset of a metric space X and $T : K \to K$ be a mapping. Denote by $F(T)$, the set of fixed points of T. We say that T is:

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for $x, y \in K$

(2) quasi-nonexpansive if $d(Tx, y) \leq d(x, y)$ for $x \in K, y \in F(T)$

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(3) said to satisfy condition(C) if $\frac{1}{2}d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq d(x,y)$ for $x,y \in K$

(4) $\alpha-$nonexpansive if $d(Tx,Ty)^2 \leq \alpha d(Tx,y)^2 + \alpha d(x,Ty)^2 + (1-2\alpha)\, d(x,y)^2$ for $x,y \in K$ and for some $\alpha < 1$.

In 2008, Suzuki[15] proposed the condition (C) and showed that it is weaker than nonexpansiveness but stronger than quasi-nonexpansiveness.

Aoyama and Kohsaka[2] introduced the class of $\alpha-$nonexpansive mappings in Banach spaces and concluded the following facts:

(i) $0-$ nonexpansive mapping is nonexpansive

(ii) $\frac{1}{2}-$nonexpansive mapping is nonspreading

(iii) $\frac{1}{3}-$nonexpansive mapping is hybrid mapping

The following example shows that $\alpha-$nonexpansive mapping and a mapping satisfying condition(C) are two different generalizations of nonexpansive mappings with a common fixed point.

Example 1.1. Take $X = \mathbb{R}, K = [0,3]$ and $T, S : K \to K$ by

$$Tx = \begin{cases} 0 & \text{if } \ x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$$

and

$$Sx = \begin{cases} 0 & \text{if } \ x \neq 3 \\ 2 & \text{if } x = 3. \end{cases}$$

Here we see that T satisfies condition(C) and S is $\frac{1}{4}-$nonexpansive with 0 as their common fixed point. Also, T is not an $\alpha-$nonexpansive and S does not satisfy condition(C). Moreover, both S and T are discontinuous mappings and therefore are not nonexpansive.

Takahashi and Tamura[17] studied the weak convergence of two nonexpansive mappings T_1 and T_2 in the setting of Banach space using the scheme

$$x_1 \in K, \ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T_1 \{\beta_n T_2 x_n + (1-\beta_n)\, x_n\} \tag{1.1}$$

where $0 < a \leq \alpha_n, \ \beta_n \leq b < 1$.

Dhompongsa et al. [5] used the scheme(1.1) to prove the weak convergence theorem of a nonspreading mapping and a mapping satisfying condition(C) in the framework of Hilbert spaces(see also [3],[10],[13]). Wattanawitoon and Khamlae[18] also considered the scheme(1.1) for proving the convergence theorem for an $\alpha-$nonexpansive mapping and a mapping satisfying the condition(C)

in Hilbert spaces. The proof of their main result depends on the following identity in Hilbert spaces

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle . \tag{1.2}$$

A nonlinear framework for the iterative construction of fixed points of certain classes of nonlinear mappings is a metric space embedded with a convex structure.

Takahashi[16] introduced a convex structure $W : X^2 \times I \to X$ on a metric space X satisfying

$$\text{(W1):} \quad d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in I = [0, 1]$.

A metric space X with a convex structure W is known as a convex metric space and is also denoted by X.

In general, convex structure W is not continuous. However, if the inequality

$$d(W(x, y, \lambda), W(x, z, \lambda)) \leq (1-\lambda)\, d(z, w)$$

holds in the convex metric space X, then it becomes continuous.

Kohlenbach [11] enriched the concept of Takahashi convex metric space as "hyperbolic space" by including the following additional conditions in the definition of a convex metric space.

$$\begin{aligned}
&\text{(W2)}: \quad d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2|\, d(x, y)\\
&\text{(W3)}: \quad W(x, y, \lambda) = W(y, x, 1-\lambda)\\
&\text{(W4)}: \quad d(W(x, z, \lambda), W(y, w, \lambda)) \leq \lambda d(x, y) + (1-\lambda)\, d(z, w)
\end{aligned}$$

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in I$.

A nonempty subset K of X is convex if and only if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

A convex metric space X is uniformly convex [14] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta > 0$ such that $d\left(W(x, y, \frac{1}{2}), u\right) \leq (1-\delta) r < r$, whenever $d(x, u) \leq r, d(y, u) \leq r$ and $d(x, y) \geq r\varepsilon$.

Let $\{x_n\}$ be a bounded sequence in X. We define $r(., \{x_n\})$ on X by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \quad x \in X.$$

The asymptotic radius $r_K(\{x_n\})$ of $\{x_n\}$ with respect to $K \subseteq X$ is defined as

$$r_K(\{x_n\}) = \inf_{x\in K} r(x, \{x_n\})$$

and the asymptotic center $A_K(\{x_n\})$ of $\{x_n\}$ with respect to K is the set

$$A_K(\{x_n\}) = \{y \in K : r(y, \{x_n\}) = r_K(\{x_n\})\}.$$

A sequence $\{x_n\}$ in (X, d) (a) is Fejér monotone with respect to a subset K of X if $d(x_{n+1}, x) \leq d(x_n, x)$ for all $x \in K$ (b) $\triangle-$converges to $x \in X$ if x is

the unique asymptotic center for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim_n x_n = x$.

In this paper, we are interested to approximate common fixed point of an $\alpha-$nonexpansive mapping and a mapping satisfying condition(C) in the convex metric space. Due to lack of the identity(1.2) in the convex metric space, we are unable to approximate common fixed point of the mappings through convex metric version of scheme(1.1). Therefore, we propose a one−step iterative scheme to approximate common fixed point of an $\alpha-$nonexpansive mapping and a mapping satisfying condition(C) in the setting of a convex metric space. Our scheme is as under

$$x_1 \in K,\ x_{n+1} = W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \tag{1.3}$$

where $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ and $\alpha_n + \beta_n < 1$.

When $S = I$ in (1.3), it reduces to Mann iterative scheme [12]

$$x_{n+1} = W\left(Tx_n, x_n, \alpha_n\right).$$

In a normed space setting, (1.3) becomes one−step iterative scheme[19]

$$x_{n+1} = \alpha_n T x_n + \beta_n S x_n + (1 - \alpha_n - \beta_n)\, x_n$$

where $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$ and $\alpha_n + \beta_n < 1$.

Here, we state some results which will be needed in the main section.

Lemma 1.2. [15] *Let T be a self-mapping on a subset K of a metric space X. If T satisfies condition(C), then*

$$d\left(x, Ty\right) \leq 3d\left(Tx, x\right) + d\left(x, y\right)$$

holds for all $x, y \in K$.

Lemma 1.3. [4] *Let K be a nonempty closed subset of a complete metric space (X, d) and $\{x_n\}$ a Fejér monotone sequence with respect to K. Then $\{x_n\}$ converges to some point $p \in K$ if and only if $\lim_{n\to\infty} d(x_n, K) = 0$.*

Lemma 1.4. [6] *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to K.*

Lemma 1.5. [7] *Let X be a uniformly convex metric space with continuous convex structure W. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in X such that $\limsup_{n\longrightarrow\infty} d(u_n, x) \leq r$, $\limsup_{n\longrightarrow\infty} d(v_n, x) \leq r$ and $\lim_{n\longrightarrow\infty} d(W\left(u_n, v_n, a_n\right), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

Lemma 1.6. [8] *Let K be a nonempty, closed and convex subset of a metric space X and T be an $\alpha-$ nonexpansive mapping on K. For any $x, y \in K$, the following assertions hold:*
(i) If $0 \leq \alpha < 1$, then $d(x,Ty)^2 \leq \frac{1+\alpha}{1-\alpha} d(x,Tx)^2 + \frac{2}{1-\alpha}\{\alpha d(x,y) + d(Tx,Ty)\} d(x,Tx) + d(x,y)^2$,
(ii) If $\alpha < 0$, then $d(x,Ty)^2 \leq d(x,Tx)^2 + \frac{2}{1-\alpha}\{d(Tx,Ty) - \alpha d(Tx,y)\} d(x,Tx) + d(x,y)^2$.

From now onwards, for an $\alpha-$*nonexpansive mapping* T on K and S a mapping on K satisfying condition(C), we set $F = F(S) \cap F(T)$.

2. Convergence Theorems

We start with the following lemma.

Lemma 2.1. *Let K be a subset of a metric space X. Let $T : K \to K$ be an $\alpha-$nonexpansive sel-mapping for some $\alpha < 1$ and S a self-mapping on K satisfying condition(C) with $F \neq \phi$. Then T and S are quasi-nonexpansive and F is closed.*

Proof. Let $x \in K$ and $z \in F$. Consider

$$\begin{aligned} d(Tx,z)^2 &= d(Tx,Tz)^2 \\ &\leq \alpha d(Tx,z)^2 + \alpha d(x,Tz)^2 + (1-2\alpha) d(x,z)^2 \\ &= \alpha d(Tx,z)^2 + \alpha d(x,z)^2 + (1-2\alpha) d(x,z)^2 \\ &= \alpha d(Tx,z)^2 + (1-\alpha) d(x,z)^2. \end{aligned}$$

That is,

$$d(Tx,z) \leq d(x,z)$$

and

$$\frac{1}{2} d(z,Sz) = 0$$

gives that

$$d(Sx,z) \leq d(x,z).$$

Therefore, both S and T are quasi-nonexpansive.
Let $\{z_n\}$ be a sequence in F such that $z_n \to z$. We claim that $z \in F$.
Since

$$d(Sz,z_n) \leq d(z,z_n) \to 0 \text{ and } d(Tz,z_n) \leq d(z,z_n) \to 0,$$

therefore $Sz = z = Tz$, proving that F is closed. □

Lemma 2.2. *Let K be a nonempty, closed and convex subset of a convex metric space X. Let T be an $\alpha-$nonexpansive self-mapping on K and S a self-mapping on K satisfying condition(C) such that $F \neq \phi$. Then for the sequence $\{x_n\}$ in (1.3), we have the followings:*
(i) $\{x_n\}$ is a Fejér monotone sequence with respect to F

(ii) $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in F$
(iii) $\lim_{n\to\infty} d(x_n, F)$ *exists.*

Proof. With the help of (W1) and the scheme(1.3), for any $p \in F$, we have

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(Tx_n, p) + (1-\alpha_n) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n} d(Sx_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p)\right] \\ &\quad + \alpha_n d(x_n, p) \\ &\leq (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p)\right] \\ &\quad + \alpha_n d(x_n, p) \\ &= \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1 - \alpha_n - \beta_n) d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

That is,

$$d(x_{n+1}, p) \leq d(x_n, p). \tag{2.1}$$

Immediately, (2.1) gives that (i): $\{x_n\}$ *is a Fejér monotone sequence with respect to* F *and* (ii): $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in F$.
Finally $\inf_{p\in F} d(x_{n+1}, p) \leq \inf_{p\in F} d(x_n, p)$ provides that (iii): $\lim_{n\to\infty} d(x_n, F)$ exists. □

Lemma 2.3. *Let* K *be a nonempty, closed and convex subset of a complete and uniformly convex metric space* X *with continuous convex structure* W. *Let* T *be an* $\alpha-$*nonexpansive self-mapping on* K *and* S *a self-mapping on* K *satisfying condition(C) such that* $F \neq \phi$. *If* $\{z_n\}$ *is any bounded sequence in* K *with* $A(\{z_n\}) = \{z\}$ *and*

$$\lim_{n\to\infty} d(z_n, Sz_n) = 0 = \lim_{n\to\infty} d(z_n, Tz_n),$$

then $z \in F$.

Proof. Let $A(\{z_n\}) = \{z\}$. We show that $z \in F$.

By Lemma 1.2, we have

$$d(z_n, Sz) \leq 3d(z_n, Sz_n) + d(z_n, z)$$

which further implies that

$$\begin{aligned} \limsup_{n\to\infty} d(z_n, Sz) &\leq 3 \limsup_{n\to\infty} d(z_n, Sz_n) + \limsup_{n\to\infty} d(z_n, z) \\ &= \limsup_{n\to\infty} d(z_n, z). \end{aligned}$$

By the uniqueness of asymptotic centers (Lemma 1.4), we have that $Sz = z$.

Next, we show that $Tz = z$.

Since $\{z_n\}$ is bounded and $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, therefore $\{Tz_n\}$ is also bounded. Set $M = \sup_{n\geq 1}\{d(z_n, z), d(Tz_n, z), d(Tz_n, Tz)\} < \infty$.

Applying Lemma 1.6 (i)-(ii) for $0 \leq \alpha < 1$ and $\alpha < 0$, respectively, we have that

$$\begin{aligned} d(z_n, Tz)^2 &\leq \frac{1+\alpha}{1-\alpha} d(z_n, Tz_n)^2 + \frac{2}{1-\alpha}(\alpha d(z_n, z) + d(Tz_n, Tz)) d(z_n, Tz_n) \\ &\quad + d(z_n, z)^2 \\ &\leq \frac{1+\alpha}{1-\alpha} d(z_n, Tz_n)^2 + \frac{2M(1+\alpha)}{1-\alpha} d(z_n, Tz_n) + d(z_n, z)^2 \end{aligned}$$

and

$$\begin{aligned} d(z_n, Tz)^2 &\leq d(z_n, Tz_n)^2 + \frac{2}{1-\alpha}(d(Tz_n, Tz) - \alpha d(Tz_n, z)) d(z_n, Tz_n) \\ &\quad + d(z_n, z)^2 \\ &\leq d(z_n, Tz_n)^2 + 2M d(z_n, Tz_n) + d(z_n, z)^2 \end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides in the above two inequalities and using the fact that $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, we have that

$$\limsup_{n\to\infty} d(z_n, Tz)^2 \leq \limsup_{n\to\infty} d(z_n, z)^2.$$

By the uniqueness of asymptotic centers (Lemma 1.4), $Tz = z$. □

Lemma 2.4. *Let K be a nonempty, closed and convex subset of a uniformly convex metric space X with continuous convex structure W. Let T be an $\alpha-$nonexpansive self-mapping on K and S be a self-mapping on K satisfying condition(C) such that $F \neq \phi$. Then for the sequence $\{x_n\}$ in (1.3), we have*

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n).$$

Proof. It follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F$. Set $\lim_{n\to\infty} d(x_n, p) = c$.

For $c > 0, \lim_{n\to\infty} d(x_{n+1}, p) = c$ can be expressed as

$$\lim_{n\to\infty} d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c. \tag{2.2}$$

As T is an $\alpha-$nonexpansive *and* $p \in F(T)$, therefore

$$\limsup_{n\to\infty} d(Tx_n, p) \leq c. \tag{2.3}$$

Since S satisfies condition(C) and $p \in F(S)$,we have

$$\begin{aligned} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) &\leq \frac{\beta_n}{1-\alpha_n} d(Sx_n, p) \\ &\quad + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

That is,

$$\limsup_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{2.4}$$

In the light of (2.2)-(2.4), we use Lemma 1.5 for the values $x = p, r = c, a_n = \alpha_n, u_n = Tx_n, v_n = W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)$ and get

$$\lim_{n\to\infty} d\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \tag{2.5}$$

With the help of (2.5) and the the inequality

$$\begin{aligned} d(x_{n+1}, Tx_n) &\leq d\left(W\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), Tx_n\right) \\ &\leq (1-\alpha_n) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), Tx_n\right) \\ &\leq (1-b) d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), Tx_n\right), \end{aligned}$$

we get that

$$\lim_{n\to\infty} d(x_{n+1}, Tx_n) = 0. \tag{2.6}$$

By $\liminf_{n\to\infty}$ on both sides in the following inequality

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, Tx_n) + d\left(Tx_n, W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right)\right) \\ &\quad + d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right), \end{aligned}$$

we have

$$c \leq \liminf_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \tag{2.7}$$

The combined effect of (2.4) and (2.7) provides that

$$\lim_{n\to\infty} d\left(W\left(Sx_n, x_n, \frac{\alpha_n}{1-\beta_n}\right), p\right) = c. \tag{2.8}$$

Again by Lemma 1.5 for values $x = p, r = c, a_n = \frac{\alpha_n}{1-\beta_n}, u_n = Sx_n, v_n = x_n$, we get

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{2.9}$$

Now with the help of (2.5), (2.6), (2.9) and the inequality

$$\begin{aligned} d(x_{n+1},x_n) &\leq d(x_{n+1},Tx_n)+d\left(Tx_n,W\left(Sx_n,x_n,\frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad +d\left(W\left(Sx_n,x_n,\frac{\alpha_n}{1-\beta_n}\right),x_n\right) \\ &\leq d(x_{n+1},Tx_n)+d\left(Tx_n,W\left(Sx_n,x_n,\frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad +\left(1-\frac{\alpha_n}{1-\beta_n}\right)d(x_n,Sx_n) \\ &\leq d(x_{n+1},Tx_n)+d\left(Tx_n,W\left(Sx_n,x_n,\frac{\alpha_n}{1-\beta_n}\right)\right) \\ &\quad +\left(\frac{1-2a}{1-b}\right)d(x_n,Sx_n), \end{aligned}$$

we get that

$$\lim_{n\to\infty} d(x_{n+1},x_n)=0. \tag{2.10}$$

Taking $\limsup_{n\to\infty}$ on both sides in the following inequality

$$d(x_n,Tx_n)\leq d\left(x_n,x_{n+1}\right)+d\left(x_{n+1},Tx_n\right)$$

and using (2.6) and (2.10), we get

$$\lim_{n\to\infty} d(x_n,Tx_n)=0.$$

Therefore

$$\lim_{n\to\infty} d(x_n,Sx_n)=0=\lim_{n\to\infty} d(x_n,Tx_n).$$

□

Here is our $\triangle-$convergence theorem.

Theorem 2.5. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let T be an $\alpha-$nonexpansive self-mapping on K, S a self-mapping on K satisfying condition(C) and $\{x_n\}$ given in (1.3). If $F\neq\phi$, then $\triangle-\lim_n x_n=x\in F$.*

Proof. Lemma 2.2 provides that $\{x_n\}$ is bounded and therefore Lemma 1.4 appeals that $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\})=\{x\}$. For any subsequence $\{u_n\}$ of $\{x_n\}$, Lemma 1.4 gives that $A(\{u_n\})=\{u\}$ and Lemma 2.4 provides that

$$\lim_{n\to\infty} d(u_n,Tu_n)=0=\lim_{n\to\infty} d(u_n,Su_n).$$

Then by Lemma 2.3, we conclude that $u \in F$. We claim that $x = u$. If not, then by the uniqueness of asymptotic centres, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction.
Therefore, $A(\{u_n : \{u_n\} \text{ is any subsequence of } \{x_n\}\}) = \{x\}$. This proves that $\triangle - \lim_n x_n = x \in F$. □

A self-mapping $T : K \to K$ is *semi-compact* if for any bounded sequence $\{x_n\}$ in K with $d(x_n, Tx_n) \to 0$, we must have that $\{x_n\}$ has a convergent subsequence in K.

Two self-mappings S and T on K with a nonempty subset F of K are said to satisfy condition (AV) if there exists a nondecreasing function f on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{2}\left[d\left(x, Tx\right) + d\left(x, Sx\right)\right] \geq f(d(x, F)) \text{ for all } x \in K.$$

Using Lemma 2.2 and Lemma 2.4, we obtain the following strong convergence theorems.

Theorem 2.6. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let T be an $\alpha-$nonexpansive self-mapping on K, S a self-mapping on K satisfying condition(C) and $\{x_n\}$ given in (1.3). If $F \neq \phi$ and either S or T is semi-compact, then the sequence $\{x_n\}$ converges strongly to an element of F.*

Proof. Suppose that T is semi-compact. Since $\{x_n\}$ is bounded and $d\left(x_n, Sx_n\right) \to 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in K$ and

$$\lim_{j\to\infty} d(x_{n_j}, Sx_{n_j}) = 0 = \lim_{j\to\infty} d(x_{n_j}, Tx_{n_j}).$$

Using $x = x_{n_j}$ and $y = q$ in Lemma 1.2 and Lemma 1.6, we get that $q \in F$.

Therefore $x_n \to q \in F$ as $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F$ (Lemma 2.2). □

Theorem 2.7. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let T be an $\alpha-$nonexpansive self-mapping on K, S a self-mapping on K satisfying condition(C) and $\{x_n\}$ given in (1.3). If $F \neq \phi$ and S and T satisfy condition (AV), then the sequence $\{x_n\}$ converges strongly to an element of F.*

Proof. By Lemma 2.1, F is closed. Using condition (AV) and Lemma 2.4, we have that $\lim_{n\to\infty} d(x_n, F) = 0$. Finally by Lemma 1.3, $x_n \to p$ for some $p \in F$. □

The followings are corollaries to our Theorems 2.5-2.7 and yet they are new in the literature.

Corollary 2.8. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let T be a nonspreading(or hybrid) self-mapping on K, S a self-mapping on K satisfying condition(C) and $\{x_n\}$ given in (1.3). If $F \neq \phi$, then $\triangle - \lim_n x_n = x \in F$.*

Proof. Choose $\alpha = \frac{1}{2}$ in Theorem 2.5 for a nonspreading mapping ($\alpha = \frac{1}{3}$ in the case of a hybrid mapping) to get the required result. □

Corollary 2.9. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let T be a nonspreading(or hybrid) self-mapping on K, S a mapping on K satisfying condition(C) and $\{x_n\}$ given in (1.3). If either S or T is semi-compact, then the sequence $\{x_n\}$ converges strongly to an element of F.*

Proof. Set $\alpha = \frac{1}{2}$ in Theorem 2.6 for a nonspreading self-mapping ($\alpha = \frac{1}{3}$ in the case of a hybrid self-mapping) to get the required result. □

Corollary 2.10. *Let K be a nonempty, closed and convex subset of a complete and uniformly convex metric space X with continuous convex structure W. Let T be a nonspreading(or hybrid) self-mapping on K, S a mapping on K satisfying condition(C) and $\{x_n\}$ given in (1.3). If S and T satisfy condition (AV), then the sequence $\{x_n\}$ converges strongly to an element of F.*

Proof. Take $\alpha = \frac{1}{2}$ in Theorem 2.7 for a nonspreading self-mapping ($\alpha = \frac{1}{3}$ in the case of a hybrid self-mapping) to get the required result. □

Remark 2.11. Observe that
(i) Hyperbolic spaces, $CAT\,(0)$ spaces, Banach spaces and Hilbert spaces are convex metric spaces, therefore our results also hold in Hyperbolic spaces, $CAT\,(0)$ spaces, Banach spaces and Hilbert spaces, simultaneously.
(ii) Every nonexpansive self-mapping is $\alpha-$nonexpansive and satisfy condition(C) also, therefore our theorems generalize the corresponding ones in [1, 7, 9, 10] etc.
(iii) Results of this paper are analogues of Theorem 3.1-Theorem 3.3 in [18].
(iv) A nonexpansive mapping is always continuous but an $\alpha-$nonexpansive mapping and a mapping satisfying condition(C) may or may not be continuous. Therefore our results also hold for discontinuous mappings.
(v) The approximation of common fixed point of an $\alpha-$ nonexpansive self-mapping$T$ on K and a self-mapping S on K satisfying condition(C) via Ishikawa

iterative scheme: $x_1 \in K, x_{n+1} = W\left(T\left(W\left(Sx_n, x_n, \beta_n\right)\right), x_n, \alpha_n\right)$ requires the extensive use of identity (1.2) (see [18]) while our scheme(1.3) does not. Therefore our scheme is better than Ishikawa iterative scheme. Also our scheme is computationally simpler than Ishikawa iterative scheme.

Remark 2.12. The essentials of hypotheses in our theorems are natural in view of the following observations: Take $X = \mathbb{R}, K = [0,3], T, S : K \to K$ as in Example 1.1.Then $F(S) \cap F(T) = \{0\}$ .If $\alpha_n = \frac{n+1}{3n}$ and $\beta_n = \frac{n+1}{4n}$, then $0 < \alpha_n, \beta_n < 1$.

Open Problem: Can we approximate common fixed point of an $\alpha-$nonexpansive mapping and a mapping satisfying condition(C) via scheme(1.1) under the hypothesis of Theorem 2.5?

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References

1. M. Abbas, S. H. Khan, Some $\triangle-$ Convergence Theorems in CAT (0) Spaces, *Hacet. J. Math. Stat.*, **40**(4), (2011), 563-569.
2. K. Aoyama, F. Kohsaka, Fixed Point Theorem for α-Nonexpansive Mappings in Banach Spaces, *Nonlinear Analysis*, **74**, (2011), 4387-4391.
3. M. Alimohammady , M. Ramazannejad, Z. Bagheric , R. J. Shahkoohid, Common Zero Points of Two Finite Families of Maximal Monotone Operators Via Iteration Methods, *Iranian Journal of Mathematical Sciences and Informatics*, **12**(2), (2017), 73-99
4. H. H. Bauschke, P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer-Verlag, New York, 2011.
5. S. Dhompongsa, W. Inthakon, W. Takahashi, A Weak Convergence Theorem for Common Fixed Points of Some Generalized Nonexpansive Mappings and non-Spreading Mappings in a Hilbert Space, *Optimization: A Journal of Mathematical Programming and Operations Research*, **60**, (2011), 769-779.
6. H. Fukhar-ud-din, Existence and Approximation of Fixed Points in Convex Metric Spaces, *Carpathian J. Math.*, **30**, (2014), 175-185.
7. H. Fukhar-ud-din, One Step Iterative Scheme for a Pair of Nonexpansive Mappings in a Convex Metric Space, *Hacet. J. Math.Stat.*, **44**, (2015), 1023–1031.
8. H. Fukhar-ud-din, A. R. Khan, Approximation of Common Fixed Point of Two Quasi-nonexpansive Mappings in Convex Metric Spaces, *Mediterr. J. Math.*, (2018), 15:77.
9. B. Gunduz, S. Akbulut, Strong and $\triangle-$ Convergence Theorems in Hyperbolic Spaces, *Miskolc Mathematical Notes*, **14**(3), (2013), 915-925.
10. M. A. Khamsi, Approximate Fixed Point Sequences of Nonlinear Semigroups in Metric Spaces, *Canad. Math. Bull.*, **58** (2), (2015), 297-305.
11. U. Kohlenbach, Some Logical Metatheorems with Applications in Functional Analysis, *Trans. Am. Math. Soc.*, **357**(1), (2005), 89-128.
12. W. R. Mann, Mean Value Methods in Iterations, *Proc.Amer. Math. Soc.*, **4**, (1953), 506-510.

13. H. R. Sahebi, A. Razani, An Explicit Viscosity Iterative Algorithm for Finding Fixed Points of Two Noncommutative Nonexpansive Mappings,*Iranian Journal of Mathematical Sciences and Informatics*, **11**(1) (2016), 69-83.
14. T. Shimizu, W. Takahashi, Fixed Points of Multivalued Mappings in Certain Convex Metric Spaces, *Topol. Methods Nonlinear Anal.*, **8**, (1996), 197-203.
15. T. Suzuki, Fixed Point Theorems and Convergence Theorems for Some Generalized Nonexpansive Mappings, *J. Math. Anal. Appl.*, **341**, (2008), 1088-1095.
16. W. Takahashi, A Convexity in Metric Spaces and Nonexpansive Mappings, *Kodai Math. Sem. Rep.*, **22**, (1970), 142-149.
17. W. Takahashi, T. Tamura, Convergence Theorems for a Pair of Nonexpansive Mappings, *J. Convex Anal.*, **5**, (1998), 45-56.
18. K. Wattanawitoon, Y. Khamlae, Weak and Strong Convergence Theorems for an $\alpha-$ Nonexpansive Mapping and a Generalized Nonexpansive Mapping in Hilbert Spaces, *Thai J. Math.*, **11**, (2013), 633-643.
19. Y. Yao, R. Chen, Weak and Strong Convergence of a Modified Mann Iteration for Asymptotically Nonexpansive Mappings, *Nonlinear Funct. Anal. Appl.*, **12**, (2007), 307-315.