Iranian Journal of Mathematical Sciences and Informatics
Vol. 14, No. 2 (2019), pp 19-32
DOI: 10.7508/ijmsi.2019.02.003

Common Fixed Point Theorems for Weakly Compatible Mapping by (CLR) Property on Partial Metric Space

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Abstract. The purpose of this paper is to obtain the common fixed point results for two pair of weakly compatible mapping by using common (CLR) property in partial metric space. Also we extend the very recent results which are presented in [19] with proofing a new version of the continuity of partial metric.

Keywords: Fixed point, Partial metric space, (CLR)-Property.

2000 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

The partial metric space (briefly PMS), which is published for the first time in 1992 by Mattews [16], is an extension of the usual metric space in which $d(x, x)$ is no necessarily zero. The existence of fixed point for mapping defined on complete metric spaces (X, d) satisfying a general contractive inequality of

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Received 01 November 2016; Accepted 25 July 2017

integral type was established by Branciari [6]. This result which involves more general contractive condition of integral type, was used by many authors to obtain some fixed point and common fixed point theorems on various spaces [2, 4, 7, 8, 9, 10, 13, 14, 15, 18, 22]. Most of the common fixed point theorems require compatibility conditions (introduced by Junck [12]) and completeness assumption of the space or subspace or continuity of mappings involved besides some contractive condition. Afterward the notion of compability was extended to PMS spaces by Mishra [17]. In the general setting, the notion of $(E.A)$ and common $(E.A)$ properties which require the closedness of the subspace was introduced by Aamri, Moutawakil [1]. The CLR and common CLR properties which is an analogue to $(E.A)$ property which never requires any condition on closedness of the space or subspace, are obtained by Sintunavarat and Kumam [21] and Imdad et.al [11].

This paper mainly aims to employ the common CLR property to obtain common fixed point results for two pair of weakly compatible mappings satisfying contractive condition of integral type on the partial metric space.

Definition 1.1. [16], [20, Definition 1.1] A partial metric space (briefly PMS) is a pair (X,p) where $p : X \times X \to \mathbb{R}^+$ is continuous map and $\mathbb{R}^+ = [0,\infty)$ such that for all $x, y, z \in X$:

(p1) $p(x,x) = p(y,y) = p(x,y) \iff x = y$,
(p2) $p(x,x) \leq p(x,y)$,
(p3) $p(x,y) = p(y,x)$,
(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Each partial metric p on X generates a T_0 topology τ_p on X which has the family of open p-balls

$$\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

as a base, where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. (1) A sequence $\{x_n\}$ in a PMS, (X,p), converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(2) A sequence $\{x_n\}$ in a PMS, (X,p), is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m,x_n)$. exists and is finite.

(3) A PMS (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x,x) = \lim_{m,n\to\infty} p(x_m,x_n).$$

The following lemma states a new version of the continuity of partial metric. And we present two proof, at first directly and second very short proof.

Lemma 1.3. *Assume that $x_n \to x$ and $y_n \to y$ in PMS (X, p). Then*

$$\lim_{n\to\infty}(p(x_n, y_n) - \min\{p(x_n, x_n), p(y_n, y_n)\}) = p(x, y) - \min\{p(x, x), p(y, y)\}). \tag{1.1}$$

Proof. Put

$$\begin{aligned} a_n : &= \min\{p(x_n, x_n), p(y_n, y_n)\}, \\ a : &= \max\{p(x, x), p(y, y)\} \\ b : &= p(x, x) + p(y, y) \\ b_n : &= p(x_n, x_n) + p(y_n, y_n) \end{aligned}$$

We note that

$$\begin{aligned} b - a &= \min\{p(x, x), p(y, y)\} \\ a &\leq b \\ a_n &\leq b_n. \end{aligned}$$

Now we show that $\limsup_{n\to\infty} p(x_n, x_n) = p(x, x)$. Since $p(x_n, x) \to p(x, x)$ as $n \to \infty$ therefore

$$\forall \varepsilon > 0 \quad \exists N_1 \quad \forall n \ (n \geq N_1 \Rightarrow |p(x_n, x) - p(x, x)| < \varepsilon).$$

So we get

$$p(x_n, x_n) \leq p(x_n, x) \leq p(x, x) + \varepsilon, \quad \forall n \geq N_1, \tag{1.2}$$

likewise

$$p(y_n, y_n) \leq p(y_n, y) \leq p(y, y) + \varepsilon, \quad \forall n \geq N_2, \tag{1.3}$$

means $\limsup_{n\to\infty} p(x_n, x_n) = p(x, x)$ and $\limsup_{n\to\infty} p(y_n, y_n) = p(y, y)$.
Also by (1.2) and (1.3)

$$a_n \leq b - a + 2\varepsilon, \quad \forall n \geq N, \tag{1.4}$$

where $N = \max\{N_1, N_2\}$.
Put

$$A_n := a_n - b_n - (b - a) = -\max\{p(x_n, x_n), p(y_n, y_n)\} - \min\{p(x, x), p(y, y)\}, \tag{1.5}$$

now if $p(x_n, x_n) \leq p(y_n, y_n)$, then by taking upper limit $p(x, x) \leq p(y, y)$ so $A_n = -p(y_n, y_n) - p(x, x)$ and if $p(y_n, y_n) \leq p(x_n, x_n)$, then $p(y, y) \leq p(x, x)$ which implies $A_n = -p(x_n, x_n) - p(y, y)$. Therefore

$$\liminf_{n\to\infty}(p(x, x_n) + p(y_n, y) + A_n) = 0. \tag{1.6}$$

Thus by (1.4)

$$\begin{array}{rcl} p(x_n,y_n) & \leq & p(x_n,x)+p(x,y_n)-p(x,x) \\ & \leq & p(x_n,x)+p(x,y)+p(y,y_n)-p(y,y)-p(x,x) \\ & - & a_n+a_n-(b-a)+(b-a) \\ p(x_n,y_n)-a_n & \leq & (p(x,y)-(b-a))+p(x_n,x)-p(x,x)+p(y,y_n)-p(y,y) \\ & - & a_n+(b-a) \\ p(x_n,y_n)-a_n & \leq & (p(x,y)-(b-a))+p(x_n,x)-p(x,x)+p(y,y_n) \\ & - & p(y,y)+2\varepsilon, \end{array}$$

for $n \geq N$. On the other hand, by (1.6)

$$\begin{array}{rcl} p(x,y) & \leq & p(x,x_n)+p(x_n,y)-p(x_n,x_n) \\ & \leq & p(x,x_n)+p(x_n,y_n)+p(y_n,y)-p(y_n,y_n)-p(x_n,x_n) \\ p(x,y)-(b-a) & \leq & p(x,x_n)+p(x_n,y_n)+p(y_n,y)-b_n-(b-a)-a_n+a_n \\ & \leq & (p(x_n,y_n)-a_n)+p(x,x_n)+p(y_n,y)+a_n-b_n-(b-a) \\ & \leq & (p(x_n,y_n)-a_n)+p(x,x_n)+p(y_n,y)+A_n \end{array}$$

Now by above inequalities we get

$$\limsup_{n\to\infty}(p(x_n,y_n)-a_n) \leq p(x,y)-(b-a), \tag{1.7}$$

$$p(x,y)-(b-a) \leq \liminf_{n\to\infty} p(x_n,y_n)-a_n). \tag{1.8}$$

By equations (1.7) and (1.8) assertion is clear. □

Example 1.4. Let $X=\{1,2,3\}$,

$$p(1,1)=1, \quad p(2,2)=2, \quad p(3,3)=3,$$

$$p(1,2)=2, \quad p(2,3)=3, \quad p(1,3)=3,$$

$$p(x,y)=p(y,x) \quad x\neq y,$$

for every $x,y\in X$. (X,p) is PMS. Assume $x_n=1$, $x=2$, $y_n=2$ and $y=3$. So $x_n\to x$ and $y_n\to y$ in PMS. $p(x_n,y_n)=2$, $p(x_n,x_n)=1$, $p(y_n,y_n)=2$, $p(x,x)=2$, $p(y,y)=3$ and $p(x,y)=3$. So Lemma 1.3 holds, but $p(x_n,y_n)\not\to p(x,y)$.

Remark 1.5. If we consider the following definition, then Lemma 1.3 has simple and short proof, since every partial metric p is m-metric by [5, Lemma 1.1] and assertion obtain by [5, Lemma 2.2].

Definition 1.6. ([5]) Let X be a non empty set. A function $m: X\times X\to\mathbb{R}^+$ is called M-metric if the following conditions are satisfied:

(m1) $m(x,x)=m(y,y)=m(x,y) \iff x=y$,
(m2) $m_{xy}\leq m(x,y)$,
(m3) $m(x,y)=m(y,x)$,

(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Where

$$m_{xy} := \min\{m(x,x), m(y,y)\} = m(x,x) \vee m(y,y),$$

Then the pair (X, m) is called a M-metric space.

Remark 1.7. Let

$$p^*(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\} \quad \forall x, y \in X. \tag{1.9}$$

Therefore by Lemma 1.3

$$\lim_{n\to\infty} p^*(x_n, y_n) = p^*(x,y),$$

when $x_n \to x$ and $y_n \to y$ in PMS.

Let $\mathcal{L}(\mathbb{R}^+)$ denote the Lebesgue integrable functions with finite integral and $USC(\mathbb{R}^+)$ denote the upper semi-continuous functions.

$$\Phi := \left\{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ : \varphi \in \mathcal{L}(\mathbb{R}^+),\ \int_0^\varepsilon \varphi(t)dt > 0, \varepsilon > 0\right\}$$

and

$$\Psi := \left\{\psi : \mathbb{R}^+ \to \mathbb{R}^+ : \psi \in USC(\mathbb{R}^+), \psi(0) = 0 \text{ and } \psi(t) < t;\ \forall t > 0\right\}.$$

Definition 1.8. A pair of self-mappings F and G on X is weakly compatible if there exists a point $x \in X$ such that $Fx = Gx$ implies $FGx = GFx$ i.e., they commute at their coincidence points.

The following definitions are partial metric version of metric ones in ([1, 11, 21]).

Definition 1.9. Let (X,p) be a partial metric space for the self mappings $F, G, S; T : X \to X$. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \in X,$$

then the pairs (F,G) and (S,T) satisfy the common $(E.A)$ property.

Definition 1.10. Let (X,p) be a partial metric space for the self mappings $F, G, S; T : X \to X$. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = t \in G(X) \cap T(X),$$

then pairs (F,G) and (S,T) satisfy the common limit range property with respect to the mappings G and T, denoted by (CLR_{GT}).

2. Common Fixed Point Theorems

In this section, we study common fixed point theorems for weakly compatible mappings using common (CLR) and common $(E.A)$ properties.

Theorem 2.1. *Let (X,p) be a partial metric space and F,G,S and T be four self-mappings on X satisfying in the following conditions:*

(1) *The pair (F,G) and (S,T) share (CLR_{GT}) property;*
(2)

$$\int_0^{p(Fx,Sy)} \varphi(t)dt \leq \psi\left(\int_0^{C^1_{F,G,S,T}(x,y)} \varphi(t)dt\right) \qquad \forall x,y \in X,$$

where $(\varphi,\psi) \in \Phi \times \Psi$ and

$$\begin{aligned} C^1_{F,G,S,T}(x,y) \quad = \quad \max\Big\{ & p(Gx,Ty), p(Gx,Fx), p^*(Ty,Sy), \\ & \frac{1}{2}\left[p^*(Fx,Ty) + p(Sy,Gx)\right], \\ & \frac{p(Fx,Gx)p^*(Sy,Ty)}{1+p(Gx,Ty)}, \\ & \frac{p^*(Fx,Ty)p(Sy,Gx) + p^*(Fx,Sy)}{1+p(Gx,Ty)}, \\ & p(Fx,Gx)\frac{1+p(Gx,Sy)+p^*(Ty,Fx)}{1+p(Gx,Fx)+p^*(Ty,Sy)}\Big\}. \end{aligned}$$

If the pairs (F,G) and (S,T) are weakly compatible, then F,S,T and G have a unique common fixed point in X.

Proof. By (CLR_{GT}) property for (F,G) and (S,T), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = z, \tag{2.1}$$

for some $z \in T(X) \bigcap G(X)$.
Since $z \in G(X)$, then there exists a point $u \in X$ such that $Gu = z$.

Now we claim that $Fu = Gu$. To prove the claim, let $Fu \neq Gu$.
By putting $x = u$ and $y = y_n$ in condition (2) of Theorem 2.1 we have

$$\int_0^{p(Fu,Sy_n)} \varphi(t)dt \leq \psi\left(\int_0^{C^1_{F,G,S,T}(u,y_n)} \varphi(t)dt\right). \tag{2.2}$$

We have

$$\begin{aligned}\lim_{n\to\infty} C^1_{F,G,S,T}(u,y_n) &= \max\Big\{p(z,z),p(z,Fu),p(z,z),\\ &\quad \frac{1}{2}[p^*(Fu,z)+p(z,z)],0,\\ &\quad p^*(Fu,z),p(z,Fu)\frac{1+p(z,z),p^*(Fu,z)}{1+p(Fu,z)+0}\Big\}\\ &= p(Fu,z),\end{aligned}$$

because

$$\begin{aligned} p(Gu,Ty_n) &= p(z,Ty_n)\to p(z,z),\\ p(Gu,Fu) &= p(z,Fu),\\ p(Sy_n,Gu) &= p(Sy_n,z)\to p(z,z),\\ p^*(Ty_n,Sy_n) &\to p^*(z,z)=0,\\ p^*(Fu,Ty_n) &\to p^*(Fu,z)\le p(Fu,z),\\ p^*(Fu,Sy_n) &\to p^*(Fu,z)\le p(Fu,z),\\ p^*(Fu,Ty_n) &\to p^*(Fu,z)\le p(Fu,z),\end{aligned}$$

also

$$p^*(Fu,Ty_n)\to p^*(Fu,z)=p(Fu,z)-\min\{p(z,z),p(Fu,Fu)\}.$$

If $p(z,z)\le p(Fu,Fu)$ then $p^*(Fu,z)=p(Fu,z)-p(z,z)$ which implies that

$$p(Fu,z)\frac{1+p(z,z)+p^*(Fu,z)}{1+p(Fu,z)}=p(Fu,z),$$

and if $p(Fu,Fu)\le p(z,z)$, then

$$p^*(Fu,z)=p(Fu,z)-p(Fu,Fu)\le p(Fu,z)-p(z,z),$$

which implies that

$$p(Fu,z)\frac{1+p(z,z)+p^*(Fu,z)}{1+p(Fu,z)}\le p(Fu,z).$$

So

$$\begin{aligned}\int_0^{p(Fu,z)} \varphi(t)dt &= \limsup_{n\to\infty} \int_0^{p(Fu,Sy_n)} \varphi(t)dt \\ &\le \limsup_{n\to\infty} \psi\left(\int_0^{C^1_{F,G,S,T}(u,y_n)} \varphi(t)dt\right) \\ &\le \psi\left(\limsup_{n\to\infty} \int_0^{C^1_{F,G,S,T}(u,y_n)} \varphi(t)dt\right) \\ &= \psi\left(\int_0^{p(Fu,z)} \varphi(t)dt\right) \\ &< \int_0^{p(Fu,z)} \varphi(t)dt,\end{aligned}$$

which is a contradiction, thus $Fu = Gu$ and hence,

$$Fu = Gu = z. \tag{2.3}$$

Similarly, it can be shown that $Sv = Tv$ and hence

$$Sv = Gv = z. \tag{2.4}$$

Therefore from (2.3) and (2.4) one can write

$$Fu = Gu = Sv = Tv = z. \tag{2.5}$$

Next, we show that z is a common fixed point of F, S, T and G. For this, since the pairs (F, G) and (S, T) are weakly compatible, then using (2.5) we have

$$Fu = Gu \quad \Rightarrow \quad GFu = FGu \quad \Rightarrow \quad Fz = Gz, \tag{2.6}$$

and

$$Sv = Tv \quad \Rightarrow \quad TSv = STv \quad \Rightarrow \quad Sz = Tz. \tag{2.7}$$

We will show next that $Fz = z$. Otherwise, if $Fz \ne z$, using condition (2) of Theorem 2.1 with $x = z$ and $y = v$, we have

$$\int_0^{(Fz,Sv)} \varphi(t)dt \le \psi\left(\int_0^{C^1_{F,G,S,T}(z,v)} \varphi(t)dt\right).$$

In the light of (2.5) and (2.6), we get

$$\begin{aligned}C^1_{F,G,S,T}(z,v) = \max\Big\{&p(Fz,z), p(Fz,Fz), p(z,z),\\ &\frac{1}{2}[p^*(Fz,z)+p(z,Fz)], 0, p^*(Fz,z),\\ &p(Fz,Fz)\frac{1+p(z,Fz)+p^*(z,Fz)}{1+p(Fz,Fz)+0}\Big\}\\ &= p(Fz,z)\end{aligned}$$

and

$$\int_0^{p(Fz,z)}\varphi(t)dt \le \psi\left(\int_0^{p(Fz,z)}\varphi(t)dt\right) < \int_0^{p(Fz,z)}\varphi(t)dt,$$

which is a contradiction. Thus $Fz = z$ and from (2.6), we can write

$$Fz = Gz = z. \tag{2.8}$$

Similarly, setting $x = u$ and $y = z$ in condition (2) of theorem 2.1 and using (2.5), (2.6), one can get

$$Sz = Tz = z. \tag{2.9}$$

Therefore from (2.8) and (2.9), it follows that

$$Fz = Sz = Tz = Gz = z, \tag{2.10}$$

that is, z is a common fixed point of F, S, T and G.
Finally, we prove the uniqueness of the common fixed point of F, S, T and G. Assume that z_1 and z_2 are two distinct common fixed points of F, S, T and G. Then replacing x by z_1 and y by z_2 in condition (2) of Theorem 2.1, we have

$$\int_0^{p(z_1,z_2)}\varphi(t)dt = \int_0^{p(Fz_1,Sz_2)}\varphi(t)dt \le \psi\left(\int_0^{C^1_{F,G,S,T}(z_1,z_2)}\varphi(t)dt\right).$$

Since $C^1_{F,G,S,T}(z_1,z_2) = p(z_1,z_2)$ So

$$\int_0^{p(z_1,z_2)}\varphi(t)dt \le \psi\left(\int_0^{p(z_1,z_2)}\varphi(t)dt\right) < \int_0^{p(z_1,z_2)}\varphi(t)dt,$$

which is a contradiction and thus $z_1 = z_2$. Hence F, S, T and G have a unique common fixed point in X. □

EXAMPLE 2.2. Suppose $X = \mathbb{R}^+$ and $p(x,y) = \max\{x,y\}$; then (X,p) is a PMS (See e.g.[3]). Define four self mappings F, S, T and G on X by

$$F(x) = \frac{x}{2}+\frac{1}{2}, \quad G(x) = x^2, \quad S(x) = x, \quad T(x) = \frac{2}{x+1}$$

Let $x_n = \{1+\frac{1}{n}\}_{n\in\mathbb{N}}$ and $y_n = \{\frac{n}{n+1}\}_{n\in\mathbb{N}}$ be two sequences, so we have

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} G(x_n) = \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} T(y_n) = 1$$

Also

$$1 \in T(X) \cap G(X) = (0,2] \cap \mathbb{R}^+,$$

Hence (F,G) and (S,T) satisfy CLR_{GT} property. It is easy to check that the pair (F,G) and (S,T) is weakly compatible at $x=1$ as a coincidence point.
To verify condition (2) of theorem 2.1, let us define $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t$ and $\psi(t) = \frac{t}{2}$.
So

$$F(2) = \frac{3}{2}, \quad G(2) = 4, \quad S(\frac{1}{2}) = \frac{1}{2}, \quad T(\frac{1}{2}) = \frac{4}{3},$$

$$\int_0^{p(F(2),S(\frac{1}{2}))} \varphi(t)dt = \int_0^{\frac{3}{2}} tdt = \frac{9}{8} \quad \text{and} \quad C_1(2,\frac{1}{2}) = 4.$$

Thus we obtain

$$\psi\left(\int_0^{C_1(2,\frac{1}{2})} \varphi(t)dt\right) = \psi\left(\int_0^4 tdt\right) = \psi(8) = 4.$$

Hence from above we have

$$\int_0^{p(F(2),S(\frac{1}{2}))} \varphi(t)dt \leq \psi\left(\int_0^{C_1(2,\frac{1}{2})} \varphi(t)dt\right).$$

So according to theorem 2.1 F,S,T and G have a common fixed point.

From Theorem 2.1, we easily deduce the following corollaries.

Corollary 2.3. *Let (X,p) be a partial metric space and F,G and T be three self-mappings on X satisfying the following condition:*

(1) *The pair (F,G) and (F,T) share (CLR_{GT}) property.*
(2)

$$\int_0^{p(Fx,Fy)} \varphi(t)dt \leq \psi\left(\int_0^{C^1_{F,G,F,T}(x,y)} \varphi(t)dt\right), \quad \forall x,y \in X$$

where $(\varphi,\psi) \in \Phi \times \Psi$.

If the pairs (F,G) and (F,T) are weakly compatible, then F,G and T have a unique common fixed point in X.

Corollary 2.4. *Let (X,p) be a partial metric space and F,T be two self-mappings on X satisfying the following condition:*

(1) *The pair (F,T) share (CLR_T) property.*
(2)

$$\int_0^{p(Fx,Fy)} \varphi(t)dt \leq \psi\left(\int_0^{C^1_{F,T,F,T}(x,y)} \varphi(t)dt\right), \quad \forall x,y \in X,$$

where $(\varphi,\psi) \in \Phi \times \Psi$.

If the pairs (F,T) are weakly compatible, then F and T have a unique common fixed point in X.

In a similar method as in Theorem 2.1 the following result can be concluded and proved.

Theorem 2.5. *Let (X,p) be a partial metric space and F,S,T and G be for self-mappings on X satisfying in following conditions:*

(1) *The pair (F,G) and (S,T) share (CLR_{GT}) property.*

(2)
$$\int_0^{P(Fx,Sy)} \varphi(t)dt \leq \psi\left(\int_0^{C^2_{F,G,S,T}(x,y)} \varphi(t)dt\right) \qquad \forall x,y \in X,$$
where $(\varphi,\psi) \in \Phi \times \Psi$ and
$$\begin{aligned} C^2_{F,G,S,T}(x,y) \quad = \quad & \max\Big\{p(Gx,Ty), p(Gx,Fx), p(Gy,Sy), \\ & \frac{1}{2}[p^*(Fx,Ty)+p(Sy,Gx)], \\ & \frac{p(Fx,Gx)p^*(Sy,Ty)}{1+p(Fx,Sy)}, \\ & \frac{p^*(Fx,Ty)p(Sy,Gx)+P^*(Fx,Sy)}{1+p(Fx,Sy)}, \\ & p(Gx,Fx)\frac{1+p(Gx,Sy)+p^*(Ty,Fx)}{1+p(Gx,Fx)+p^*(Ty,Sy)}\Big\}. \end{aligned}$$

If the pairs (F,G) and (S,T) are weakly compatible, then F,S,T and G have a unique common fixed point in X.

Obviously, (CLR_{GT}) property implies the common property $(E.A)$ but the converse is not true in general. So replacing (CLR_{GT}) property by common property $(E.A)$ in Theorem 2.1 and Theorem 2.5, we get the following results, the proofs of which can be easily done by following the lines of the proof of Theorem 2.1, because the $(E.A)$ property together with the closedness property of a suitable subspace gives rise to the closed range property.

Corollary 2.6. *Let (X,p) be a partial metric space and F,S,T and G be for self-mappings on X satisfying:*

(1) *The pair (F,G) and (S,T) share $(E.A)$ property such that $T(X)$ (or $G(X)$) is closed subspace of X;*

(2)
$$\int_0^{p(Fx,Sy)} \varphi(t)dt \leq \psi\left(\int_0^{C^1_{F,G,S,T}(x,y)} \varphi(t)dt\right) \qquad \forall x,y \in X$$
where $(\varphi,\psi) \in \Phi \times \Psi$.

If the pairs (F,G) and (S,T) are weakly compatible, then F,S,T and G have a unique common fixed point in X.

Corollary 2.7. *Let (X,p) be a partial metric space and F,S,T and G be for self-mappings on X satisfying:*

(1) *The pair (F,G) and (S,T) share common $(E.A)$ property such that $T(X)$ (or $G(X)$) is closed subspace of X.*

(2)
$$\int_0^{p(Fx,Sy)} \varphi(t)dt \le \psi\left(\int_0^{C^2_{F,G,S,T}(x,y)} \varphi(t)dt\right) \qquad \forall x,y\in X,$$

where $(\varphi,\psi)\in\Phi\times\Psi$.

If the pairs (F,G) and (S,T) are weakly compatible, then F,S,T and G have a unique common fixed point in X.

One can obtained other consequences from Theorem 2.5 and Corollaries 2.6 and 2.7 in a similar way as obtained from Theorem 2.1.

Remark 2.8. Theorem 2.1 and 2.6 are still valid, if we replace $C^1_{F,G,S,T}(x,y)$ by $C^3_{F,G,S,T}(x,y)$. Similarly, Theorem 2.5 and Corollary 2.7 are still valid, if we replace $C^2_{F,G,S,T}(x,y)$ by $C^4_{F,G,S,T}(x,y)$, where

$$\begin{aligned} C^3_{F,G,S,T}(x,y) \quad = \quad & \max\Big\{p(Gx,Ty), p(Gx,Fx), p(Ty,Sy), \\ & \frac{1}{2}[p^*(Fx,Ty)+p(Sy,Gx)], \\ & \min\{\frac{p(Fx,Gx)p^*(Sy,Ty)}{1+p(Gx,Ty)}, \frac{p^*(Fx,Ty)p(Sy,Gx)+p^*(Fx,Sy)}{1+p(Gx,Ty)}, \\ & p(Gx,Fx)\frac{1+p(Gx,Sy)+p^*(Ty,Fx)}{1+p(Gx,Fx)+p^*(Ty,Sy)}\}\Big\} \end{aligned}$$

and

$$\begin{aligned} C^4_{F,G,S,T}(x,y) \quad = \quad & \max\Big\{p(Gx,Ty), p(Gx,Fx), p(Ty,Sy), \\ & \frac{1}{2}[p^*(Fx,Ty)+p(Sy,Gx)], \\ & \min\{\frac{p(Fx,Gx)p^*(Sy,Ty)}{1+p(Fx,Sy)}, \frac{p^*(Fx,Ty)p(Sy,Gx)+p^*(Fx,Sy)}{1+p(Fx,Sy)}, \\ & p(Gx,Fx)\frac{1+p(Gx,Sy)+p^*(Ty,Fx)}{1+p(Gx,Fx)+p^*(Ty,Sy)}\}\Big\}. \end{aligned}$$

Finally, by choosing $F=S$ and G and T as identity mappings, we conclude some fixed point theorems for integral type contraction from our main Theorem 2.1 which can be listed as follows:

Corollary 2.9. *Let (X,p) be a partial metric space and $F : X \to X$be a self mapping satisfying:*

$$\int_0^{p(Fx,Fy)} \varphi(t)dt \leq \psi\left(\int_0^{C^1_{F,id,F,id}(x,y)} \varphi(t)dt\right) \quad \forall x,y \in X,$$

where $(\varphi,\psi) \in \Phi \times \Psi$. Then F has a unique fixed point in X.

Corollary 2.10. *Let (X,p) be a partial metric space and $F : X \to X$ be a self mapping satisfying:*

$$\int_0^{p(Fx,Fy)} \varphi(t)dt \leq \psi\left(\int_0^{C^2_{F,id,F,id}(x,y)} \varphi(t)dt\right) \quad \forall x,y \in X$$

Where $(\varphi,\psi) \in \Phi \times \Psi$. Then F has a unique fixed point in X.

Remark 2.11. Replacing the partial metric p in (X,p) by metric d we can get the similar results which are given in [19].

Remark 2.12. Notice that several fixed point theorems such as the celebrated Banach fixed point theorem, fixed point theorems for Kannan, Chatterjee and Reich type mappings and others can be deduced as particular cases of Corollary 2.9.

Acknowledgement

The authors are thankful to the learned referee for his/her valuable suggestions for improving the content of the paper.

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