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# Graded $r$-Ideals

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Abstract. Let $G$ be a group with identity $e$ and $R$ be a commutative $G$-graded ring with nonzero unity 1. In this article, we introduce the concept of graded $r$-ideals. A proper graded ideal $P$ of a graded ring $R$ is said to be a graded $r$-ideal if whenever $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$. We study and investigate the behavior of graded $r$-ideals to introduce several results. We introduced several characterizations for graded $r$-ideals; we proved that $P$ is a graded $r$-ideal of $R$ if and only if $aP = aR \bigcap P$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Also, $P$ is a graded $r$-ideal of $R$ if and only if $P = (P : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Moreover, $P$ is a graded $r$-ideal of $R$ if and only if whenever $A, B$ are graded ideals of $R$ such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$, then $B \subseteq P$. In this article, we introduce the concept of a $huz$-rings. A graded ring $R$ is said to be a $huz$-ring if every homogeneous element of $R$ is either a zero divisor or a unit. In fact, we proved that $R$ is a $huz$-ring if and only if every graded ideal of $R$ is a graded $r$-ideal. Moreover, assuming that $R$ is a graded domain, we proved that $\{0\}$ is the only graded $r$-ideal of $R$.

**Keywords:** Graded ideals, Graded prime ideals, Graded $r$-ideals.

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## 1. Introduction

Let $G$ be a group with identity $e$. A ring $R$ is said to be a $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of $R_g$ are called homogeneous of degree $g$ and $R_e$ (the identity component of $R$) is a subring of $R$ and $1 \in R_e$. For $x \in R$, $x$ can be written uniquely as $\sum_{g\in G} x_g$ where $x_g$ is the component of $x$ in $R_g$. Also we write $h(R) = \bigcup_{g\in G} R_g$ and $supp(R, G) = \{g \in G : R_g \neq 0\}$. For more details, see [3].

Let $R$ be a $G$-graded ring and $P$ be an ideal of $R$. Then $P$ is called a $G$-graded ideal if $P = \bigoplus_{g\in G}(P \cap R_g)$, i.e., if $x \in P$ and $x = \sum_{g\in G} x_g$, then $x_g \in P$ for all $g \in G$. An ideal of a graded ring need not be graded; see the following example.

Example 1.1. Consider $R = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then $R$ is $G$-graded by $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Now, $P = \langle 1+i\rangle$ is an ideal of $R$ with $1+i \in P$. If $P$ is a graded ideal, then $1 \in P$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbf{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, $P$ is not graded ideal.

Throughout this article, $R$ will be a commutative ring with nonzero unity 1. For $a \in R$, we define $Ann(a) = \{r \in R : ra = 0\}$. An element $a \in R$ is said to be a regular element if $Ann(a) = \{0\}$, the set of all regular elements of $R$ is denoted by $r(R)$. If $A$ is a subset of $R$ and $P$ is an ideal of $R$, then we define $(P : A) = \{r \in R : rA \subseteq P\}$.

The notion of $r$-ideals was introduced and studied by Rostam Mohamadian in [2]. A proper ideal $P$ of $R$ is said to be an $r$-ideal (resp. $pr$-ideal) if whenever $a, b \in R$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbf{N}$).

In this article, we introduce the concept of graded $r$-ideals. A proper graded ideal $P$ of a graded ring $R$ is said to be a graded $r$-ideal (resp. graded $pr$-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbf{N}$). We study and investigate the behavior of graded $r$-ideals to introduce several results.

We introduce several characterizations for graded $r$-ideals; we prove that $P$ is a graded $r$-ideal of $R$ if and only if $aP = aR \bigcap P$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Also, $P$ is a graded $r$-ideal of $R$ if and only if $P = (P : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$. Moreover, $P$ is a graded $r$-ideal of $R$ if and only if whenever $A, B$ are graded ideals of $R$ such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$, then $B \subseteq P$.

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A proper graded ideal of a graded ring $R$ is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in P$, then either $a \in P$ or $b \in P$ ([1]). We prove that the intersection of two graded $r$-ideals is a graded $r$-ideal. On the other hand, if the intersection of two non-comparable graded prime ideals is a graded $r$-ideal, then both ideals are graded $r$-ideals. Moreover, we prove that every graded maximal $r$-ideal is graded prime.

If $P$ is a graded $r$-ideal of $R$, we prove that $P_e$ is an $r$-ideal of $R_e$ and $(P : a)$ is a graded $r$-ideal of $R$ for all $a \in h(R) - P$. Also, we prove that if $R$ is $\mathbb{Z}$-graded, then $P$ is a graded $pr$-ideal of $R$ if and only if $\sqrt{P}$ is a graded $r$-ideal of $R$.

In this article, we introduce the concept of $huz$-rings. A graded ring $R$ is said to be a $huz$-ring if every homogeneous element of $R$ is either a zero divisor or a unit. In fact, we prove that $R$ is a $huz$-ring if and only if every graded ideal of $R$ is a graded $r$-ideal. Moreover, assuming that $R$ is a graded domain, we prove that $\{0\}$ is the only graded $r$-ideal of $R$.

## 2. Graded $r$-Ideals

In this section, we introduce and study the concept of graded $r$-ideals.

**Definition 2.1.** Let $R$ be a $G$-graded ring. A proper graded ideal $P$ of $R$ is said to be a graded $r$-ideal (resp. graded $pr$-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbf{N}$).

Note that for a graded ideal $P$ of a $G$-graded ring $R$, $P_g = P \bigcap R_g$ for $g \in G$.

**Theorem 2.2.** *Let $R$ be a $G$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded $r$-ideal if and only if $aP = aR \bigcap P$ for every $a \in h(R)$ with $Ann(a) = \{0\}$.*

*Proof.* ($\Rightarrow$) Let $a \in h(R)$ such that $Ann(a) = \{0\}$. Then $aP \subseteq P$ and $aP \subseteq aR$, i.e., $aP \subseteq aR \bigcap P$. Let $x \in aR \bigcap P$. Then $x = az \in P$ for some $z \in R$. Since $R$ is $G$-graded, $z = \sum_{g \in G} z_g$ and then $x = \sum_{g \in G} az_g \in P$ and since $P$ is a graded ideal, $az_g \in P$ for all $g \in G$. Since $P$ is a graded $r$-ideal, $z_g \in P$ for all $g \in G$ and then $z = \sum_{g \in G} z_g \in P$ which implies that $x = az \in aP$. Hence, $aP = aR \bigcap P$. ($\Leftarrow$) Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $ab \in aR \bigcap P = aP$ and then $ab = ax$ for some $x \in P$ which implies that $a(b - x) = 0$. Since $Ann(a) = \{0\}$, $b - x = 0$, i.e., $b = x \in P$. Hence, $P$ is a graded $r$-ideal. □

**Theorem 2.3.** *Let $R$ be a $G$-graded ring and $P$ be a graded ideal of $R$. If $aP_g = aR_h \bigcap P_g$ for all $g, h \in G$ and for all $a \in h(R)$ with $Ann(a) = \{0\}$, then $P$ is a graded $r$-ideal of $R$.*

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*Proof.* Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then there exist $g, h \in G$ such that $a \in R_g$ and $b \in R_h$ and then $ab \in R_gR_h \bigcap P \subseteq R_{gh} \bigcap P = P_{gh}$. Now, $ab \in aR_h \bigcap P_{gh} = aP_{gh}$, i.e., $ab = ay$ for some $y \in P_{gh}$ and then $a(b - y) = 0$. Since $Ann(a) = \{0\}$, $b = y \in P_{gh} \subseteq P$. Hence, $P$ is a graded $r$-ideal of $R$. □

**Theorem 2.4.** *Let $R$ be a $G$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded $r$-ideal if and only if $P = (P : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$.*

*Proof.* Suppose that $P$ is a graded $r$-ideal of $R$. Let $a \in h(R)$ with $Ann(a) = \{0\}$. Clearly, $P \subseteq (P : a)$. Let $y \in (P : a)$. Then $ya \in P$. Since $R$ is $G$-graded, $y = \sum_{g \in G} y_g$ and then $ya = \sum_{g \in G} y_g a \in P$ and since $P$ is graded, $y_g a \in P$ for all $g \in G$. Since $P$ is a graded $r$-ideal, $y_g \in P$ for all $g \in G$ and then $y = \sum_{g \in G} y_g \in P$. Hence, $P = (P : a)$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $b \in (P : a) = P$. Hence, $P$ is a graded $r$-ideal of $R$. □

**Theorem 2.5.** *Let $R$ be a $G$-graded ring and $P$ be a graded ideal of $R$. If $P_g = (P_g :_{R_h} a)$ for all $g, h \in G$ and for all $a \in h(R)$ with $Ann(a) = \{0\}$, then $P$ is a graded $r$-ideal of $R$.*

*Proof.* Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $a \in R_g$ and $b \in R_h$ for some $g, h \in G$ and then $ab \in R_gR_h \bigcap P \subseteq R_{gh} \bigcap P = P_{gh}$, i.e., $b \in (P_{gh} :_{R_h} a) = P_{gh} \subseteq P$. Hence, $P$ is a graded $r$-ideal of $R$. □

**Theorem 2.6.** *Let $R$ be a $G$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded $r$-ideal if and only if whenever $A, B$ are graded ideals of $R$ such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$, then $B \subseteq P$.*

*Proof.* Suppose that $P$ is a graded $r$-ideal of $R$. Let $A, B$ be two graded ideals of $R$ such that $AB \subseteq P$ and $A \bigcap r(h(R)) \neq \phi$. Since $A \bigcap r(h(R)) \neq \phi$, there exists $a \in A \bigcap r(h(R))$. Let $g \in G$ and $b \in B_g$. Then $ab \in AB_g \subseteq AB \subseteq P$. Since $P$ is a graded $r$-ideal, $b \in P$. So, $B_g \subseteq P$ for all $g \in G$ which implies that $B \subseteq P$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $A = \langle a \rangle$ and $B = \langle b \rangle$ are graded ideals of $R$ such that $AB \subseteq P$ and $a \in A \bigcap r(h(R))$. By assumption, $B \subseteq P$ and then $b \in P$. Hence, $P$ is a graded $r$-ideal of $R$. □

**Theorem 2.7.** *If $R$ is a $G$-graded domain, then $\{0\}$ is a unique graded $r$-ideal of $R$.*

*Proof.* Let $P$ be a nonzero proper graded ideal of $R$. Then there exists $0 \neq a = \sum_{g \in G} a_g \in P$ and then $a_g \in P$ for all $g \in G$ since $P$ is graded. Since $R$ is

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a domain, $Ann(a_g) = \{0\}$ with $1.a_g \in P$. If $P$ is a graded $r$-ideal, then $1 \in P$ which is a contradiction. Hence, $\{0\}$ is the only graded $r$-ideal of $R$. □

**Lemma 2.8.** *If $R$ is a $G$-graded ring, then $R_e$ contains all homogeneous idempotent elements of $R$.*

*Proof.* Let $0 \neq x \in h(R)$ be an idempotent. Then $x \in R_g$ for some $g \in G$ and then $x = x^2 \in R_g \bigcap R_{g^2}$. Since $0 \neq x \in R_g \bigcap R_{g^2}$, $g^2 = g(\in G)$ which implies that $g = e$. Hence, $x \in R_e$. □

**Theorem 2.9.** *Let $R$ be a $G$-graded ring. Suppose that $\{x_i : i \in \Gamma\}$ is a set of homogeneous idempotent elements in $R_e$. Then $P = \sum_{i\in\Gamma} R_e x_i$ is an $r$-ideal of $R_e$.*

*Proof.* Let $a, b \in R_e$ such that $ab \in P$ and $Ann(a) = \{0\}$. Let $z = \prod_{k=1}^{n}(1 - x_{i_k})$ where $ab = \sum_{j=1}^{n} r_j x_{i_j}$ for some $r_1.....r_n \in R_e$. Then $abz = 0$. Since $Ann(a) = \{0\}$, $bz = 0$. On the other hand, there exists $r \in P$ such that $z = 1 - r$ and then $b(1 - r) = 0$ which implies that $b = br \in P$. Hence, $P$ is an $r$-ideal of $R_e$. □

The next lemma is well known and clear; so we omit the proof.

**Lemma 2.10.** *If $P_1$ and $P_2$ are graded ideals of a graded ring $R$, then $P_1 \bigcap P_2$ is a graded ideal of $R$.*

**Theorem 2.11.** *Let $R$ be a $G$-graded ring. If $P_1$ and $P_2$ are graded $r$-ideals of $R$, then $P_1 \bigcap P_2$ is a graded $r$-ideal of $R$.*

*Proof.* By Lemma 2.10, $P_1 \bigcap P_2$ is a graded ideal of $R$. Let $a, b \in h(R)$ such that $ab \in P_1 \bigcap P_2$ and $Ann(a) = \{0\}$. Then $ab \in P_1$. Since $P_1$ is a graded $r$-ideal, $b \in P_1$. Similarly, $b \in P_2$ and hence $b \in P_1 \bigcap P_2$. Therefore, $P_1 \bigcap P_2$ is a graded $r$-ideal of $R$. □

**Theorem 2.12.** *Let $R$ be a $G$-graded ring and $P_1, P_2$ be graded prime ideals of $R$ which are not comparable. If $P_1 \bigcap P_2$ is a graded $r$-ideal of $R$, then $P_1$ and $P_2$ are graded $r$-ideals of $R$.*

*Proof.* Let $a, b \in h(R)$ such that $ab \in P_1$ and $Ann(a) = \{0\}$. Suppose that $y \in P_2 - P_1$. Then $aby \in P_1 \bigcap P_2$. Since $P_1 \bigcap P_2$ is graded $r$-ideal, $by \in P_1 \bigcap P_2$ and then $by \in P_1$. Since $P_1$ is graded prime and $y \notin P_1$, $b \in P_1$. Hence, $P_1$ is a graded $r$-ideal of $R$. Similarly, $P_2$ is a graded $r$-ideal of $R$. □

If $P$ is a graded ideal of a $G$-graded ring $R$, then $\sqrt{P}$ need not to be a graded ideal of $R$; see ([4], Exercises 17 and 13 on pp. 127-128). We introduce the following.

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**Lemma 2.13.** *If $P$ is a graded ideal of a $\mathbb{Z}$-graded ring $R$, then $\sqrt{P}$ is a graded ideal of $R$.*

*Proof.* Clearly, $\sqrt{P}$ is an ideal of $R$. Let $x \in \sqrt{P}$ and write $x = \sum_{i=1}^{t} x_i$ where $x_i \in R_{n_i}$ and $n_1 < n_2 < ..... < n_t$. Then $x^k \in P$ for some positive integer $k$. Of course, $x^k = x_1^k +$ (higher terms) and as $P$ is graded, we should have that $x_1^k \in P$. Thus, $x_1 \in \sqrt{P}$ which implies that $x - x_1 \in \sqrt{P}$. Now, induct on the number of homogeneous components to conclude that $x_i \in \sqrt{P}$ for all $1 \leq i \leq t$. Hence, $\sqrt{P}$ is a graded ideal of $R$. □

**Theorem 2.14.** *Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded pr-ideal of $R$ if and only if $\sqrt{P}$ is a graded r-ideal of $R$.*

*Proof.* Suppose that $P$ is a graded *pr*-ideal of $R$. By Lemma 2.13, $\sqrt{P}$ is a graded ideal of $R$. Let $a, b \in h(R)$ such that $ab \in \sqrt{P}$ and $Ann(a) = \{0\}$. Then $a^n b^n = (ab)^n \in P$ for some $n \in \mathbf{N}$. Since $a, b \in h(R)$, there exist $g, h \in G$ such that $a \in R_g$ and $b \in R_h$ and then $a^n \in R_{g^n}$ and $b^n \in R_{h^n}$ which implies that $a^n, b^n \in h(R)$ such that $a^n b^n \in P$. Clearly, $Ann(a^n) = \{0\}$ and since $P$ is a graded *pr*-ideal, $b^{nm} = (b^n)^m \in P$ for some $m \in \mathbf{N}$ which implies that $b \in \sqrt{P}$. Hence, $\sqrt{P}$ is a graded *r*-ideal of $R$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $ab \in \sqrt{P}$ and since $\sqrt{P}$ is a graded *r*-ideal, $b \in \sqrt{P}$ which implies that $b^n \in P$ for some $n \in \mathbf{N}$. Hence, $P$ is a graded *pr*-ideal of $R$. □

Using Theorem 2.14 and Theorem 2.2, we have the next corollary.

**Corollary 2.15.** *Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded pr-ideal if and only if $a\sqrt{P} = aR \bigcap \sqrt{P}$ for every $a \in h(R)$ with $Ann(a) = \{0\}$.*

Using Theorem 2.14 and Theorem 2.4, we have the next corollary.

**Corollary 2.16.** *Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is graded pr-ideal if and only if $\sqrt{P} = (\sqrt{P} : a)$ for all $a \in h(R)$ with $Ann(a) = \{0\}$.*

**Theorem 2.17.** *If $P$ is a graded r-ideal of a $G$-graded ring $R$, then $(P : a)$ is a graded r-ideal of $R$ for all $a \in h(R) - P$.*

*Proof.* Let $a \in h(R) - P$. Clearly, $(P : a)$ is an ideal of $R$. Let $x \in (P : a)$. Then $x \in R$ such that $xa \in P$. Since $R$ is graded, $x = \sum_{g \in G} x_g$ where $x_g \in R_g$. Since $a \in h(R)$, $a \in R_h$ for some $h \in G$ and then $x_g a \in R_g R_h \subseteq R_{gh}$, i.e., $x_g a \in h(R)$ for all $g \in G$. Now, $xa = \sum_{g \in G} x_g a \in P$. Since $P$ is a graded, $x_g a \in P$ for all $g \in G$, i.e., $x_g \in (P : a)$ for all $g \in G$. Hence, $(P : a)$ is a graded ideal of $R$.

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Let $b, c \in h(R)$ such that $bc \in (P : a)$ and $Ann(b) = \{0\}$. Then $bca \in P$. Since $P$ is a graded $r$-ideal, $ca \in P$ which implies that $c \in (P : a)$. Therefore, $(P : a)$ is a graded $r$-ideal of $R$. □

**Theorem 2.18.** *Every graded maximal $r$-ideal of a graded ring $R$ is graded prime.*

*Proof.* Let $P$ be a graded maximal $r$-ideal of $R$. Suppose that $a, b \in h(R)$ such that $ab \in P$ and $a \notin P$. Then by Theorem 2.17, $(P : a)$ is a graded $r$-ideal of $R$. Clearly, $P \subseteq (P : a)$ and $b \in (P : a)$. By maximality of $P$, $P = (P : a)$ and then $b \in P$. Hence, $P$ is a graded prime ideal of $R$. □

**Definition 2.19.** A graded ring $R$ is said to be an *huz*-ring if every homogeneous element of $R$ is either a zero divisor or a unit.

The next theorem gives an example on *huz*-rings.

**Theorem 2.20.** *Every graded finite ring is an huz-ring.*

*Proof.* Let $R$ be a $G$-graded finite ring. Assume that $a \in h(R)$. Then $a \in R_g$ for some $g \in G$. Define $\phi : R_{g^{-1}} \to R_e$ by $\phi(x) = ax$. If $\phi$ is injective, then since $R$ is finite, $\phi$ is surjective and as $1 \in R_e$, $1 = ax$ for some $x \in R_{g^{-1}}$ and then $a$ is a unit. Suppose that $\phi$ is not injective. Then there exist $x, y \in R_{g^{-1}}$ with $x \neq y$ such that $ax = ay$. But then $a(x - y) = 0$ and $x - y \neq 0$, so $a$ is a zero divisor. □

If we drop the finite condition in Theorem 2.20, then the result is not true in general. See the following example.

Example 2.21. Let $G = \mathbb{Z}$. Then clearly, the semigroup ring $R[X; \mathbb{Z}]$ is a $\mathbb{Z}$-graded ring. If $R$ is a field, then $R[X; \mathbb{Z}]$ is a *huz*-ring; and if $R = \mathbb{Z}$, then $R[X; \mathbb{Z}]$ is not a *huz*-ring.

Finally, we prove that a graded ring $R$ is an *huz*-ring if and only if every proper graded ideal of $R$ is a graded $r$-ideal.

**Theorem 2.22.** *A graded ring $R$ is a huz-ring if and only if every proper graded ideal of $R$ is a graded $r$-ideal.*

*Proof.* Suppose that $R$ is an *huz*-ring. Let $P$ be a proper graded ideal of $R$. Assume that $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Since $Ann(a) = \{0\}$, $a$ is not zero divisor and since $R$ is *huz*, $a$ is a unit and then $b = a^{-1}(ab) \in P$. Hence, $P$ is a graded $r$-ideal of $R$. Conversely, let $a \in h(R)$ such that $a$ is not a zero divisor. Then $Ann(a) = \{0\}$. Suppose that $P = \langle a \rangle$. If $P$ is proper, then $P$ is a graded $r$-ideal of $R$ by assumption. Let $b \in h(R)$. Then $ab \in P$ and then $b \in P$ since $P$ is a graded $r$-ideal. So, $h(R) \subseteq P$. Since $1 \in R_e \subseteq h(R)$, $1 \in P$ which is a contradiction. So, $P = R$, then $1 \in P$ and then $1 = xa$ for some $x \in R$ which implies that $a$ is a unit and hence $R$ is an *huz*-ring. □

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