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# $z_R$ -Ideals and $z_R^{\circ}$ -Ideals in Subrings of $\mathbb{R}^X$

Ali Rezaei Aliabad and Mehdi Parsinia\*

Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran.

> E-mail: aliabady\_r@scu.ac.ir E-mail: parsiniamehdi@gmail.com

ABSTRACT. Let X be a topological space and R be a subring of  $\mathbb{R}^X$ . By determining some special topologies on X associated with the subring R, characterizations of maximal fixed and maximal g-ideals in R of the form  $M_x(R)$  are given. Moreover, the classes of  $z_R$ -ideals and  $z_R^{\circ}$ -ideals are introduced in R which are topological generalizations of z-ideals and  $z^{\circ}$ -ideals of C(X), respectively. Various characterizations of these ideals are established. Also, coincidence of  $z_R$ -ideals with z-ideals and  $z_R^{\circ}$ -ideals with  $z^{\circ}$ -ideals in R are investigated. It turns out that some fundamental statements in the context of C(X) are extended to the subrings of  $\mathbb{R}^X$ .

**Keywords:** Z(R)-topology, Coz(R)-topology, g-ideal,  $z_R$ -ideal,  $z_R$ -ideal, invertible subring.

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### 1. Introduction

For a topological space X,  $\mathbb{R}^X$  denotes the algebra of all real-valued functions and C(X) (resp.,  $C^*(X)$ ) denotes the subalgebra of  $\mathbb{R}^X$  consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use R to denote a unital subring of  $\mathbb{R}^X$ . Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each  $f \in \mathbb{R}^X$ ,

<sup>\*</sup>Corresponding Author

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 $Z(f) = \{x \in X : f(x) = 0\}$  denotes the zero-set of f and Coz(f) denotes the complement of Z(f) with respect to X. We denote by Z(R) the collection of all the zero-sets of elements of R, we use Z(X) instead of Z(C(X)). We denote by  $M_x(R)$  the set  $\{f \in R : x \in Z(f)\}, M_x(C(X))$  is denoted by  $M_x$ . The subring R is called invertible, if  $f \in R$  and  $Z(f) = \emptyset$  implies that f is invertible in R. Moreover, R is called a lattice-ordered subring if it is a sublattice of  $\mathbb{R}^X$  (i.e.,  $f \wedge g$  and  $f \vee g$  are in R for each  $f, g \in R$ ). It is clear that C(X) is an invertible lattice-orderd subring of  $\mathbb{R}^X$ . However, the same statement does not hold for  $C^*(X)$ . A proper ideal I of R is called a growing ideal, briefly, a g-ideal, if contains no invertible element of  $\mathbb{R}^X$ , i.e.,  $Z(f) \neq \emptyset$ for each  $f \in I$ . It is evident that a subring R is invertible if and only if every ideal every ideal of R is a g-ideal. Clearly,  $M^{*p}$ , for each  $p \in \beta X \setminus vX$ , is not a g-ideal of  $C^*(X)$ . An ideal I of R is called fixed if  $\bigcap_{f\in I} Z(f) \neq \emptyset$ , otherwise, it is called free. By a maximal fixed ideal of R, we mean a fixed ideal which is maximal in the set of all fixed ideals of R. An ideal I in a commutative ring S is called a z-ideal (resp.,  $z^{\circ}$ -ideal) if  $M_a(S) \subseteq I$  (resp.,  $P_a(S) \subseteq I$ ), for each  $a \in I$ , where  $M_a(S)$  (resp.,  $P_a(S)$ ) denotes the intersection of all the maximal (resp., minimal prime) ideals of S containing a. It is well-known that in C(X)an ideal I is a z-ideal (resp.,  $z^{\circ}$ -ideal) if and only if whenever  $Z(f) \subseteq Z(g)$ (resp.,  $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$ ),  $f \in I$  and  $g \in C(X)$ , then  $g \in I$ .

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on X which the subring R generate, namely, Z(R)-topology and Coz(R)-topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in R, specially, maximal fixed and maximal g-ideals. Using the Z(R)-topology, characterizations of maximal fixed ideals of R, which are of the form  $M_x(R)$ , are given. Moreover, relations between mapping " $x \longrightarrow M_x(R)$ " and the separation properties of the topological space  $(X, \tau_{Z(R)})$  will be found. In section 3, we introduce the notion of  $z_R$ -ideal in a subring R as a natural topological generalization of the notion of z-ideal in C(X). Various characterizations of these ideals via Z(R)-topology are given and relations between  $z_R$ -ideals and z-ideals in R (by their algebraic descriptions) are discussed. Section 4 deals with  $z_R^{\circ}$ -ideals of R which are natural topological generalizations of  $z^{\circ}$ -ideals of

**Definition 1.1.** For each subring R of  $\mathbb{R}^X$ , clearly, Z(R) and Coz(R) constitute bases for some topologies on X. The induced topologies are called Z(R)-topology and Coz(R)-topology, respectively, and are denoted by  $\tau_{Z(R)}$  and  $\tau_{Coz(R)}$ , respectively.

In the next three statements we compare these topologies. Note that two subsets  $S_1, S_2$  of  $\mathbb{R}^X$  are called zero-set equivalent, if  $Z(S_1) = Z(S_2)$ .

**Proposition 1.2.** Let R be a subring of  $\mathbb{R}^X$ , if S and  $C(\mathbb{R})$  are zero-set equivalent subsets of  $\mathbb{R}^{\mathbb{R}}$  and gof  $\in R$  for each  $f \in R$  and each  $g \in S$ , then  $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$  and the equality does not hold, in general.

*Proof.* We are to show that  $Coz(R) \subseteq \tau_{Z(R)}$ . If  $x \notin Z(f)$  where  $f \in R$ , then there is a g in S such that  $f(x) \in Z(g)$  and  $f^{-1}(Z(g)) \cap Z(f) = \emptyset$ . Therefore,  $gof \in R, x \in Z(gof) \text{ and } Z(gof) \cap Z(f) = \emptyset \text{ which proves the inclusion. Now,}$ we show that the inclusion may be proper. Let  $(X, \tau_X)$  be a Tychonoff space which has at least one non-open zero-set Z. Set R = C(X), then  $\tau_{Coz(R)} = \tau_X$ , whereas  $Z \notin \tau_X$  and hence,  $\tau_{Coz(R)} \subsetneq \tau_{Z(R)}$ .

Proof of the following proposition is standard.

**Proposition 1.3.** The following statements are equivalent.

- (a)  $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$ .
- (b) Every  $Z \in Z(R)$  is clopen under Z(R)-topology.

The annihilator of  $f \in R$  in R is defined to be the set  $\{g \in R : fg = 0\}$  and is denoted by  $Ann_R(f)$ . A simple reasoning shows that if X is equipped with the Coz(R)-topology, then  $Ann_R(f) = \{g \in R : Coz(g) \subseteq int_X Z(f)\} = \{g \in R : Coz(g) \subseteq int_X Z(f)\}$  $R: \operatorname{cl}_X(Coz(g)) \subseteq Z(f)$ .

**Proposition 1.4.** The following statements are equivalent.

- (a)  $\tau_{Z(R)} \subseteq \tau_{Coz(R)}$ .
- (b) Z(f) is clopen in  $(X, \tau_{Coz(R)})$  for every  $f \in R$ .
- (c) For each  $f \in R$ ,  $Z(f) = \bigcup_{g \in Ann_R(f)} Coz(g)$ .
- (d) For each  $f \in R$ ,  $(Ann_R(f), f)$  is a free ideal.

*Proof.* The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

- (c) $\Rightarrow$ (d). This clear by the hypothesis and the fact that whenever  $f \in R$ and I is an ideal of R, then  $\bigcap_{h\in(I,f)}Z(h)=\bigcap_{g\in I}(Z(f)\cap Z(g)).$
- (d) $\Rightarrow$ (a). Let  $f \in R$  and  $x \in Z(f)$ . By (d), there exists  $g \in Ann_R(f)$ such that  $x \notin Z(f) \cap Z(g)$ . Hence,  $x \notin Z(g)$  and  $x \in Coz(g) \subseteq Z(f)$  and so  $Z(f) \in \tau_{Coz(R)}$ .

An immediate consequence of Propositions 1.3 and 1.4 is that  $\tau_{Coz(R)}$  $\tau_{Z(R)}$  if and only if Z(f) is clopen under both Z(R)-topology and Coz(R)topology, for each  $f \in R$ .

### 2. Characterization of Maximal Fixed Ideals in Subrings

We remind that maximal fixed ideals of C(X) coincide with its fixed maximal ideals and are of the form  $M_x = \{ f \in C(X) : f(x) = 0 \}$ , where  $x \in X$ . This fact is generalized for some special subalgebras of C(X), such as intermediate subalgebras (subalgebras of C(X) containing  $C^*(X)$ , see [7]),  $C_c(X)$  (the subalgebra of C(X) consisting of all functions with countable image, see [9]) and the subalgebras of the form  $\mathbb{R} + I$  where I is an ideal of C(X), see [13].

We will show that the same statement does not hold for arbitrary subrings of  $\mathbb{R}^X$ , in general.

- Remark 2.1. (a) Every maximal fixed ideal and fixed maximal ideal of R is of the form  $M_x(R) = \{f \in R : f(x) = 0\}$  for some  $x \in X$ . However, parts (1) and (2) of Example 2.2 show that the ideals  $M_x(R)$  are not necessarily maximal ideals or even maximal fixed ideals in R.
- (b) Every fixed maximal ideal is both a maximal fixed ideal and a maximal g-ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.
  - (c) A maximal fixed ideal need not be a maximal g-ideal, see Example 2.3.
  - (d) Every fixed maximal g-ideal is a maximal fixed ideal.
- EXAMPLE 2.2. (1) Let X be a Tychonoff space,  $x \in X$  and  $R = \mathbb{Z} + M_x$ . Then  $M_x(R) = M_x$  is not a maximal ideal in R, since  $2\mathbb{Z} + M_x$  is a proper ideal of R and  $M_x \subseteq 2\mathbb{Z} + M_x$ . Therefore,  $M_x(R)$  is a maximal fixed ideal and a maximal g-ideal which is not a maximal ideal.
- (2) Let X be a topological space with more than one point and  $a \in X$ . Also, let  $t \in \mathbb{R}$  be a transcendental number and define  $f: X \longrightarrow \mathbb{R}$  by f(a) = 0 and f(x) = t, for every  $x \neq a$ . Set  $R = \{\sum_{i=0}^{n} m_i f^i : n \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z}\}$ . Evidently,  $M_a(R) = \{f\}$  and  $M_x(R) = \{0\}$ , for every  $x \neq a$ . Therefore,  $M_x(R)$  is not a maximal fixed ideal for any  $x \neq a$ .

In the next example we construct a subring R such that, for some  $x \in X$ ,  $M_x(R)$  is a maximal fixed ideal which is not a maximal g-ideal.

EXAMPLE 2.3. Let  $X = \mathbb{R}$ ,  $a \in \mathbb{R} \setminus \mathbb{Q}$ ,  $b \in \mathbb{R} \setminus \{0\}$  and t be a transcendental number. For every  $\epsilon > 0$ , define  $f_{\epsilon} : X \longrightarrow \mathbb{R}$  by  $f_{\epsilon}(x) = 0$ , if  $|x - a| < \epsilon$  and  $f_{\epsilon}(x) = b$ , if  $|x - a| \ge \epsilon$ . Also, define  $f : X \longrightarrow \mathbb{R}$  by f(x) = 0, if  $x \in \mathbb{Q}$  and f(x) = t, if  $x \in \mathbb{R} \setminus \mathbb{Q}$ . Let R be the algebra over  $\mathbb{Q}$  generated by  $\{f_{\epsilon} : \epsilon > 0\} \cup \{f, 1\}$ . Evidently, R is a subring of  $\mathbb{R}^{X}$ , and  $M_{a}(R)$  equales to  $(f_{a})$  which is not a maximal ideal. It is easy to see that  $M_{a}(R)$  is a maximal fixed ideal and  $M_{a}(R) = I$ , where I is the ideal generated by  $\{f_{\epsilon} : \epsilon > 0\}$ . Clearly,  $Z(f) \cap Z(g) \neq \emptyset$ , for all  $g \in I$ . Hence J = (I, f) is a g-ideal which strictly contains I. Therefore, I is not a maximal g-ideal.

**Proposition 2.4.** The following statements hold for a subring R of  $\mathbb{R}^X$ .

- (a)  $M_x(R)$  is a maximal g-ideal if and only if whenever  $Z \in Z(R)$  and  $x \notin Z$ , then  $x \notin cl_{\tau_{Z(R)}}Z$ .
- (b) For each  $x \in X$ ,  $M_x(R)$  is a maximal g-ideal if and only if every  $Z \in Z(R)$  is clopen under Z(R)-topology.

*Proof.* (a  $\Rightarrow$ ). Let  $f \in R$  and  $x \notin Z(f)$ , thus, the ideal  $(M_x(R), f)$  contains an invertible element of  $\mathbb{R}^X$ . Hence, there are  $g \in M_x(R)$  and  $h \in R$  such that  $Z(g + fh) = \emptyset$ . Consequently,  $x \in Z(g)$  and  $Z(f) \cap Z(g) = \emptyset$ .

(a  $\Leftarrow$ ). Assume that  $f \notin M_x(R)$ . Then there is some  $g \in R$  such that  $x \in Z(g)$  and  $Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset$ . Hence,  $(M_x(R), f)$  contains an invertible element of  $\mathbb{R}^X$ . Also, clearly,  $M_x(R)$  is a g-ideal. Thus,  $M_x(R)$  is a maximal g-ideal.

(b). An easy consequence of (a).

**Corollary 2.5.** If  $M_x(R)$  is a maximal ideal for each  $x \in X$ , then every  $Z \in Z(R)$  is clopen under Z(R)-topology.

Corollary 2.6. Let R be an invertible subring. Then every  $Z \in Z(R)$  is clopen under Z(R)-topology if and only if  $M_x(R)$  is a maximal ideal for each  $x \in X$ .

*Proof.* By our hypothesis and Proposition 2.4, this is clear.

The following lemma is a restatement of the fact that the transcendental degree of  $\mathbb{R}$  over  $\mathbb{Q}$  is unountable, see [14].

**Lemma 2.7.** Let  $S = \mathbb{Q}[y_1, ..., y_n]$  be the ring of n-variable polynomials with rational coefficients. Then there exists an uncountable set X of transcendental numbers for which  $F(a_1, \cdots, a_n) \neq 0$ , for every distinct elements  $a_1, \cdots, a_n$  of X and every  $F \in S$ .

The following example shows that the converse of Corollary 2.5 does not hold, in general.

EXAMPLE 2.8. Let S be the polynomial ring  $\mathbb{Q}[y_1,...,y_n]$ , where  $n \in \mathbb{N}$  and n > 1. By Lemma 2.7, there exists an infinite set of transcendental numbers X for which  $F(a_1, \dots, a_n) \neq 0$ , for every  $a_1, \dots, a_n \in X$  and every  $F \in S$ . For each  $a \in X$ , define the function  $f_a : X \longrightarrow \mathbb{R}$  by  $f_a(a) = 0$  and  $f_a(x) = x$  for each  $x \neq a$ . Now, set

$$R = \{ F(f_{a_1}, ..., f_{a_n}) : F \in S, n \in \mathbb{N}, a_1, ..., a_n \in X \}.$$

Hence,  $M_a(R) = (f_a)$ , for each  $a \in X$ , which is not a maximal ideal. However, every  $Z \in Z(R)$  is clopen under Z(R)-topology.

**Proposition 2.9.** If R is a subalgebra of  $\mathbb{R}^X$ , then  $M_x(R)$  is a maximal g-ideal and a maximal fixed ideal for every  $x \in X$ .

*Proof.* It suffices to prove that every element of Z(R) is closed under Z(R)-topology. To this aim, suppose that  $a \in X$  and  $a \notin Z(f)$ , for some  $f \in R$ . Put g = f - f(a). Clearly,  $Z(g) \in Z(R)$ ,  $a \in Z(g)$  and  $Z(g) \cap Z(f) = \emptyset$ .

**Corollary 2.10.** If R is an invertible subalgebra of  $\mathbb{R}^X$ , then  $M_x(R)$  is a maximal ideal for each  $x \in X$ .

The converse of Corollary 2.10 does not hold, in general. For example, let R denote the collection of all single variable polynomials over  $\mathbb{R}$ . Then,  $M_r(R)$  is the maximal ideal (x-r) for each  $r \in \mathbb{R}$ . However,  $f = x^2 + 1$  is invertible in

 $\mathbb{R}^{\mathbb{R}}$  which is not invertible in R. Note that the subalgebras  $C_c(X)$  and  $\mathbb{R}+I$ , for each ideal I in C(X), satisfy Corollary 2.10 and so  $M_x(C_c(X))$  and  $M_x(\mathbb{R}+I)$  are maximal ideals of  $C_c(X)$  and  $\mathbb{R}+I$ , respectively, for each  $x \in X$ . Remark that in parts (b) and (e) of the following proposition we assume that "=" is a partial order on X.

**Proposition 2.11.** For a subring R of  $\mathbb{R}^X$ , the following statements hold.

- (a) The mapping  $x \longrightarrow M_x(R)$  is a one-one correspondence if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space.
- (b) The mapping  $x \longrightarrow M_x(R)$  is an order isomorphism between X and the set of all maximal fixed ideals of R if and only if  $(X, \tau_{Z(R)})$  is a  $T_1$ -space.
- (c) For every two distinct elements  $x, y \in X$ ,  $M_x(R) + M_y(R)$  is not a g-ideal if and only if  $(X, \tau_{Z(R)})$  is a  $T_2$ -space.
- (d) The mapping  $x \longrightarrow M_x(R)$  is an order embedding between X and the set of all maximal g-ideals of R if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space and every element of Z(R) is clopen under Z(R)-topology.
- *Proof.* (a). Let x, y be distinct points of X, so  $M_x(R) \neq M_y(R)$ , say  $M_x(R) \not\subseteq M_y(R)$ . Hence, there exists  $f \in M_x(R) \setminus M_y(R)$ . Thus  $x \in Z(f)$  and  $y \notin Z(f)$ . It is clear that the above reasoning is reversible and hence we are done.
- (b  $\Rightarrow$ ). Suppose that x and y are two distinct points of X. Since  $M_x(R) \nsubseteq M_y(R)$ , there exists  $f \in M_x(R) \setminus M_y(R)$ . Consequently,  $x \in Z(f)$  and  $y \notin Z(f)$ .
- (b  $\Leftarrow$ ). Suppose that  $x \in X$  and I is a fixed ideal in R containing  $M_x(R)$ . Take  $y \in \cap_{f \in I} Z(f)$ . Clearly,  $M_x(R) \subseteq I \subseteq M_y(R)$ . It suffices to show x = y. Suppose that  $x \neq y$  and seek a contradiction. By our hypothesis, there exists  $f \in R$  such that  $x \in Z(f)$  and  $y \notin Z(f)$ . Therefore,  $M_x(R) \nsubseteq M_y(R)$  and this is a contradiction. Now, by part (a), the proof is complete.
- (c). For any two distinct points  $x, y \in X$ , clearly,  $M_x(R) + M_y(R)$  is not a g-ideal if and only if there exist  $f \in M_x(R)$  and  $g \in M_y(R)$  such that  $Z(f) \cap Z(g) = \emptyset$ .
- $(d \Rightarrow)$ . By part (a), clearly,  $(X, \tau_{Z(R)})$  is a  $T_0$ -space. Now, Suppose that  $f \in R$  and  $x \notin Z(f)$ . Since  $M_x(R)$  is a maximal g-ideal, it follows that  $(M_x(R), f)$  has an invertible element of  $\mathbb{R}^X$  and so there exists  $g \in M_x(R)$ , such that  $Z(g) \cap Z(f) = \emptyset$ . Thus, Z(f) is closed and hence is clopen under Z(R)-topology.
- (d  $\Leftarrow$ ). Suppose that  $x \in X$ , it suffices to show that  $M_x(R)$  is a maximal g-ideal. Assume that I is an ideal which properly contains  $M_x(R)$ . Hence, there exists  $f \in I$  such that  $x \notin Z(f)$ . By our hypothesis, there is  $g \in R$  such that  $x \in Z(g)$  and  $Z(g) \cap Z(f) = \emptyset$ . Therefore,  $Z(f^2 + g^2) = \emptyset$  and  $f^2 + g^2 \in I$ , hence, I is not a g-ideal.

It is easy to see that  $M_x(R)$ , for each  $x \in X$ , is a prime ideal of R and thus the hull-kernel topology may be defined on the family  $\{M_x(R) : x \in X\}$ .

By considering this space, the next statement gives a relation between Z(R)topology on X and points of X.

**Proposition 2.12.** Let R be a subring of  $\mathbb{R}^X$  and X equipped with the Coz(R)topology. Then the mapping  $\Phi: X \to \{M_x(R) : x \in X\}$  defined by  $x \mapsto M_x(R)$ is a homeomorphism if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space.

*Proof.* By part (a) of Theorem 2.12,  $\Phi$  is a one-one correspondence if and only if  $(X, \tau_{Z(R)})$  is a  $T_0$ -space. Also, if  $f \in R$  and  $x \in Z(f)$ , then  $f \in M_x(R)$ which means that basic closed sets of X equipped with the Coz(R)-topology are mapped to the basic closed sets in  $\{M_x(R): x \in X\}$  equipped with the hullkernel topology by the mapping  $\Phi$  and therefore, it is a homeomrohpism.

#### 3. $z_R$ -Ideals and z-Ideals in Subrings

In this section we introduce  $z_R$ -ideals in a subring R and via the Z(R)topology and maximal g-ideals of R, various characterizations of these ideals are given.

**Definition 3.1.** A subset  $\mathcal{F}$  of Z(R) is called  $z_R$ -filter on X, if

- (b) If  $Z_1, Z_2 \in \mathcal{F}$ , then  $Z_1 \cap Z_2 \in \mathcal{F}$ .
- (c) If  $Z_1 \in \mathcal{F}$ ,  $Z_2 \in Z(R)$  and  $Z_1 \subseteq Z_2$ , then  $Z_2 \in \mathcal{F}$ .

Moreover,  $\mathcal{F}$  is called a prime  $z_R$ -filter, if whenever  $Z_1 \cup Z_2 \in \mathcal{F}$ , then  $Z_1 \in \mathcal{F}$ or  $Z_2 \in \mathcal{F}$  for each  $Z_1, Z_2 \in Z(R)$ . Also,  $\mathcal{F}$  is called a  $z_R$ -ultrafilter, if  $\mathcal{F}$  is maximal among  $z_R$ -filters on X.

The following proposition immediately follows from Definition 3.1.

**Proposition 3.2.** For any subring R, the following statements hold.

- (a)  $I \subseteq R$  is a g-ideal in R if and only if  $Z_R(I) = \{Z(f) : f \in I\}$  is a
- (b)  $\mathcal{F}$  is a  $z_R$ -filter on X if and only if  $Z_R^{-1}(\mathcal{F}) = \{f \in R : Z(f) \in \mathcal{F}\}$  is a g-ideal.
  - (c)  $\mathcal{F}$  is a prime  $z_R$ -filter on X if and only if  $Z_R^{-1}(\mathcal{F})$  is a prime g-ideal.
  - (d) A is a  $z_R$ -ultrafilter on X if and only if  $Z_R^{-1}(A)$  is a maximal g-ideal.
  - (e) If M is a maximal g-ideal in R, then  $Z_R(M)$  is a  $z_R$ -ultrafilter on X.

It is easy to see that for an ideal I of R we always have  $I \subseteq Z_R^{-1}Z_R(I)$  and the inclusion may be proper. We call an ideal I in R a  $z_R$ -ideal, if  $I = Z_R^{-1} Z_R(I)$ . It follows that every  $z_R$ -ideal is semiprime and arbitrary intersections of  $z_R$ ideals is a  $z_R$ -ideal. Also, the zero ideal, the ideals of the form  $M_x(R)$ , maximal g-ideals and  $Z^{-1}(\mathcal{F})$ , for each  $z_R$ -filter  $\mathcal{F}$ , are all  $z_R$ -ideals of R. For each  $f \in R$ , the intersection of all the maximal ideals, maximal g-ideals and maximal fixed ideals of R containing f are denoted by  $M_f(R)$ ,  $MG_f(R)$  and  $MF_f(R)$ , respectively. It is easy to observe that  $MG_f(R)$  is a  $z_R$ -ideal for each  $f \in R$ .

Obviously,  $MG_f \cap MG_g = MG_{fg}$ ,  $MF_f \cap MF_g = MF_{fg}$ ,  $MG_{f^2+g^2} = MG_{(f,g)}$  and  $MF_{f^2+g^2} = MF_{(f,g)}$  for all  $f, g \in R$ .

**Proposition 3.3.** Let  $(X, \tau_{Z(R)})$  be a  $T_1$ -space. Then the following statements hold.

- (a) The following statements are equivalent.
- (1)  $g \in MF_f(R)$ .
- (2)  $MF_g(R) \subseteq MF_f(R)$ .
- (3)  $Z(f) \subseteq Z(g)$ .
- (b)  $MF_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}.$
- (c) An ideal I of R is a  $z_R$ -ideal if and only if  $MF_f(R) \subseteq I$  for every  $f \in I$ .

*Proof.* (a:  $1 \Rightarrow 2$ ). Evident.

(a:  $2 \Rightarrow 3$ ). Let  $x \in Z(f)$ . Then  $f \in M_x(R)$  and thus  $MF_g(R) \subseteq MF_f(R) \subseteq M_x(R)$ . This implies  $g \in M_x(R)$  and hence  $x \in Z(g)$ .

(a:  $3 \Rightarrow 1$ ). If  $g \notin MF_f(R)$ , then there exists  $x \in X$  such that  $f \in M_x(R)$  and  $g \notin M_x(R)$ . Therefore,  $x \in Z(f) \setminus Z(g)$  and so  $Z(f) \subsetneq Z(g)$ .

(b) and (c) obviously follow from part (a).

**Lemma 3.4.** Assume that every  $Z \in Z(R)$  is clopen under Z(R)-topology. Then  $MG_f(R) = MF_f(R)$ , for every  $f \in R$ .

Proof. Suppose that  $f \in R$ . By part (b) of Proposition 2.4,  $M_x(R)$  is a maximal g-ideal for each  $x \in X$ . Consequently,  $MG_f(R) \subseteq MF_f(R)$ . Now, assume that  $g \notin MG_f(R)$ . Hence, there exists a maximal g-ideal M in R such that  $f \in M$  and  $g \notin M$ . Thus, there exists  $h \in M$  such that  $Z(g) \cap Z(h) = \emptyset$ . Since  $f^2 + h^2 \in M$  and M is a g-ideal, there is a point  $x \in Z(f^2 + h^2) = Z(f) \cap Z(h)$ . Clearly,  $g \notin M_x(R)$  and  $f \in M_x(R)$ . Therefore,  $g \notin MF_f(R)$ .

Proposition 3.3 and Lemma 3.4 imply the next statement.

**Proposition 3.5.** Let  $(X, \tau_{Z(R)})$  be a  $T_1$ -space and every  $Z \in Z(R)$  be a clopen set under Z(R)-topology. Then the following statements hold.

- (a) The following statements are equivalent.
- (1)  $g \in MG_f(R)$ .
- (2)  $MG_g(R) \subseteq MG_f(R)$ .
- (3)  $Z(f) \subseteq Z(g)$ .
- (b)  $MG_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}.$
- (c) An ideal I of R is  $z_R$ -ideal if and only if  $MG_f(R) \subseteq I$  for every  $f \in I$ .

The following corollary follows from Corollary 2.6 and Proposition 3.5.

**Corollary 3.6.** Let R be an invertible subalgebra of  $\mathbb{R}^X$ . Then the following statements hold.

- (a) The following conditions are equivalent;
- (1)  $g \in M_f(R)$ .
- (2)  $M_g(R) \subseteq M_f(R)$ .
- (3)  $Z(f) \subseteq Z(g)$ .
- (b)  $M_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}.$
- (c) An ideal I of R is  $z_R$ -ideal if and only if  $M_f(R) \subseteq I$  for every  $f \in I$ .

It follows from Corollary 3.6 that for an invertible subalgebra R, the notion of  $z_R$ -ideal coincides with the notion of z-ideal. The next statement extend this fact and shows that this coincidence is equivalent to invertibility of R.

**Theorem 3.7.** Let R be a subring of  $\mathbb{R}^X$ . The following statements are equivalent.

- (a) Every maximal ideal in R is a g-ideal.
- (b) Every maximal g-ideal of R is a maximal ideal and if J is a maximal ideal of R, then every maximal element in the set of g-ideals contained in J is a prime ideal.
  - (c) Every maximal ideal in R is a g-ideal.
  - (d) R is an invertible subring.
  - (e) Every z-ideal of R is a  $z_R$ -ideal.

Moreover, if R is a subalgebra and one of (a)-(c) holds, then every  $z_R$ -ideal is a z-ideal.

*Proof.* (a)  $\Rightarrow$  (b). This is clear.

- (b) $\Rightarrow$ (c). Suppose that M is a maximal ideal and P is a maximal element of  $G_M$ , where  $G_M$  is the set of all g-ideals contained in M. Assume that J is a maximal ideal of R containing P. Then  $M \cap J = P$ . As  $M \cap J$  is prime and both M and J are maximal ideal, we have M = J. Hence, M is a maximal g-ideal.
- (c) $\Rightarrow$ (d). Suppose that  $Z(f) = \emptyset$  for  $f \in R$  and, on the contrary, f is a non-unit element of R. Clearly, there exists a maximal ideal M of R containing f. By our hypothesis, M is a g-ideal which contradicts with  $Z(f) = \emptyset$ .
- $(d)\Rightarrow(e)$ . Suppose that I is a z-ideal and  $Z(f)\subseteq Z(g)$  where  $f\in I$  and  $g\in R$ . Since I is a z-ideal, it follows that  $M_f\subseteq I$ . It suffices to prove that  $g\in M_f$ . To see this, suppose that M is a maximal ideal containing f. As R is invertible, M is a g-ideal and so it is a maximal g-ideal. Obviously, M is a  $z_R$ -ideal and so  $g\in M$ .
- (e) $\Rightarrow$ (a). Suppose that M is a maximal ideal and, on the contrary, M is not a g-ideal. Thus, there exists  $f \in M$  such that  $Z(f) = \emptyset$ . By (e), M is a  $z_R$ -ideal and since  $f \in M$ , it follows that M = R, which is a contradiction.

Now, suppose that one of (a)-(c) holds, R is a subalgebra and I is a  $z_R$ -ideal of R. By our hypothesis,  $MF_f(R) = M_f(R)$  for every  $f \in R$ , and thus we are done.

It is well-known that every minimal prime ideals over a z-ideal is also a z-ideal, see [10, Theorem 14.7]. The same statement holds for  $z_R$ -ideals as the following proposition shows.

**Proposition 3.8.** Let I be a  $z_R$ -ideal of R and P a prime ideal in R minimal over I. Then P is a  $z_R$ -ideal.

*Proof.* Assume that Z(f) = Z(g) and  $f \in P$ . Thus, there exists  $h \notin P$ , such that  $fh \in I$ . Since Z(fh) = Z(gh) and I is a  $z_R$ -ideal, it follows that  $gh \in I \subseteq P$ . As  $h \notin P$ , clearly, this implies that  $g \in P$ .

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring R is a  $z_R$ -ideal. By the following statement, we extend some fundamental statements about z-ideals in the literature of C(X) to the subrings of  $\mathbb{R}^X$ , namely, [10, 2.9, 5.3 and 5.5]. The proofs are left to the reader.

**Proposition 3.9.** Let R be a lattice-ordered subring of  $\mathbb{R}^X$  and I be a  $z_R$ -ideal in R. Then the following statements hold.

- (a) The following statements are equivalent
  - (1) I is a prime ideal;
  - (2) I contains a prime ideal;
  - (3) if fg = 0, then  $f \in I$  or  $g \in I$ ;
  - (4) for each  $f \in R$ , there is a  $Z \in Z_R(I)$  on which f does not change sign.
- (b) Every prime g-ideal of R is contained in a unique maximal g-ideal.
- (c) If P is a prime ideal of R, then  $Z_R(P)$  is a prime  $z_R$ -filter on X.
- (d) If  $\mathcal{P}$  is a prime  $z_R$ -filter on X, then  $Z_R^{-1}(\mathcal{P})$  is a prime ideal in R.
- (e) Every  $z_R$ -ideal of R is absolutely convex.

Thus, if I is an absolutely convex ideal of R, then R/I is a lattice ring.

- (f)  $I(f) \ge 0$  if and only if  $f \ge 0$  on some  $Z \in Z_R(I)$ .
- (g) Suppose that there exists  $Z \in Z_R(I)$  such that f(x) > 0, for every  $x \in Z$ , then I(f) > 0. The converse is true whenever I is a maximal q-ideal.

4. 
$$z_{R}^{\circ}$$
-IDEALS AND  $z^{\circ}$ -IDEALS IN SUBRINGS

In this section we generalize the concept of  $z^{\circ}$ -ideals of C(X) to the subrings of  $\mathbb{R}^X$  and introduce  $z_R^{\circ}$ -ideal. Coincidence of  $z_R^{\circ}$ -ideals with  $z^{\circ}$ -ideals of R is discussed. Note that, for each element f of a commutative rings S, we use  $P_f(S)$  to denote the intersection of all the minimal prime ideals in S containing f.

**Definition 4.1.** An ideal I of a subring R of  $\mathbb{R}^X$  is called a  $z_R^{\circ}$ -ideal, if  $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$ , where  $f \in I$  and  $g \in R$ , implies  $g \in I$ .

The following statement investigates some characterizations of  $z_R^{\circ}$ -ideals in subrings.

**Theorem 4.2.** Let R be a subing of  $\mathbb{R}^X$  and I be an ideal in R. The following statements are equivalent.

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- (a) I is a  $z_R^{\circ}$ -ideal.
- (b) Whenever  $Ann_C(f) \subseteq Ann_C(g)$  where  $f \in I$  and  $g \in R$ , then  $g \in I$ .
- (c)  $R \cap P_f(C) \subseteq I$  for each  $f \in I$ .
- (d) Whenever  $P_g(C) \cap R \subseteq P_f(C) \cap R$ , where  $f \in I$  and  $g \in R$ , then  $g \in I$ .

*Proof.* (a $\Rightarrow$ b). First note that by [3, Lemma 2.1] we have  $Ann_C(f) \subseteq Ann_C(g)$  if and only if  $int_X Z(f) \subseteq int_X Z(g)$  for each  $f, g \in C(X)$ . Now, let I be a  $z_R^{\circ}$ -ideal in R and  $Ann_C(f) \subseteq Ann_C(g)$  where  $f \in I$  and  $g \in R$ . Thus, by our hypothesis, we have  $int_X Z(f) \subseteq int_X Z(g)$  which implies that  $g \in I$ .

(b $\Rightarrow$ c). By [3, Proposition 2.3], we have  $P_f(C) = \{g \in C(X) : Ann_C(f) \subseteq Ann_C(g)\}$ . Thus the proof is evident.

(c $\Rightarrow$ d). Let  $P_g(C) \cap R \subseteq P_f(C) \cap R$ , where  $f \in I$  and  $g \in R$ . As  $f \in I$ , by our hypothesis,  $P_f(C) \cap R \subseteq I$  and thus  $P_g(C) \cap R \subseteq I$  which implies that  $g \in I$ .

 $(d\Rightarrow a)$ . Let  $\operatorname{int}_X Z(f)\subseteq \operatorname{int}_X Z(g)$  where  $f\in I$  and  $g\in R$ . Therefore, by [3, Lemma 2.1], we have  $P_f(C)\subseteq P_g(C)$  and hence  $P_f(C)\cap R\subseteq P_g(C)\cap R$ . Thus we are done by our hypothesis.

**Lemma 4.3.** Let R be a subring of  $\mathbb{R}^X$ , then for each  $f \in R$  we have  $P_f(C) \subseteq P_f(R)$ .

Proof. Let  $g \in P_f(C)$ . By [3, Proposition 2.3.], we have  $Ann_C(f) \subseteq Ann_C(g)$ . Therefore,  $Ann_R(f) = Ann_C(f) \cap R \subseteq Ann_C(g) \cap R = Ann_R(g)$ . Thus, by [2, Proposition 1.5] we are done.

**Theorem 4.4.** Let R be a subring of  $\mathbb{R}^X$ . Then every  $z_R^{\circ}$ -ideal in R is a  $z^{\circ}$ -ideal if and only if  $P_f(R) = P_f(C)$  for each  $f \in R$ .

*Proof.* ( $\Rightarrow$ ). Assumme on the contrary that there exists some  $f \in R$  such that  $P_f(R) \neq P_f(C)$ . Thus, using Theorem 4.2 we have  $P_f(C) \subseteq P_f(R)$ . Again by Theorem 4.2,  $P_f(C) \cap R$  is a  $z_R^{\circ}$ -ideal in R. Also, it is clear that this ideal is not a  $z^{\circ}$ -ideal, since,  $P_f(R) \not\subseteq P_f(C) \cap R$ .

(⇐). Let I be a  $z_R^{\circ}$ -ideal in R and  $f \in I$ . By Theorem 4.2,  $P_f(C) \cap R \subseteq I$ . Thus, by our hypothesis,  $P_f(R) \subseteq I$  which means that I is a  $z^{\circ}$ -ideal in R.  $\square$ 

From Theorem 4.2 it follows that every  $z^{\circ}$ -ideal in a subring R is a  $z_{R}^{\circ}$ -ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring R which has a  $z_{R}^{\circ}$ -ideal that is not a  $z^{\circ}$ -ideal.

Example 4.5. Let  $f: \mathbb{R} \longrightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$ . It is clear

that  $f \in C(\mathbb{R})$ . Now, let  $R = \{\sum_{i=0}^n r_i f^i : r_i \in \mathbb{R}, n = 0, 1, ...\}$ . It is easy to see that  $P_f(R) = R$ , however,  $P_f(C) \cap R \neq R$ . Also, by Theorem 4.2,  $P_f(C) \cap R$  is  $z_R^\circ$ -ideal and it is clear that this ideal is not a  $z^\circ$ -ideal.

The next theorem gives a sufficent conditions on X in order that  $z_R^{\circ}$ -ideals in a subring R coincide with  $z^{\circ}$ -ideals of R.

**Theorem 4.6.** Let R be a subring of  $\mathbb{R}^X$  and X be equipped with the Coz(R)-topology. Then an ideal I in R is a  $z^{\circ}$ -ideal if and only if it is a  $z^{\circ}_R$ -ideal.

Proof. Let I be a  $z_R^{\circ}$ -ideal in R and  $f \in I$ . As X is equipped with the Coz(R)-topology, we have  $g \in Ann_R(f)$  if and only if  $Coz(g) \subseteq \operatorname{int}_X Z(f)$  for each  $f,g \in R$ . Therefore,  $P_f(R) = Ann_R Ann_R(f) = \{g \in R : Coz(g) \cap \operatorname{int}_X Z(f) = \emptyset\} = \{g \in R : Ann_R(f) \subseteq Ann_R(g)\}$ . Hence,  $P_f(R) \subseteq I$  which means that I is a  $z^{\circ}$ -ideal in R. This completes the proof, since, as former stated, every  $z^{\circ}$ -ideal in R is a  $z_R^{\circ}$ -ideal.

Note that the condition that X is equipped with the Coz(R)-topology is a sufficient condition for coincidence of  $z_R^{\circ}$ -ideals with  $z^{\circ}$ -ideals in a given subring R. The next example shows that this condition is not necessary.

EXAMPLE 4.7. Let  $X = \mathbb{R} \setminus \{0\}$  with the topology inherits from the usual topology on  $\mathbb{R}$ . Also, let  $f: X \longrightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ . It

is clear that  $f \in C(X)$  and  $f^2 = f$ . Now, set  $R = \{r + sf : r, s \in \mathbb{R}\}$ . It is clear that R is a subring of C(X). Also, by a routine reasoning, one can proves that the only ideals of R are the ideals (0), (f), (1-f) and R. Moreover, the minimal prime ideals of R are only the ideals (f) and (1-f). These imply that every  $z_R^{\circ}$ -ideal is a  $z^{\circ}$ -ideal in R. However, clearly, X is not equipped with the Coz(R)-topology.

It follows from Theorem 4.6 that for an intermediate subalgebra A(X) of C(X),  $z_A^{\circ}$ -ideals coincide with  $z^{\circ}$ -ideals of A(X). However, the same statement does not true for  $z_A$ -ideals and z-ideals in A(X), in general, see [6, Theorem 2.2]. Moreover, Theorem 3,7 together with Theorem 4.6 imply that in the subalgebras of C(X) which are of the form  $\mathbb{R} + I$ , where I is a free ideal in C(X),  $z_{\mathbb{R}+I}$ -ideals concide with z-ideals of  $\mathbb{R} + I$  and  $z_{\mathbb{R}+I}^{\circ}$ -ideals coincide with  $z^{\circ}$ -ideals, too. Note that whenever I is a free ideal in C(X), then  $\mathbb{R} + I$  determines the topology of X.

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