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Some Generalizations of Locally Closed Sets

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ABSTRACT. Arenas et al. [1] introduced the notion of λ -closed sets as a generalization of locally closed sets. In this paper, we introduce the notions of λ -locally closed sets, Λ_{λ} -closed sets and λg -closed sets and obtain some decompositions of closed sets and continuity in topological spaces.

Keywords: λ -Open set, λ -Locally closed set, Λ_{λ} -Closed set, λg -Closed set, Decompositions of continuity.

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1. Introduction and Preliminaries

The study of locally closed sets was introduced by Bourbaki [3] in 1966 then the authors Ganster and Reilly [6] have studied it extensively. A subset A of a topological space X is called locally closed if $A = U \cap F$, where U is open and F is closed. It is interesting that a locally closed set is a generalization of both open sets and closed sets. The generalization has also been discussed in completely regular Hausdorff spaces [5] and has also been done on algebra with topology in [12] and [2].

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In this paper we consider a new type of sets in the topological space which is called λ -open sets. A set is said to be λ -open if it contains a nonempty open set. This idea is not a new idea. In literature, semi-open sets [7] and α -sets [11] are examples of that type of sets although preopen sets [10] is not an example of it. Because: let \mathbf{R} be the usual real line and Q the rational numbers. Then $\mathrm{Cl}(Q) = \mathbf{R}$ and $Q \subseteq \mathrm{Int}(\mathrm{Cl}(Q)) = \mathbf{R}$ (where 'Cl' and 'Int' denote the closure and interior operators, respectively). But Q does not contain nonempty open set. However Dontechev [4] has introduced an S-space: A topological space X is called an S-space if every subset which contains a non-void open subset is open. But the concept of λ -open sets is different from Dontechev's S-spaces.

Definition 1.1. A subset A of a topological space X is said to be λ -open if A contains a nonempty open set. The complement of a λ -open set is said to be λ -closed.

For a subset A of a topological space X, $\operatorname{Int}_{\lambda}(A)$ and $\operatorname{Cl}_{\lambda}(A)$ are defined as follows:

Definition 1.2. Let X be a topological space and A be a subset of X.

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Int<sub>\(\lambda\)</sub>(A) = \(\begin{aligned} \{U : U \subseteq A, U \text{ is } \lambda\)-open in X\}; \\
\text{Cl}_\(\lambda(A) = \cap \{F : A \subseteq F, F \text{ is } \lambda\)-closed in X\}.
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Lemma 1.3. Let X be a topological space and A, B subsets of X.

- (1) if $A \subseteq B$, then $\operatorname{Int}_{\lambda}(A) \subseteq \operatorname{Int}_{\lambda}(B)$ and $\operatorname{Cl}_{\lambda}(A) \subseteq \operatorname{Cl}_{\lambda}(B)$,
- (2) $X \setminus \operatorname{Int}_{\lambda}(A) = \operatorname{Cl}_{\lambda}(X \setminus A),$
- (3) For any index set Δ , if A_{α} is λ -open (resp. λ -closed), then $\cup \{A_{\alpha} : \alpha \in \Delta\}$ is λ -open (resp. $\cap \{A_{\alpha} : \alpha \in \Delta\}$ is λ -closed),
 - (4) $\operatorname{Int}_{\lambda}(A)$ is λ -open and $\operatorname{Cl}_{\lambda}(A)$ is λ -closed.

Remark 1.4. The finite intersection of λ -open sets need not be λ -open. Let \mathbf{R} be the usual real line, A=(-1,0] and B=[0,1). The A and B are λ -open but $A\cap B=\{0\}$ is not λ -open.

We generalize the locally closed set by using λ -open sets.

2. λ -Locally Closed Sets

Definition 2.1. A subset A of a topological space X is said to be λ -locally closed if $A = U \cap F$, where U is λ -open and F is closed.

Corollary 2.2. Let $f: X \to Y$ be a continuous function. If L is a λ -locally closed subset of Y, then $f^{-1}(L)$ is λ -locally closed in X.

From Definition 1.1 it is obvious that every locally closed set is λ -locally closed. But the converse need not hold in general.

EXAMPLE 2.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\}$. Then C(X) (all closed sets in X) = $\{\emptyset, X, \{b, c, d\}\}$. And λ -open sets are: \emptyset , X, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, c\}$,

 $\{a,d\}, \{a,b,d\}, \{a,c,d\}$. Therefore, $\{d\} = \{a,d\} \cap \{b,c,d\}$ is a λ -locally closed set but it is not a locally closed set in X.

Remark 2.4. A subset A of a topological space X is λ -locally closed if and only if $X \setminus A$ is the union of a λ -closed set and an open set.

Remark 2.5. For a subset of a topological space, the following hold:

- (1) Every λ -open set is λ -locally closed,
- (2) Every closed set is λ -locally closed.

Theorem 2.6. For a subset A of a topological space X, the following are equivalent:

- (1) A is λ -locally closed;
- (2) $A = U \cap Cl(A)$ for some λ -open set U;
- (3) $A \cup (X \setminus Cl(A))$ is λ -open;
- (4) $A \subseteq \operatorname{Int}_{\lambda}[A \cup (X \setminus \operatorname{Cl}(A))];$
- (5) $Cl(A) \setminus A$ is λ -closed.
- *Proof.* (1) \Rightarrow (2): Suppose A is λ -locally closed. Then $A = U \cap F$ where U is λ -open and F is closed. Then $\mathrm{Cl}(A) = \mathrm{Cl}(U \cap F) \subseteq \mathrm{Cl}(F) = F$. Then $A \subseteq U \cap \mathrm{Cl}(A) \subseteq U \cap F = A$ and hence $A = U \cap \mathrm{Cl}(A)$.
- $(2) \Rightarrow (3) \colon X \setminus [A \cup (X \setminus \operatorname{Cl}(A))] = (X \setminus A) \cap \operatorname{Cl}(A) = \operatorname{Cl}(A) \setminus A = \operatorname{Cl}(A) \setminus (U \cap \operatorname{Cl}(A)) = \operatorname{Cl}(A) \setminus U = \operatorname{Cl}(A) \cap (X \setminus U).$ Since U is λ -open, $\operatorname{Cl}(A) \cap (X \setminus U)$ is λ -closed and hence $A \cup (X \setminus \operatorname{Cl}(A))$ is λ -open.
- (3) \Rightarrow (4): Since $A \cup (X \setminus \operatorname{Cl}(A))$ is a λ -open set containing A, it is obvious that $A \subset \operatorname{Int}_{\lambda}[A \cup (X \setminus \operatorname{Cl}(A))]$.
- $(4) \Rightarrow (1)$: $A = A \cap \operatorname{Cl}(A) \subseteq \operatorname{Int}_{\lambda}[A \cup (X \setminus \operatorname{Cl}(A))] \cap \operatorname{Cl}(A) \subseteq [A \cup (X \setminus \operatorname{Cl}(A))] \cap \operatorname{Cl}(A) = A \cap \operatorname{Cl}(A) = A$. Therefore, $A = \operatorname{Int}_{\lambda}[A \cup (X \setminus \operatorname{Cl}(A))] \cap \operatorname{Cl}(A)$ and A is λ -locally closed.
 - $(3) \Leftrightarrow (5)$: It is obvious.

The union of two λ -locally closed sets need not be λ -locally closed.

EXAMPLE 2.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then $C(X) = \{\emptyset, X, \{c, d\}, \{a, b\}\}$ and λ -open sets are: \emptyset , X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. λ -locally closed sets are: \emptyset , X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{c\}$, $\{d\}$, $\{a\}$, $\{b\}$. Therefore, $\{a\}$ and $\{c\}$ are λ -locally closed sets but their union $\{a, c\}$ is not a λ -locally closed set.

3. Λ_{λ} -Closed Sets

Locally closed sets in a topological space are introduced and investigated in [3] and [6]. As a generalization of locally closed sets, Arenas et al. [1] introduced the notion of λ -closed sets in a topological space. In this section, we introduce the notion of Λ_{λ} -closed sets which is a generalization of λ -closed sets. We obtain some characterizations of Λ_{λ} -closed sets and obtain decompositions of closed sets.

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Definition 3.1. Let X be a topological space and A a subset of X. The subset $\Lambda_{\lambda}(A)$ is defined as follows: $\Lambda_{\lambda}(A) = \cap \{U : A \subseteq U, U \text{ is } \lambda\text{-open }\}.$

A subset A is called a Λ_{λ} -set if $A = \Lambda_{\lambda}(A)$. If U is open in Definition 3.1, then a Λ_{λ} -set A is called a Λ -set [9].

Lemma 3.2. For any subsets A and B of a topological space X, the following hold:

- (1) $A \subseteq \Lambda_{\lambda}(A)$,
- (2) If $A \subseteq B$, then $\Lambda_{\lambda}(A) \subseteq \Lambda_{\lambda}(B)$,
- (3) $\Lambda_{\lambda}(\Lambda_{\lambda}(A)) = \Lambda_{\lambda}(A)$,
- (4) $\Lambda_{\lambda}(\cap_{\alpha\in\Delta}A_{\alpha})\subseteq\cap_{\alpha\in\Delta}\Lambda_{\lambda}(A_{\alpha})$ for any index set Δ .

Lemma 3.3. For any subset A of a topological space X, the following hold:

- (1) $\Lambda_{\lambda}(A)$ is a Λ_{λ} -set,
- (2) If A is λ -open, then A is a Λ_{λ} -set,
- (3) If A_{α} is a Λ_{λ} -set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_{\alpha}$ is a Λ_{λ} -set.

Remark 3.4. The converse of Lemma 3.3 (2) need not hold as shown by the following example: Let **R** be the usual real line and $A = \{0\}$. Then A is a Λ_{λ} -set but it is not λ -open. Because $\{0\} \subseteq \Lambda_{\lambda}(\{0\}) \subseteq (-1,0] \cap [0,1) = \{0\}$ and hence $\Lambda_{\lambda}(\{0\}) = \{0\}$. Therefore, $A = \{0\}$ is a Λ_{λ} -set but it is not λ -open.

Definition 3.5. A subset A of a topological space X is said to be Λ_{λ} -closed (resp. λ -closed [1]) if $A = L \cap F$, where L is a Λ_{λ} -set (resp. Λ -set) and F is a closed set.

Lemma 3.6. For a subset of a topological space X, the following properties hold:

- (1) Every λ -locally closed set is Λ_{λ} -closed,
- (2) Every λ -closed set is Λ_{λ} -closed.

Proof. (1) By Lemma 3.3, every λ -open set is a Λ_{λ} -set and (1) holds.

(2) Let U be a Λ -set. Then,

$$U = \cap \{V : U \subseteq V, V \text{ is open }\} \supseteq \cap \{V : U \subset V, V \text{ is } \lambda\text{-open }\} \supseteq U$$

and hence U is a Λ_{λ} -set. Therefore, (2) holds.

Remark 3.7. By Lemma 3.6, we obtain the following diagram.

DIAGRAM I

$$\begin{array}{ccc} \text{locally closed} \Rightarrow \lambda\text{-locally closed} \\ & & & \downarrow \\ & & \lambda\text{-closed} \Rightarrow \Lambda_{\lambda}\text{-closed} \end{array}$$

Theorem 3.8. For a subset A of a topological space X, the following are equivalent:

- (1) A is Λ_{λ} -closed;
- (2) $A = U \cap Cl(A)$ for some Λ_{λ} -set U;
- (3) $A = \Lambda_{\lambda}(A) \cap \operatorname{Cl}(A)$.

Proof. (1) \Rightarrow (2): Let A be a Λ_{λ} -closed set. Then $A = U \cap F$, where U is a Λ_{λ} -set and F is a closed set. Thus, we have $A \subseteq U \cap \operatorname{Cl}(A) \subseteq U \cap \operatorname{Cl}(F) = U \cap F = A$. Therefore, $A = U \cap \operatorname{Cl}(A)$.

- $(2) \Rightarrow (3)$: Let $A = U \cap \operatorname{Cl}(A)$ for some Λ_{λ} -set U. Since $A \subseteq U$, by Lemma 3.2 $\Lambda_{\lambda}(A) \subseteq \Lambda_{\lambda}(U) = U$ and hence $A \subseteq \Lambda_{\lambda}(A) \cap \operatorname{Cl}(A) \subseteq U \cap \operatorname{Cl}(A) = A$. Therefore, we obtain $A = \Lambda_{\lambda}(A) \cap \operatorname{Cl}(A)$.
- $(3) \Rightarrow (1)$: Let $A = \Lambda_{\lambda}(A) \cap \operatorname{Cl}(A)$. By Lemma 3.3, $\Lambda_{\lambda}(A)$ is a Λ_{λ} -set and $\operatorname{Cl}(A)$ is closed. Therefore, A is Λ_{λ} -closed.

Definition 3.9. Let X be a topological space. A subset A of X is said to be $\lambda g\text{-}closed$ (resp. g-closed [8]) if $\mathrm{Cl}(A)\subseteq U$ whenever $A\subseteq U$ and U is a λ -open (resp. open) set.

Theorem 3.10. For a subset A of a topological space X, the following are equivalent:

- (1) A is closed;
- (2) A is λ -locally closed and λg -closed;
- (3) A is Λ_{λ} -closed and λg -closed.

Proof. (1) \Rightarrow (2): Let A be closed in X. Since $A = X \cap A$ and X is a Λ_{λ} -set, A is λ -locally closed. Let U be any λ -open set containing A. Then $\operatorname{Cl}(A) = A \subseteq U$ and hence A is λg -closed.

- (2) \Rightarrow (3): By Lemma 3.6, every λ -locally closed set is Λ_{λ} -closed.
- $(3)\Rightarrow (1)$: Let A be Λ_{λ} -closed and λg -closed. Since A is Λ_{λ} -closed, $A=P\cap L$, where P is a Λ_{λ} -set and L is closed in X. Let V be any λ -open set containing A. Since A is λg -closed, $\operatorname{Cl}(A)\subseteq V$ and hence $\operatorname{Cl}(A)\subseteq \cap \{V:A\subseteq V,V\text{ is }\lambda\text{-open }\}=\Lambda_{\lambda}(A)$. Therefore, $\operatorname{Cl}(A)\subseteq \Lambda_{\lambda}(A)\subseteq \Lambda_{\lambda}(P)=P$. On the other hand, $A\subseteq L$ and $\operatorname{Cl}(A)\subseteq\operatorname{Cl}(L)=L$. Therefore, we obtain $\operatorname{Cl}(A)\subseteq P\cap L=A$. Thus A is closed.

Theorem 3.11. Let X be a topological space. If A_{α} is a Λ_{λ} -closed set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_{\alpha}$ is Λ_{λ} -closed.

Proof. Let A_{α} be a Λ_{λ} -closed set for each $\alpha \in \Delta$. Then $A_{\alpha} = U_{\alpha} \cap F_{\alpha}$, where U_{α} is a Λ_{λ} -set and F_{α} is a closed set for each $\alpha \in \Delta$. By Lemma 3.3, $\bigcap_{\alpha \in \Delta} U_{\alpha}$ is a Λ_{λ} -set, $\bigcap_{\alpha \in \Delta} F_{\alpha}$ is closed and $\bigcap_{\alpha \in \Delta} A_{\alpha} = (\bigcap_{\alpha \in \Delta} U_{\alpha}) \cap (\bigcap_{\alpha \in \Delta} F_{\alpha})$. Therefore, $\bigcap_{\alpha \in \Delta} A_{\alpha}$ is Λ_{λ} -closed.

4. Decompositions of Continuity

In this section, we obtain the decompositions of continuity.

Definition 4.1. A function $f: X \to Y$ is said to be

- (1) λ -LC-continuous if $f^{-1}(V)$ is λ -locally closed in X for any closed set V of Y,
 - (2) Λ_{λ} -continuous if $f^{-1}(V)$ is Λ_{λ} -closed in X for any closed set V of Y,
 - (3) λg -continuous if $f^{-1}(V)$ is λg -closed in X for any closed set V of Y.

Theorem 4.2. For a function $f: X \to Y$, the following are equivalent:

- (1) f is continuous;
- (2) f is λ -LC-continuous and λg -continuous;
- (3) f is Λ_{λ} -continuous and λg -continuous.

Proof. This is an immediate consequence of Theorem 3.10

Remark 4.3. The following facts are shown by Examples 4.4 and 4.5 and Remark 4.6:

- (1) λ -LC-continuity and λg -continuity are independent of each other,
- (2) Λ_{λ} -continuity and λg -continuity are independent of each other.

EXAMPLE 4.4. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}\}$. Then $C(X) = C(Y) = \{\emptyset, \{b, c, d\}\}$ and λ -open sets in X (resp. Y) are: \emptyset , X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$. λ -locally closed sets in X (resp. Y) are: \emptyset , X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{b, c\}$, $\{c\}$, $\{d\}$. Define a function $f: X \to Y$ by f(a) = c, f(b) = b, f(c) = d, f(d) = a. Then we have the following:

- (1) Since $f^{-1}(\{b,c,d\}) = \{a,b,c\}$, then f is not continuous.
- (2) Since $f^{-1}(\{b,c,d\}) = \{a,b,c\}$, then f is λ -LC-continuous.
- (3) Since $Cl(\{a,b,c\})=X$ (i.e. $\{a,b,c\}$ is not λg -closed), then f is not λg -continuous.
- (4) Since $\{a,b,c\} \subseteq \cap \{U: \{a,b,c\} \subseteq U, U \text{ is } \lambda\text{-open }\} = \{a,b,c\}$ and $\{a,b,c\} = \{a,b,c\} \cap X = \{a,b,c\}$, then $\{a,b,c\}$ is Λ_{λ} -closed. Thus f is Λ_{λ} -continuous.

EXAMPLE 4.5. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then $C(X) = C(Y) = \{\emptyset, X, \{a, b\}, \{c, d\}\}$ and λ -open sets in X (resp. Y) are: \emptyset , X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. And λ -locally closed sets in X (resp. Y) are: \emptyset , X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$. Define $g: X \to Y$ by g(a) = c, g(b) = b, g(c) = a, g(d) = d. Then we have the following:

- (1) Since $g^{-1}(\{c,d\}) = \{a,d\}$, then g is not a continuous function.
- (2) Since $g^{-1}(\{c,d\}) = \{a,d\}$, it is not a λ -locally closed set in X. Then g is not a λ -LC-continuous function.
 - (3) Since $g^{-1}(\{a,b\}) = \{b,c\} \subseteq \cap \{U : \{b,c\} \subseteq U, U \text{ is } \lambda \text{ -open in } X\} =$

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- $\{b,c\}\cap X = \{b,c\} \text{ and } g^{-1}(\{c,d\}) = \{a,d\} = \cap \{U: \{a,d\} \subseteq U, U \text{ is } \lambda\text{-open in } X\}$ $= \{a, d\} \cap X = \{a, d\}$ are Λ_{λ} -closed, then Λ_{λ} -continuous.
- Remark 4.6. (1) If every λg -continuous function is λ -LC-continuous, then it is continuous from Theorem 4.2 This is not true from Example 4.4(1).
- (2) If every λq -continuous function is Λ_{λ} -continuous, then it is continuous from Theorem 4.2. This not true from Example 4.5(1).

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