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# A Submodule-Based Zero Divisor Graph for Modules

Sakineh Babaei<sup>a</sup>, Shiroyeh Payrovi<sup>\*,a</sup>, Esra Sengelen Sevim<sup>b</sup>

<sup>a</sup>Department of Mathematics, Imam Khomeini International University, Postal Code: 34149-1-6818, Qazvin, Iran.

<sup>b</sup>Department of Mathematics, Istanbul Bilgi University, Kazim Karabekir Cad. No: 2/13, 34060 Eyup-Istanbul, Turkey.

E-mail: sbabaei@edu.ikiu.ac.ir E-mail: shpayrovi@sci.ikiu.ac.ir E-mail: esra.sengelen@bilgi.edu.tr

ABSTRACT. Let R be a commutative ring with identity and M be an Rmodule. The zero divisor graph of M is denoted by  $\Gamma(M)$ . In this study,
we are going to generalize the zero divisor graph  $\Gamma(M)$  to submodulebased zero divisor graph  $\Gamma(M,N)$  by replacing elements whose product
is zero with elements whose product is in some submodule N of M. The
main objective of this paper is to study the interplay of the properties of
submodule N and the properties of  $\Gamma(M,N)$ .

**Keywords:** Zero divisor graph, Submodule-based zero divisor graph, Semisimple module.

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## 1. Introduction

Let R be a commutative ring with identity. The zero divisor graph of R, denoted  $\Gamma(R)$ , is an undirected graph whose vertices are the nonzero zero divisor of R with two distinct vertices x and y are adjacent by an edge if and only

<sup>\*</sup>Corresponding Author

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if xy = 0. The idea of a zero divisor graph of a commutative ring was introduced by Beck in [3] where he was mainly interested with colorings of rings. The definition above first is appeared in [2], which contains several fundamental results concerning  $\Gamma(R)$ . The zero-divisor graph of a commutative ring is further examined by Anderson, Levy and Shapiro, Mulay in [1, 9]. Also, the ideal-based zero divisor graph of R is defined by Redmond, in [12].

The zero divisor graph for modules over commutative rings has been defined by Behboodi in [4] as a generalization of zero divisor graph of rings. Let R be a commutative ring and M be an R-module, for  $x \in M$ , we denote the annihilator of the factor module M/Rx by  $I_x$ . An element  $x \in M$  is called a zero divisor, if either x = 0 or  $I_x I_y M = 0$  for some  $y \neq 0$  with  $I_y \subset R$ . The set of zero divisors of M is denoted by Z(M) and the associated graph to M with vertices in  $Z^*(M) = Z(M) \setminus \{0\}$  is denoted by  $\Gamma(M)$ , such that two different vertices x and y are adjacent provided  $I_x I_y M = 0$ .

In this paper, we introduce the submodule-based zero divisor graph that is a generalization of zero divisor graph for modules. Let R be a commutative ring, M be an R-module and N be a proper submodule of M. An element  $x \in M$  is called zero divisor with respect to N, if either  $x \in N$  or  $I_x I_y M \subseteq N$  for some  $y \in M \setminus N$  with  $I_y \subset R$ . We denote Z(M, N) for the set of zero divisors of M with respect to N. Also, we denote the associated graph to M with vertices  $Z^*(M, N) = Z(M, N) \setminus N$  by  $\Gamma(M, N)$ , and two different vertices x and y are adjacent provided  $I_x I_y M \subseteq N$ .

In the second section, we define a submodule-based zero divisor graph for a module and we study basic properties of this graph. In the third section, if M is a finitely generated semisimple R-module such that its homogenous components are simple and N is a submodule of M, we determine some relations between  $\Gamma(M,N)$  and  $\Gamma(M/N)$ , where M/N is the quotient module of M, we show that the clique number and chromatic number of  $\Gamma(M,N)$  are equal. Also, we determine some submodule of M such that  $\Gamma(M,N)$  is an empty or a complete bipartite graph.

Let  $\Gamma$  be a (undirected) graph. We say that  $\Gamma$  is *connected* if there is a path between any two distinct vertices. For vertex x the number of graph edges which touch x is called the degree of x and is denoted by  $\deg(x)$ . For vertices x and y of  $\Gamma$ , we define d(x,y) to be the length of a shortest path between x and y, if there is no path, then  $d(x,y) = \infty$ . The diameter of  $\Gamma$  is  $\dim(\Gamma) = \sup\{d(x,y)|x \text{ and } y \text{ are vertices of } \Gamma\}$ . The girth of  $\Gamma$ , denoted by  $\operatorname{gr}(\Gamma)$ , is the length of a shortest cycle in  $\Gamma$  ( $\operatorname{gr}(\Gamma) = \infty$  if  $\Gamma$  contains no cycle).

A graph  $\Gamma$  is complete if any two distinct vertices are adjacent. The complete graph with n vertices is denoted by  $K^n$  (we allow n to be an infinite cardinal). The clique number,  $\omega(\Gamma)$ , is the greatest integer n > 1 such that  $K^n \subseteq \Gamma$ , and  $\omega(\Gamma) = \infty$  if  $K^n \subseteq \Gamma$  for all  $n \ge 1$ . A complete bipartite graph is a graph  $\Gamma$  which may be partitioned into two disjoint nonempty vertex sets  $V_1$  and  $V_2$ 

such that two distinct vertices are adjacent if and only if they are in different vertex sets. If one of the vertex sets is a singleton, then we call that  $\Gamma$  is a *star graph*. We denote the complete bipartite graph by  $K^{m,n}$ , where  $|V_1| = m$  and  $|V_2| = n$  (again, we allow m and n to be infinite cardinals); so a star graph is  $K^{1,n}$ , for some  $n \in \mathbb{N}$ .

The chromatic number,  $\chi(\Gamma)$ , of a graph  $\Gamma$  is the minimum number of colors needed to color the vertices of  $\Gamma$ , so that no two adjacent vertices share the same color. A graph  $\Gamma$  is called *planar* if it can be drawn in such a way that no two edges intersect.

Throughout this study, R is a commutative ring with nonzero identity, M is a unitary R-module and N is a proper submodule of M. Given any subset S of M, the annihilator of S is denoted by  $\operatorname{ann}(S) = \{r \in R | rs = 0 \text{ for all } s \in S\}$  and the cardinal number of S is denoted by |S|.

#### 2. Submodule-based Zero Divisor Graph

Recall that R is a commutative ring, M is an R-module and N is a proper submodule of M. For  $x \in M$ , we denote  $\operatorname{ann}(M/Rx)$  by  $I_x$ .

**Definition 2.1.** Let M be an R-module and N be a proper submodule of M. An  $x \in M$  is called a zero divisor with respect to N if  $x \in N$  or  $I_xI_yM \subseteq N$  for some  $y \in M \setminus N$  with  $I_y \subset R$ .

We denote the set of zero divisors of M with respect to N by Z(M,N) and  $Z^*(M,N) = Z(M,N) \setminus N$ . The submodule-based zero divisor graph of M with respect to N,  $\Gamma(M,N)$ , is an undirected graph with vertices  $Z^*(M,N)$  such that distinct vertices x and y are adjacent if and only if  $I_x I_y M \subseteq N$ .

The following example shows that Z(M/N) and Z(M,N) are different from each other.

EXAMPLE 2.2. Let  $M = \mathbb{Z} \oplus \mathbb{Z}$  and  $N = 2\mathbb{Z} \oplus 0$ . Then  $I_{(m,n)} = 0$ , for all  $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$ . But  $I_{(m,n)+N} = 2n\mathbb{Z}$  whenever  $m \in 2\mathbb{Z}$  and  $I_{(m,n)+N} = 2\mathbb{Z}$  whenever  $m \notin 2\mathbb{Z}$ . Thus  $(1,0),(1,1) \in Z^*(M,N)$  are adjacent in  $\Gamma(M,N)$ , but  $(1,0)+N,(1,1)+N \notin Z^*(M/N)$ .

**Proposition 2.3.** If  $Z^*(M, N) = \emptyset$ , then ann(M/N) is a prime ideal of R.

*Proof.* Suppose that  $\operatorname{ann}(M/N)$  is not prime. Then there are ideals I and J of R such that  $IJM \subset N$  but  $IM \not\subseteq N$  and  $JM \not\subseteq N$ . Let  $x \in IM \setminus N$  and  $y \in JM \setminus N$ . Then  $I_xJ_yM \subseteq IJM \subseteq N$  and  $I_y \subset R$ . Thus  $x \in Z^*(M,N)$ , a contradiction. Hence,  $\operatorname{ann}(M/N)$  is a prime ideal of R.

**Lemma 2.4.** Let  $x, y \in Z^*(M, N)$ . If x - y is an edge in  $\Gamma(M, N)$ , then for each  $0 \neq r \in R$ , either  $ry \in N$  or x - ry is also an edge in  $\Gamma(M, N)$ .

*Proof.* Let  $x, y \in Z^*(M, N)$  and  $r \in R$ . Assume that x - y is an edge in  $\Gamma(M, N)$  and  $ry \notin N$ . Then  $I_x I_y M \subseteq N$ . It is clear that  $I_{rx} \subseteq I_x$ . So that  $I_x I_{ry} M \subseteq I_x I_y M \subseteq N$  and therefore, x - ry is an edge in  $\Gamma(M, N)$ .

It is shown that the graphs are defined in [12] and [4], are connected with diameter less than or equal to three. Moreover, it shown that if those graphs contain a cycle, then they have the girth less than or equal to four. In the next theorems, we extend these results to a submodule-based zero divisor graph.

**Theorem 2.5.**  $\Gamma(M, N)$  is a connected graph and diam $(\Gamma(M, N)) \leq 3$ .

Proof. Let x and y be distinct vertices of  $\Gamma(M,N)$ . Then, there are  $a,b \in Z^*(M,N)$  with  $I_aI_xM \subseteq N$  and  $I_bI_yM \subseteq N$  (we allow  $a,b \in \{x,y\}$ ). If  $I_aI_bM \subseteq N$ , then x-a-b-y is a path, thus  $d(x,y) \leq 3$ . If  $I_aI_bM \nsubseteq N$ , then  $Ra \cap Rb \not\subseteq N$ , and for every  $d \in (Ra \cap Rb) \setminus N$ , x-d-y is a path of length 2,  $d(x,y) \leq 2$ , by Lemma 2.4. Hence, we conclude that  $diam(\Gamma(M,N)) \leq 3$ .  $\square$ 

**Theorem 2.6.** If  $\Gamma(M, N)$  contains a cycle, then  $gr(\Gamma(M, N)) \leq 4$ .

*Proof.* We have  $\operatorname{gr}(\Gamma(M,N)) \leq 7$ , by Proposition 1.3.2 in [7] and Theorem 2.5. Assume that  $x_1 - x_2 - \cdots - x_7 - x_1$  is a cycle in  $\Gamma(M,N)$ . If  $x_1 = x_4$  then it is clear that  $\operatorname{gr}(\Gamma(M,N)) \leq 3$ . So, suppose that  $x_1 \neq x_4$ . Then we have the following two cases:

Case 1. If  $x_1$  and  $x_4$  are adjacent in  $\Gamma(M, N)$ , then  $x_1 - x_2 - x_3 - x_4 - x_1$  is a cycle and  $gr(\Gamma(M, N)) \leq 4$ .

Case 2. Suppose that  $x_1$  and  $x_4$  are not adjacent in  $\Gamma(M, N)$ . Then  $I_{x_1}I_{x_4}M \nsubseteq N$  and so there is a  $z \in (Rx_1 \cap Rx_4) \setminus N$ . If  $z = x_1$ , then  $z \neq x_4$  and  $x_3 - x_4 - x_5 - z - x_3$  is a cycle in  $\Gamma(M, N)$ , by Lemma 2.4. If  $z \neq x_1$ , then by Lemma 2.4,  $x_1 - x_2 - z - x_7 - x_1$  is a cycle and  $\operatorname{gr}(\Gamma(M, N)) \leq 4$ .

For cycles with length 5 or 6, by using a similar argument as above, one can shows that  $gr(\Gamma(M, N)) \leq 4$ .

EXAMPLE 2.7. Assume that  $M = \mathbb{Z}$  and p, q are two prime numbers. If  $N = p\mathbb{Z}$ , then  $\Gamma(M, N) = \emptyset$ . If  $N = pq\mathbb{Z}$ , then  $\Gamma(M, N)$  is an infinite complete bipartite graph with vertex set  $V_1 \cup V_2$ , where  $V_1 = p\mathbb{Z} \setminus pq\mathbb{Z}$  and  $V_2 = q\mathbb{Z} \setminus pq\mathbb{Z}$  and so  $\operatorname{gr}(\Gamma(M, N)) = 4$ .

Corollary 2.8. If N is a prime submodule of M, then  $\operatorname{diam}(\Gamma(M,N)) \leq 2$  and  $\operatorname{gr}(\Gamma(M,N)) = 3$ , whenever it contains a cycle.

*Proof.* Let x,y be two distinct vertices which are not adjacent in  $\Gamma(M,N)$ . Thus there is an  $a \in M \setminus N$  such that  $I_aI_xM \subseteq N$ . Since N is a prime submodule, then  $I_aM \subseteq N$ . Thus  $I_aI_yM \subseteq N$ , and then x-a-y is a path in  $\Gamma(M,N)$ . Then  $\operatorname{diam}(\Gamma(M,N)) \leq 2$ .

**Lemma 2.9.** Let  $|\Gamma(M, N)| \geq 3$ ,  $\operatorname{gr}(\Gamma(M, N)) = \infty$  and  $x \in Z^*(M, N)$  with  $\operatorname{deg}(x) > 1$ . Then  $Rx = \{0, x\}$  and  $\operatorname{ann}(x)$  is a prime ideal of R.

*Proof.* First we show that  $Rx = \{0, x\}$ . Let u - x - v be a path in  $\Gamma(M, N)$ . Then u - v is not an edge in  $\Gamma(M, N)$  since  $\operatorname{gr}(\Gamma(M, N)) = \infty$ . If  $x \neq rx$  for some  $r \in R$  and  $rx \notin N$ , then by Lemma 2.4, rx - u - x - v - rx is a cycle in

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 $\Gamma(M,N)$ , that is a contradiction. So, for every  $r \in R$  either rx = x or  $rx \in N$ . If there is an  $r \in R$  such that  $rx \in N$ , then we have either  $(1+r)x \in N$  or (1+r)x = x. These imply that  $x \in N$  or rx = 0. Therefore, we have shown that  $Rx = \{0, x\}$ .

Let  $a, b \in R$  and abx = 0. Then bx = 0 or bx = x. Hence, bx = 0 or ax = 0. So, ann(x) is a prime ideal of R.

**Theorem 2.10.** If N is a nonzero submodule of M and  $gr(\Gamma(M, N)) = \infty$ , then  $\Gamma(M, N)$  is a star graph.

*Proof.* Suppose that  $\Gamma(M,N)$  is not a star graph. Then there is a path in  $\Gamma(M,N)$  such as u-x-y-v. By Lemma 2.9, we have  $Ry=\{0,y\}$  and by assumption u and y are not adjacent, thus  $I_yM\neq 0$ . So that  $I_yM=Ry$ . Also, x-y-v is a path, thus  $I_vI_yM\subseteq N$  and  $I_xI_yM\subseteq N$ . Hence,  $I_vRy\subseteq N$  and  $I_xRy\subseteq N$ . On the other hand, for every nonzero  $n\in N$ , we have

$$I_v I_{y+n} M \subseteq I_v R(y+n) \subseteq I_v (Ry+N) \subseteq N$$

and similarly  $I_x I_{y+n} M \subseteq N$ . So that x - y - v - (y + n) - x is a cycle in  $\Gamma(M, N)$ , a contradiction. Therefore,  $\Gamma(M, N)$  is a star graph.

**Theorem 2.11.** Let N be a nonzero submodule of M,  $|\Gamma(M, N)| \geq 3$  and  $\Gamma(M, N)$  is a star graph. Then the following statements are true:

- (i) If x is the center vertex, then  $I_x = \text{ann}(M)$ .
- (ii)  $\Gamma(M, N)$  is a subgraph of  $\Gamma(M)$ .

Proof. (i) By Lemma 2.9, we have  $Rx = \{0, x\}$ . Thus either  $I_xM = 0$  or  $I_xM = Rx$ . Assume that  $I_xM = Rx$ . If y is a vertex of  $\Gamma(M, N)$  such that  $y \neq x$ , then  $\deg(y) = 1$  and  $I_xI_yM \subseteq N$ . Thus  $I_yRx \subseteq N$ . Since  $I_{x+n}I_yM \subseteq I_yR(x+n) \subseteq N$  for every nonzero element  $n \in N$  it concludes that y = x + n. In this case, every other vertices of  $\Gamma(M, N)$  are adjacent to y, a contradiction. Hence,  $I_xM = 0$  and  $I_x = \operatorname{ann}(M)$ .

(ii) It is obvious. 
$$\Box$$

**Theorem 2.12.** If  $|N| \ge 3$  and  $\Gamma(M, N)$  is a complete bipartite graph which is not a star graph, then  $I_x^2M \subseteq N$ , for every  $x \in Z^*(M, N)$ .

Proof. Let  $Z^*(M,N) = V_1 \cup V_2$ , where  $V_1 \cap V_2 = \emptyset$ . Suppose that  $I_x^2M \subseteq N$  for some  $x \in Z^*(M,N)$ . Without loss of generality, we can assume that  $x \in V_1$ . By a similar argument with Lemma 2.9, either  $Rx = \{0,x\}$  or there is an  $r \in R$  such that  $x \neq rx$  and  $rx \in N$ . If  $Rx = \{0,x\}$ , then  $I_xM = Rx$ . Thus  $I_xRx \subseteq N$ . Now, for every  $y \in V_2$  and  $n \in N$  we get

$$I_y I_{x+n} M \subseteq I_y R(x+n) \subseteq I_y (Rx+N) \subseteq N$$

and  $I_xI_{x+n}M\subseteq N$ . Then,  $x+n\in V_1\cap V_2$ , a contradiction. So, assume that  $x\neq rx$  and  $rx\in N$  for some  $r\in R$ . Since  $I_{rx+x}\subseteq I_x$ , then  $I_xI_{rx+x}M\subseteq N$  and for all  $y\in V_2$ ,  $I_yI_{rx+x}M\subseteq N$ . Thus  $rx+x\in V_1\cap V_2$ , a contradiction.  $\square$ 

An R-module X is called a *multiplication-like* module if, for each nonzero submodule Y of X,  $\operatorname{ann}(X) \subset \operatorname{ann}(X/Y)$ . Multiplication-like module have been studied in [8, 13].

A vertex x of a connected graph G is a *cut-point*, if there are vertices u, v of G such that x is in every path from u to v and  $x \neq u, x \neq v$ . For a connected graph G, an edge E of G is defined to be a *bridge* if  $G - \{E\}$  is disconnected, see [6].

**Theorem 2.13.** Let M be a multiplication-like module and N be a nonzero submodule of M. Then  $\Gamma(M, N)$  has no cut-points.

*Proof.* Suppose that x is a cut-point of  $\Gamma(M,N)$ . Then there exist vertices  $u,v\in M\setminus N$  such that x lies on every path from u to v. By Theorem 2.5, the shortest path from u to v has length 2 or 3.

Case 1. Suppose that u-x-v is a path of shortest length from u to v. Since x is a cut point x, u, v aren't in a cycle. By a similar argument to that of Lemma 2.9, we have  $Rx = \{0, x\}$ . On the other hand,  $I_xM \subseteq Rx$  and M is a multiplication-like module, so we have  $I_xM = Rx$ . Hence  $I_uRx \subseteq N$  and  $I_vRx \subseteq N$ . Also, for every nonzero  $n \in N$ , we have  $I_uI_{x+n}M \subseteq I_u(Rx+N) \subseteq N$  and  $I_vI_{x+n}M \subseteq N$ . Therefore, u-(x+n)-v is a path from u to v, a contradiction.

Case 2. Suppose that u-x-y-v is a path in  $\Gamma(M,N)$ . Then, we have  $I_xM=Rx$  and for every nonzero  $n \in N$ , we have  $I_yI_{x+n}M \subseteq N$  and  $I_uI_{x+n}M \subseteq N$ . Thus u-(x+n)-y-v is a path from u to v, a contradiction.

**Theorem 2.14.** Let M be a multiplication-like module and N be a nonzero submodule of M. Then  $\Gamma(M,N)$  has a bridge if and only if  $\Gamma(M,N)$  is a graph on two vertices.

Proof. If  $|\Gamma(M,N)| = 3$ , then  $\Gamma(M,N) = K^3$ , by Theorem 2.11, and it has no bridge. Assume that  $|\Gamma(M,N)| \ge 4$  and x-y is a bridge. Thus there is not a cycle containing x-y. Without loss of generality, we can assume that  $\deg(x) > 1$ . Thus, there exists a vertex  $z \ne y$  such that z-x is an edge of  $\Gamma(M,N)$ . Then  $Rx = \{0,x\}$  and  $I_xM = Rx$ . Hence, for every  $n \in N$ ,  $I_zI_{x+n}M \subseteq N$  and  $I_yI_{x+n}M \subseteq N$ , a contradiction. Therefore,  $\Gamma(M,N)$  has not a bridge. The converse is clear.

## 3. Submodule-based Zero Divisor Graph of Semisimple Modules

A nonzero R-module X is called simple if its only submodules are (0) and X. An R-module X is called semisimple if it is a direct sum of simple modules. Also, X is called homogenous semisimple if it is a direct sum of isomorphic simple modules.

In this section, R is a commutative ring and M is a finitely generated semisimple R-module such that its homogenous components are simple and

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N is a submodule of M. The following theorem has a crucial role in this section.

**Theorem 3.1.** Let  $x, y \in M \setminus N$ . Then x, y are adjacent in  $\Gamma(M, N)$  if and only if  $Rx \cap Ry \subseteq N$ .

Proof. Let  $M=\bigoplus_{i\in I}M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of M. By assumption N is a submodule of M, so there exists a subset A of I such that  $M=N\oplus(\bigoplus_{i\in A}M_i)$  and so  $\operatorname{ann}(M/N)=\operatorname{ann}(\bigoplus_{i\in A}M_i)=\bigcap_{i\in A}\operatorname{ann}(M_i)$ . Assume that  $x,y\in M\setminus N$  are adjacent in  $\Gamma(M,N)$  and  $Rx\cap Ry\not\subseteq N$ . Thus there exists  $\alpha\in I$  such that  $M_\alpha\subseteq(Rx\cap Ry)\setminus N$ . Also, there exist subsets  $B\subset I$  and  $C\subset I$  such that  $M=Rx\oplus(\bigoplus_{i\in B}M_i)$  and  $M=Ry\oplus(\bigoplus_{i\in C}M_i)$ . Therefore,  $I_x=\bigcap_{i\in B}\operatorname{ann}(M_i)$  and  $I_y=\bigcap_{i\in C}\operatorname{ann}(M_i)$ . Since  $I_xI_yM\subseteq N$ , we have  $I_xI_y\subseteq\operatorname{ann}(M/N)$ . For every  $i,j\in I$ ,  $\operatorname{ann}(M_i)$  and  $\operatorname{ann}(M_j)$  are coprime, then

$$I_x I_y = \left[\bigcap_{i \in B} \operatorname{ann}(M_i)\right] \left[\bigcap_{i \in C} \operatorname{ann}(M_i)\right] = \prod_{i \in B \cup C} \operatorname{ann}(M_i)$$

$$\subseteq \bigcap_{i \in A} \operatorname{ann}(M_i) \subseteq \operatorname{ann}(M_r),$$

for all  $r \in A$ . Thus for any  $r \in A$  there exists  $j_r \in B \cup C$  such that  $\operatorname{ann}(M_{j_r}) \subseteq \operatorname{ann}(M_r)$ . So that  $\operatorname{ann}(M_{j_r}) = \operatorname{ann}(M_r)$  implies that  $M_{j_r} \cong M_r$  and by hypothesis  $M_{j_r} = M_r$ . Hence,

$$M_{\alpha} \subseteq \bigoplus_{i \in A} M_i \subseteq \bigoplus_{i \in B \cup C} M_i$$
.

Thus there exists  $\gamma \in B \cup C$  such that  $M_{\alpha} = M_{\gamma}$ , also

$$M_{\alpha} \subseteq Rx \cap Ry = (\bigoplus_{i \in I \setminus B} M_i) \cap (\bigoplus_{i \in I \setminus C} M_i).$$

Therefore,  $\alpha \in I \setminus (B \cup C)$ , a contradiction. The converse is obvious.

**Corollary 3.2.** Let  $x, y \in M \setminus N$  be such that  $x + N \neq y + N$ . Then

- (i) x and y are adjacent in  $\Gamma(M,N)$  if and only if x+N and y+N are adjacent in  $\Gamma(M/N)$ .
- (ii) if x and y are adjacent in  $\Gamma(M, N)$ , then all distinct elements of x + N and y + N are adjacent in  $\Gamma(M, N)$ .

*Proof.* (i) Let  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of M. Suppose that x and y are adjacent in  $\Gamma(M, N)$ ,  $Rx = \bigoplus_{i \in A} M_i$ ,  $Ry = \bigoplus_{i \in B} M_i$  and  $N = \bigoplus_{i \in C} M_i$ . Then  $Rx + N = \bigoplus_{i \in A \cup C} M_i$  and  $Ry + N = \bigoplus_{i \in B \cup C} M_i$ . Thus,

$$(Rx+N)\cap (Ry+N)=\bigoplus_{i\in (A\cup C)\cap (B\cup C)}M_i=\bigoplus_{i\in (A\cap B)\cup C}M_i=(Rx\cap Ry)+N.$$

By Theorem 3.1, we have  $Rx \cap Ry \subseteq N$  hence,

$$I_{x+N}I_{y+N}M \subset (Rx+N) \cap (Ry+N) = (Rx \cap Ry) + N = N.$$

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Therefore, x+N and y+N are adjacent in  $\Gamma(M/N)$ . The converse is obvious. (ii) Let  $x,y\in Z^*(M,N)$  be adjacent in  $\Gamma(M,N)$ . Then  $Rx\cap Ry\subseteq N$  by Theorem 3.1. So for every  $n,n'\in N$  we have

$$I_{x+n}I_{y+n'}M \subseteq R(x+n) \cap R(y+n') \subseteq (Rx+N) \cap (Ry+N) = N.$$

Hence, x + n and y + n' are adjacent in  $\Gamma(M, N)$ .

In the following theorem, we prove that the clique number of graphs  $\Gamma(M,N)$  and  $\Gamma(M/N)$  are equal.

**Theorem 3.3.** If N is a nonzero submodule of M, then  $\omega(\Gamma(M/N)) = \omega(\Gamma(M,N))$ .

*Proof.* First we show that  $I_{m+N}^2M \not\subseteq N$  for each  $0 \neq m+N \in M/N$ . Assume that  $N=\bigoplus_{i\in A}M_i$  and  $m=(m_i)_{i\in I}\in M\setminus N$ . Then  $I_{m+N}=\bigcap_{i\not\in A,m_i=0}\operatorname{ann}(M_i)$ . Hence,  $I_{m+N}=I_{m+N}^2$ . Thus  $I_{m+N}^2M\not\subseteq N$  since there is at least one  $j\in I\setminus A$  such that  $m_j\neq 0$ .

Now, Corollary 3.2 implies that  $\omega(\Gamma(M/N)) \leq \omega(\Gamma(M,N))$ . Thus, it is enough to consider the case where  $\omega(\Gamma(M/N)) = d < \infty$ . Assume that G is a complete subgraph of  $\Gamma(M,N)$  with vertices  $m_1, m_2, \cdots, m_{d+1}$ , we provide a contradiction. Consider the subgraph  $G_*$  of  $\Gamma(M/N)$  with vertices  $m_1 + N, \cdots, m_{d+1} + N$ . By Corollary 3.2,  $G_*$  is a complete subgraph of  $\Gamma(M,N)$ . Thus  $m_j + N = m_k + N$  for some  $1 \leq j,k \leq d+1$  with  $j \neq k$  since  $\omega(\Gamma(M/N)) = d$ . We have  $I_{m_j}I_{m_k}M \subseteq N$ . Therefore,  $Rm_j \cap Rm_k \subseteq N$  and so  $I_{m_j+N}I_{m_k+N}M \subseteq N$ . Hence,  $I_{m_j+N}^2M \subseteq N$ , that is a contradiction.  $\square$ 

In the following theorem, we show that there is a relation between  $\omega(\Gamma(M,N))$  and  $\chi(\Gamma(M,N))$ .

**Theorem 3.4.** Assume that  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of M and  $N = \bigoplus_{i \in A} M_i$  is a submodule of M for some  $A \subset I$ . Then  $\omega(\Gamma(M, N)) = \chi(\Gamma(M, N)) = |I| - |A|$ .

*Proof.* Suppose that  $I \setminus A = \{1, \dots, n\}$  so  $M_1, \dots, M_n \not\subseteq N$ . Let for  $1 \leq k \leq n-1$ 

$$L^k = \{ m \in M : m \text{ has } k \text{ nonzero components } \}$$

and let for  $1 \le s \le n$ 

$$L_s^1 = \{ m \in L^1 : \text{the } s^{\text{th}} \text{ component of } m \text{ is nonzero} \}.$$

If  $m \in L^1_s$  and  $m' \in L^1_t$  for some  $1 \leq s, t \leq n$  with  $s \neq t$ , then m and m' are adjacent and so  $K^n$  is a subgraph of  $\Gamma(M,N)$ . Thus  $\omega(\Gamma(M,N)) \geq n$ . If  $m,m' \in L^1_s$  for some  $1 \leq s \leq n$ , then m,m' are not adjacent because  $\operatorname{ann}(M_s) \not\subseteq I_m I_{m'}$  and so the elements of  $L^1_s$  have same color. On the other hand, if  $x \in L^t$  with t > 1, then there is not a complete subgraph  $K^h$  of  $\Gamma(M,N)$  containing x, such that  $h \geq n$ . Thus  $\omega(\Gamma(M,N)) = n \leq \chi(\Gamma(M,N))$ . Also, if  $x \in L^t$  with t > 1, then there is an s with  $1 \leq s \leq n$  such that x is not

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adjacent to each element of  $L_s^1$ . Thus the color of x is same as the elements of  $L_s^1$ . Thus  $\chi(\Gamma(M,N))=n$ .

The Kuartowski's Theorem states: A graph G is planar if and only if it contains no subgraph homeomorphic to  $K^5$  or  $K^{3,3}$ .

**Theorem 3.5.** Let N be a nonzero proper submodule of M such that N is not prime. Then  $\Gamma(M, N)$  is not planar.

Proof. Assume that  $M=\bigoplus_{i\in I}M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of M and  $N=\bigoplus_{i\in A}M_i$  for some  $A\subset I$ . Let  $I\setminus A=\{i,j\}$ . Then  $\Gamma(M,N)$  is a complete bipartite graph  $K^{n,m}$ , where  $n=(|M_i|-1)(\prod_{k\in I-\{i,j\}}|M_k|)$  and  $m=(|M_j|-1)(\prod_{k\in I-\{i,j\}}|M_k|)$ . By hypotheses N is a nonzero and  $M_i$ 's are non-isomorphic, so we have  $n,m\geq 3$ . Hence  $\Gamma(M,N)$  has a subgraph homeomorphic to  $K^{3,3}$ . The cases  $|I\setminus A|\geq 3$  are similar to that of the case  $|I\setminus A|=2$ .

**Theorem 3.6.** A nonzero submodule N of M is prime if and only if  $Z^*(M, N) = \emptyset$ .

Proof. Let  $M=\bigoplus_{i\in I}M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of M and N is prime. Then  $N=\bigoplus_{i\in I\setminus\{k\}}M_i$ , for some  $k\in I$ . If  $x\in Z^*(M,N)$ , then there exists a  $y\in M\setminus N$  such that  $I_xI_yM\subseteq N$ . If  $x\neq y$ , then  $Rx\cap Ry\subseteq N$ , by Theorem 3.1. Thus either  $M_k\not\subseteq Rx$  or  $M_k\not\subseteq Ry$ . Hence, either  $Rx\subseteq N$  or  $Ry\subseteq N$ , a contradiction. Now, suppose that x=y so by  $I_x^2M\subseteq N$  and hypotheses  $I_xM\subseteq N$ . Thus  $I_{x+n}I_xM\subseteq N$  for every  $0\neq n\in N$ . By a similar argument, we have either  $x\in N$  or  $x+n\in N$ , a contradiction. Hence,  $Z^*(M,N)=\emptyset$ .

Conversely, assume that  $Z^*(M,N) = \emptyset$ . Then  $\operatorname{ann}(M/N)$  is prime ideal of R by Proposition 2.3 and there exists a  $k \in I$  such that  $\operatorname{ann}(M/N) = \operatorname{ann}(M_k)$ . Hence,  $N = \bigoplus_{i \in I \setminus \{k\}} M_i$  is a prime submodule of M.

A proper submodule N of M is called 2-absorbing if whenever  $a,b \in R$ ,  $m \in M$  and  $abm \in N$ , then  $am \in N$  or  $bm \in N$  or  $ab \in \text{ann}(M/N)$ , see [10, 11]. In the following results, we study the behavior of  $\Gamma(M,N)$  whenever N is a 2-absorbing submodule of M.

**Theorem 3.7.** A submodule N of M is 2-absorbing if and only if at most two components of M are zero in N.

Proof. Let  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of M. Suppose that N is a 2-absorbing submodule of M and  $N = \bigoplus_{i \in A} M_i$ , where  $A = I \setminus \{s, t, k\}$ . Since for all  $i \in I$ ,  $\operatorname{ann}(M_i)$  is prime, there are  $a \in \operatorname{ann}(M_s) \setminus (\operatorname{ann}(M_t) \cup \operatorname{ann}(M_k))$ ,  $b \in \operatorname{ann}(M_t) \setminus (\operatorname{ann}(M_s) \cup \operatorname{ann}(M_k))$  and  $c \in \bigcap_{j \in I \setminus (A - \{s, t\})} \operatorname{ann}(M_j) \setminus (\operatorname{ann}(M_s) \cup \operatorname{ann}(M_t))$ . Now,  $abc \in \operatorname{ann}(M/N)$  but  $ab \notin \operatorname{ann}(M/N)$ ,  $ac \notin \operatorname{ann}(M/N)$  and  $bc \notin \operatorname{ann}(M/N)$ . This contradict with

Theorem 2.3 in [10]. Thus  $|A| \ge |I| - 2$  and at most two components of M are zero in N.

Conversely, if one component of M is zero in N, then N is a prime submodule of M. Suppose that  $N = \bigoplus_{i \in A} M_i$ , where  $A = I \setminus \{i, j\}$ . Thus  $M_i, M_j \not\subseteq N$ . Suppose that  $a, b \in R$ ,  $(m_i)_{i \in I} = m \in M \setminus N$  and  $abm \in N$ . Then either  $m_i \neq 0$  or  $m_j \neq 0$ . If  $m_i \neq 0$  and  $m_j \neq 0$ , then  $ab \in \operatorname{ann}(M_i) \cap \operatorname{ann}(M_j) = \operatorname{ann}(M/N)$ . If  $m_i \neq 0$  and  $m_j = 0$ , then  $ab \in \operatorname{ann}(M_i)$  and so either  $a \in \operatorname{ann}(M_i)$  or  $b \in \operatorname{ann}(M_i)$ . Hence,  $am \in N$  or  $bm \in N$ . The case  $m_i = 0$  and  $m_j \neq 0$ , is similar to the previous case. Therefore, N is a 2-absorbing submodule of M.

**Theorem 3.8.** N is a 2-absorbing submodule of M if and only if  $Z^*(M, N) = \emptyset$  or  $\Gamma(M, N)$  is a complete bipartite graph.

Proof. Let N be a 2-absorbing submodule of M. If N is prime, then  $Z^*(M,N)=\emptyset$ , by Theorem 3.6. Now, assume that  $N=\bigoplus_{i\in I\setminus\{j,k\}}M_i$  for some  $j,k\in I$  and  $(m_i)_{i\in I}=m\in M\setminus N$ . Thus  $I_m=\bigcap_{\{i\in I: m_i=0\}}\operatorname{ann}(M_i)$ . If  $m_j\neq 0$  and  $m_k\neq 0$ , then  $m\notin Z(M,N)$ . Let  $V_1=\{(m_i)_{i\in I}\in M\setminus N: m_j=0\}$  and  $V_2=\{(m_i)_{i\in I}\in M\setminus N: m_k=0\}$ . Thus m-m' is an edge of  $\Gamma(M,N)$  for every  $m\in V_1$  and  $m'\in V_2$ . Also, every vertices in  $V_1$  and  $V_2$  are not adjacent. Hence,  $\Gamma(M,N)$  is a complete bipartite graph.

Now, suppose that  $\Gamma(M,N)$  is a complete bipartite graph and N is not 2-absorbing. By Theorem 3.7, there are at least three components  $M_s, M_t, M_k$  such that  $M_s, M_t, M_k \not\subseteq N$ . For i = s, t, k let  $v_i = (m_i)_{i \in I}$ , where  $m_i \neq 0$  and  $m_j = 0$  for all  $j \neq i$ . Then  $v_s - v_t - v_k - v_s$  is a cycle in  $\Gamma(M,N)$ . Thus  $\operatorname{gr}(\Gamma(M,N)) = 3$  and so  $\Gamma(M,N)$  is not bipartite graph, by Theorem 1 of Sec. 1.2 in [5]. Hence, N is a 2-absorbing submodule of M.

EXAMPLE 3.9. Let  $M = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$ . Then every nonzero submodule N of M is 2-absorbing. Thus either  $Z^*(M,N) = \emptyset$  or  $\Gamma(M,N)$  is a complete bipartite graph. In particular, if  $N = \mathbb{Z}_7$ , then  $\Gamma(M,N) = K^{7,28}$ .

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