

## A Shorter and Simple Approach to Study Fixed Point Results via b-Simulation Functions

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**ABSTRACT.** The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on b-metric spaces via b-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].

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## 1. INTRODUCTION AND PRELIMINARIES

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping  $\zeta : [0, +\infty)^2 \rightarrow \mathbb{R}$  is called a simulation function if it satisfies the following:

$$(\zeta_1) \zeta(0, 0) = 0;$$

$$(\zeta_2) \zeta(t, s) < s - t \text{ for all } t, s > 0;$$

$$(\zeta_3) \text{ if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0,$$

then  $\overline{\lim}_{n \rightarrow \infty} \zeta(t_n, s_n) < 0$ .

Also, they denoted the set of all simulation functions by  $\mathcal{Z}$ .

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition  $(\zeta_1)$  (also, see [7]). Also, Roldan et al. [8] revised  $(\zeta_3)$  by taking  $t_n < s_n$ . Hence, we can say that a mapping  $\zeta : [0, +\infty)^2 \rightarrow \mathbb{R}$  is called a simulation function if it satisfies:

$$(\zeta_2) \zeta(t, s) < s - t \text{ for all } t, s > 0;$$

$$(\zeta_3) \text{ if } \{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$$

and  $t_n < s_n$  for all  $n \in \mathbb{N}$ , then  $\overline{\lim}_{n \rightarrow \infty} \zeta(t_n, s_n) < 0$ .

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

**Definition 1.1.** [4] Let  $(X, d)$  be a metric space and  $\zeta \in \mathcal{Z}$ . Then a mapping  $T : X \rightarrow X$  is called a  $\mathcal{Z}$ -contraction with respect to  $\zeta$  if the following condition is satisfied:

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \forall x, y \in X. \quad (1.1)$$

Now, it is clear that  $\zeta(t, t) < 0$  when  $t > 0$ ; further (1.1) implies that  $d(Tx, Ty) < d(x, y)$  when  $x \neq y$  for each  $x, y \in X$ . This means that each  $\mathcal{Z}$ -contraction with respect to  $\zeta$  is continuous.

**Theorem 1.2.** [4] Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be a  $\mathcal{Z}$ -contraction with respect to  $\zeta$ . Then  $T$  has a unique fixed point in  $X$  and for every  $x_0 \in X$ , the Picard sequence  $\{x_n\}$ , where  $x_n = Tx_{n-1}$  for all  $n \in \mathbb{N}$ , converges to the fixed point of  $T$ .

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely,  $(X, d)$  is b-metric space if  $X \neq \emptyset$  and  $d : X \times X \rightarrow [0, +\infty)$  be a mapping such that for all  $x, y, z \in X$  hold:  $d(x, y) = 0 \Leftrightarrow x = y$ ;  $d(x, y) = d(y, x)$  and  $d(x, y) \leq b(d(x, y) + d(y, z))$  for  $b \geq 1$ . Then  $d$  is called  $b$ -metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows.

**Definition 1.3.** Let  $(X, d)$  be a b-metric space. A b-simulation function is a function  $\zeta : [0, +\infty)^2 \rightarrow \mathbb{R}$  satisfying the following:

- ( $\xi_1$ )  $\xi(t, s) < s - t$  for all  $t, s > 0$ ;  
 ( $\xi_2$ ) if  $\{t_n\}, \{s_n\}$  are sequences in  $(0, +\infty)$  such that

$$0 < \lim_{n \rightarrow +\infty} t_n \leq \underline{\lim}_{n \rightarrow +\infty} s_n \leq \overline{\lim}_{n \rightarrow +\infty} s_n \leq b \lim_{n \rightarrow +\infty} t_n < +\infty, \quad (1.2)$$

then  $\overline{\lim}_{n \rightarrow +\infty} \xi(bt_n, s_n) < 0$ .

It is clear if  $b = 1$ , then b-simulation function is in the fact the simulation function in the framework of (standard) metric spaces.

EXAMPLE 1.4. [2] Let  $\xi : [0, +\infty)^2 \rightarrow \mathbb{R}$  be defined by

- (i)  $\xi(t, s) = \lambda s - t$  for all  $t, s \in [0, +\infty)$ , where  $\lambda \in [0, 1)$ .
- (ii)  $\xi(t, s) = \psi(s) - \varphi(t)$  for all  $t, s \in [0, +\infty)$ , where  $\varphi, \psi : [0, +\infty) \rightarrow [0, +\infty)$  are two continuous functions such that  $\psi(t) = \varphi(t) = 0$  if and only if  $t = 0$  and  $\psi(t) < t \leq \varphi(t)$  for all  $t > 0$ .
- (iii)  $\xi(t, s) = s - \frac{f(t, s)}{g(t, s)}t$  for all  $t, s \in [0, +\infty)$ , where  $f, g : [0, +\infty)^2 \rightarrow (0, +\infty)$  are two continuous functions with respect to each variable such that  $f(t, s) > g(t, s)$  for all  $t, s > 0$ .
- (iv)  $\xi(t, s) = s - \varphi(s) - t$  for all  $t, s \in [0, +\infty)$ , where  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a lower semi-continuous function such that  $\varphi(t) = 0$  if and only if  $t = 0$ .
- (v)  $\xi(t, s) = s\varphi(s) - t$  for all  $t, s \in [0, +\infty)$ , where  $\varphi : [0, +\infty) \rightarrow [0, 1)$  is such that  $\lim_{t \rightarrow r^+} \varphi(t) < 1$  for all  $r > 0$ .

Each of the function considered in (i)-(v) is a b-simulation function.

The following important and very interesting results are proved in [2].

**Lemma 1.5.** *Let  $(X, d)$  be a b-metric space and  $f : X \rightarrow X$  be a mapping. Suppose that there exists a b-simulation function  $\xi$  such that following condition holds.*

$$\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X. \quad (1.3)$$

*Let  $\{x_n\}$  be a sequence of Picard of initial at point  $x_0 \in X$  and  $x_{n-1} \neq x_n$  for all  $n \in \mathbb{N}$ . Then*

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = 0.$$

**Lemma 1.6.** *Let  $(X, d)$  be a b-metric space and  $f : X \rightarrow X$  be a mapping. Suppose that there exists a b-simulation function  $\xi$  such that (1.3) holds. Let  $\{x_n\}$  be a sequence of Picard of initial at point  $x_0 \in X$  and  $x_{n-1} \neq x_n$  for all  $n \in \mathbb{N}$ . Then  $\{x_n\}$  is a bounded sequence.*

**Lemma 1.7.** *Let  $(X, d)$  be a b-metric space and  $f : X \rightarrow X$  be a mapping. Suppose that there exists a b-simulation function  $\xi$  such that (1.3) holds. Let  $\{x_n\}$  be a sequence of Picard of initial at point  $x_0 \in X$  and  $x_{n-1} \neq x_n$  for all  $n \in \mathbb{N}$ . Then  $\{x_n\}$  is a Cauchy sequence.*

**Theorem 1.8.** *Let  $(X, d)$  be a complete  $b$ -metric space and let  $f : X \rightarrow X$  be a mapping. Suppose that there exists a  $b$ -simulation function  $\xi$  such that (1.3) holds; that is,*

$$\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X.$$

*Then  $f$  has a unique fixed point.*

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

## 2. MAIN RESULTS

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from  $b$ -metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of  $b$ -metric spaces in [3].

**Lemma 2.1.** *Let  $\{y_n\}$  be a sequence in a  $b$ -metric space  $(X, d)$  such that*

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \quad (2.1)$$

*for some  $\lambda$ ,  $0 \leq \lambda < \frac{1}{b}$  and each  $n = 1, 2, \dots$ . Then  $\{y_n\}$  is a Cauchy sequence in  $(X, d)$ .*

By utilizing Lemma 2.1, Jovanović et al. [3] proved following result.

**Theorem 2.2.** *Let  $(X, d)$  be a complete  $b$ -metric space and  $f : X \rightarrow X$  be a map such that*

$$d(fx, fy) \leq \lambda d(x, y) \quad (2.2)$$

*holds for all  $x, y \in X$ , where  $0 \leq \lambda < \frac{1}{b}$ . Then  $f$  has a unique fixed point  $z$  and for every  $x_0 \in X$ , the sequence  $\{f^n x_0\}$  converges to  $z$ .*

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

**Theorem 2.3.** *Let  $(X, d)$  be a complete  $b$ -metric space and  $f : X \rightarrow X$  be a mapping. Suppose that there exists a  $b$ -simulation function  $\xi$  such that (1.3) holds; that is,*

$$\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X. \quad (2.3)$$

*Then  $f$  has a unique fixed point.*

*Proof.* It is enough clear that (2.3) implies

$$bd(fx, fy) \leq d(x, y) \quad \forall x, y \in X. \quad (2.4)$$

Indeed, (2.4) holds if  $x = y$ . In the case that  $x \neq y$  there are two possibilities, either  $fx = fy$  or  $fx \neq fy$ . In the first case we have that  $b \cdot d(fx, fy) = 0 < d(x, y)$ , while in second case the result follows from  $(\xi_1)$ . This means that (2.3) implies (2.4) for all  $x, y \in X$ . Further, obviously, (2.4) implies that

$$d(f^2x, f^2y) \leq \frac{1}{b^2} d(x, y) = \lambda d(x, y). \quad (2.5)$$

Since  $\lambda = \frac{1}{b^2} \in [0, \frac{1}{b})$ , then according to Theorem 2.2,  $f^2$  has a unique fixed point (say  $z$ ) in  $X$ . This further means that  $f$  has a unique fixed point  $z$  in  $X$ . Now, the proof of this theorem is complete.  $\square$

Obviously, our proof is much shorter than the corresponding ones from Demma et al.'s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

**Corollary 2.4.** *Let  $(X, d)$  be a complete b-metric space and let  $f : X \rightarrow X$  be a mapping. Suppose that*

- (i)  $\lambda \in [0, 1)$  such that  $bd(fx, fy) \leq \lambda d(x, y)$ ;
- (ii) a lower semi-continuous function  $\varphi : [0, +\infty) \rightarrow [0, \infty)$  with  $\varphi^{-1}(0) = \{0\}$  such that  $bd(fx, fy) \leq d(x, y) - \varphi(d(x, y))$ ;
- (iii)  $\varphi : [0, +\infty) \rightarrow [0, 1)$  with  $\lim_{t \rightarrow r^+} \varphi(t) < 1$  for all  $r > 0$  such that  $bd(fx, fy) \leq \varphi(d(x, y)) d(x, y)$ ;
- (iv)  $\eta : [0, +\infty) \rightarrow [0, \infty)$  with  $\eta(t) < t$  for all  $t > 0$  and  $\eta(0) = 0$  such that  $bd(fx, fy) \leq \eta(d(x, y))$

for all  $x, y \in X$ . Then  $f$  has a unique fixed point in each one of above condition.

*Proof.* Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate b-simulation function in Example 1.4. Hence, we obtain that  $bd(fx, fy) \leq d(x, y)$  for all  $x, y \in X$ . The result then follows according to Theorem 2.3.  $\square$

EXAMPLE 2.5. Now, we consider Example 4.5 from [2]. Let  $X = [0, 1]$  and  $d : X \times X \rightarrow \mathbb{R}$  be defined by  $d(x, y) = |x - y|^2$ . Then  $(X, d)$  is a complete b-metric space with  $b = 2$ . Consider a mapping  $f : X \rightarrow X$  by

$$fx = \frac{ax}{1+x}$$

for all  $x \in X$ , where  $a \in [0, \frac{1}{\sqrt{2}}]$ . Now, we have

$$2d(fx, fy) = 2 \left| \frac{ax}{1+x} - \frac{ay}{1+y} \right|^2 = 2a^2 \frac{|x-y|^2}{(1+x)^2(1+y)^2} \leq |x-y|^2 = d(x, y) \quad (2.6)$$

for all  $x, y \in X$ . Further, (2.6) implies that

$$d(f^2x, f^2y) \leq \frac{1}{4}d(x, y);$$

that is,  $f^2$  has a unique fixed point according to Theorem 2.2. This means that  $f$  has a unique fixed point. Here it is  $z = 0$ .

The next result is probably known, but our proof is very condensed.

**Theorem 2.6.** Let  $(X, d)$  be a complete b-metric space and let  $f : X \rightarrow X$  be a mapping such that

$$d(fx, fy) \leq \lambda d(x, y) \quad (2.7)$$

for all  $x, y \in X$ , where  $\lambda \in [0, 1)$ . Then  $f$  has a unique fixed point (say  $z$ ) in  $X$  and for  $x_0 \in X$  the sequence  $\{f^n x_0\}_{n \in \mathbb{N}}$  converges to  $z$ .

*Proof.* The condition (2.7) implies that

$$d(f^n x, f^n y) \leq \lambda d(f^{n-1} x, f^{n-1} y) \leq \cdots \leq \lambda^n d(x, y)$$

for all  $x, y \in X$  and  $n \in \mathbb{N}$ . Since  $\lambda^n \rightarrow 0$  as  $n \rightarrow \infty$ , there is  $k \in \mathbb{N}$  such that  $\lambda^k < \frac{1}{b}$ . Therefore, we have

$$d(f^{k+1} x, f^{k+1} y) \leq \frac{1}{b^2} d(x, y).$$

The result now follows by Theorem 2.2.  $\square$

**Question 1.** Does Theorem 2.3 holds if  $\xi(d(fx, fy), d(x, y)) \geq 0$  for all  $x, y \in X$ , where  $(X, d)$  is a given complete b-metric space and  $f : X \rightarrow X$  be a mapping and  $\xi$  a given b-simulation function?

**Question 2.** Can you obtain this results by considering ordered b-metric spaces or cone b-metric spaces instead of b-metric spaces?

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