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# Domination and Signed Domination Number of Cayley Graphs

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ABSTRACT. In this paper, we investigate domination number as well as signed domination numbers of Cay(G:S) for all cyclic group G of order n, where  $n \in \{p^m, pq\}$  and  $S = \{k < n : gcd(k, n) = 1\}$ . We also introduce some families of connected regular graphs  $\Gamma$  such that  $\gamma_S(\Gamma) \in \{2, 3, 4, 5\}$ .

**Keywords:** Cayley graph, Cyclic group, Domination number, Signed domination number.

2000 Mathematics subject classification: 05C69, 05C25

## 1. Introduction

By a graph  $\Gamma$  we mean a simple graph with vertex set  $V(\Gamma)$  and edge set  $E(\Gamma)$ . A graph is said to be *connected* if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining  $v_i$  and  $v_j$  is called the distance between  $v_i$  and  $v_j$  and denoted by  $d(v_i, v_j)$ . A graph  $\Gamma$  is said to be regular of degree k or, k-regular if every vertex has degree k. A subset P of vertices of  $\Gamma$  is a k-packing if d(x, y) > k for all pairs of distinct vertices x and y of P [9].

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Let G be a non-trivial group, S be an inverse closed subset of G which does not contain the identity element of G, i.e.  $S = S^{-1} = \{s^{-1} : s \in S\}$ . The Cayley graph of G denoted by Cay(G:S), is a graph with vertex set G and two vertices a and b are adjacent if and only if  $ab^{-1} \in S$ . The Cayley graph Cay(G:S) is connected if and only if S generates G.

A set  $D \subseteq V$  of vertices in a graph  $\Gamma$  is a dominating set if every vertex  $v \in V$ is an element of D or adjacent to an element of D. The domination number  $\gamma(\Gamma)$  of a graph  $\Gamma$  is the minimum cardinality of a dominating set of  $\Gamma$ .

For a vertex  $v \in V(\Gamma)$ , the closed neighborhood N[v] of v is the set consisting v and all of its neighbors. For a function  $f:V(\Gamma)\to\{-1,1\}$  and a

subset W of V we define  $f(W) = \sum_{u \in W} f(u)$ . A signed dominating function of  $\Gamma$  is a function  $f: V(\Gamma) \to \{-1,1\}$  such that f(N[v]) > 0 for all  $v \in V(\Gamma)$ . The weight of a function f is  $\omega(f) = \sum_{v \in V} f(v)$ . The signed domination number

 $\gamma_{\rm s}(\Gamma)$  is the minimum weight of a signed dominating function of  $\Gamma.$  A signed dominating function of weight  $\gamma_s(\Gamma)$  is called a  $\gamma_s(\Gamma)$ -function. We denote f(N[v]) by f[v]. Also for  $A \subseteq V(\Gamma)$  and signed dominating function f, set  $\{v \in A : f(v) = -1\}$  is denoted by  $A_f^-$ .

Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance [2, 3, 5, 6, 11, 13])

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders  $p^n$ , pq, where p and q are prime numbers.

## 2. Cayley Graphs of Order $p^n$

In this section p is a prime number and  $B(1,n) = \{k < n : gcd(k,n) = 1\}.$ 

**Lemma 2.1.** Let G be a group and H be a proper subgroup of G such that [G:H]=t. If  $S=G\setminus H$ , then Cay(G:S) is a complete t-partite graph.

*Proof.* One can see  $G = \langle S \rangle$  and  $e \notin S = S^{-1}$ . Let  $a \in G$ . If  $x, y \in Ha$ , then  $x = h_1 a, y = h_2 a$ . Since  $xy^{-1} \in H$ ,  $xy \notin E(Cay(G:S))$ . So induced subgraph on every coset of H is empty. Let Ha and Hb two disjoint cosets of H and  $x \in Ha, y \in Hb$ . Hence,  $xy^{-1} \in S$ . So  $xy \in E(Cay(G:S))$ . Therefore,  $Cay(G:S) = K_{|H|,|H|,\cdots,|H|}.$ 

**Lemma 2.2.** Let G be a group of order n and  $G = \langle S \rangle$ , where  $S = S^{-1}$  and  $0 \notin S$ . Then  $\gamma(Cay(G:S)) = 1$  if and only if  $S = G \setminus \{0\}$ .

*Proof.* The proof is straightforward.

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**Theorem 2.3.** [13] Let  $K_{a,b}$  be a complete bipartite graph with  $b \leq a$ . Then

$$\gamma_{\scriptscriptstyle S}(K_{a,b}) = \left\{ \begin{array}{ll} a+1 & \textit{if } b=1,\\ b & \textit{if } 2 \leq b \leq 3 \textit{ and } a \textit{ is even,}\\ b+1 & \textit{if } 2 \leq b \leq 3 \textit{ and } a \textit{ is odd },\\ 4 & \textit{if } b \geq 4 \textit{ and } a, b \textit{ are both even,}\\ 6 & \textit{if } b \geq 4 \textit{ and } a, b \textit{ are both odd,}\\ 5 & \textit{if } b \geq 4 \textit{ and } a, b \textit{ have different parity.} \end{array} \right.$$

**Theorem 2.4.** Let  $\mathbb{Z}_{2^n} = \langle S \rangle$  and  $S = B(1, 2^n)$ . Then

- i.  $Cay(\mathbb{Z}_{2^n}:S)=K_{2^{n-1},2^{n-1}}$
- ii.  $\gamma(Cay(\mathbb{Z}_{2^n}:S))=2.$

iii.

$$\gamma_S(Cay(\mathbb{Z}_{2^n}:S)) = \begin{cases} 2 & \text{if } n = 1, 2, \\ 4 & \text{if } n \ge 3. \end{cases}$$

i. Let  $H = \mathbb{Z}_{2^n} \setminus S$ . Then  $H = \{i : 2 \mid i\}$ . It is not hard to see Proof. that H is a subgroup of  $\mathbb{Z}_{2^n}$  and  $[\mathbb{Z}_{2^n}:H]=2$ . Hence, by Lemma 2.1,  $Cay(\mathbb{Z}_{2^n}:S)=K_{2^{n-1}\cdot 2^{n-1}}.$ 

ii. By part i.  $Cay(\mathbb{Z}_{2^n}:S)$  is a complete bipartite graph. So

$$\gamma(Cay(\mathbb{Z}_{2^n}:S))=2.$$

iii. The proof is straightforward by Theorem 2.3.

**Corollary 2.5.** For any integer n > 2, there is a  $2^{n-1}$ -regular graph  $\Gamma$  with  $2^n$  vertices such that  $\gamma_s(\Gamma) = 4$ .

**Theorem 2.6.** Let  $\mathbb{Z}_{p^n} = \langle S \rangle$  (p odd prime) and  $S = B(1, p^n)$ . Then following statments hold:

- i.  $Cay(\mathbb{Z}_{p^n}:S)$  is a complete p-partite graph.
- ii.  $\gamma(Cay(\mathbb{Z}_{p^n}:S))=2.$
- iii.  $\gamma_S(Cay(\mathbb{Z}_{p^n}:S))=3.$

Proof. i. Let  $H = \mathbb{Z}_{p^n} \setminus S$ . Then  $H = \{i : p \mid i\}$ . H is a subgroup of  $\mathbb{Z}_{p^n}$ and  $|H| = p^n - \Phi(p^n) = p^{n-1}$ . So  $[\mathbb{Z}_{p^n} : H] = p$ . Hence, by Lemma 2.1,  $Cay(\mathbb{Z}_{p^n}:S)$  is a complete p-partite graph of size  $p^{n-1}$ .

- ii. Since  $Cay(\mathbb{Z}_{p^n}:S)$  is a complete p-partite graph,  $D=\{a,b\}$  is a
- minimal dominating set where a,b are not in the same partition. iii. Let  $\Gamma = Cay(\mathbb{Z}_{p^n}:S)$ . Let  $V(\Gamma) = \bigcup_{i=1}^p A_i$  where  $A_i = \{v_{ij}: 1 \leq j \leq p^{n-1}\}$ . Define  $f:V(\Gamma) \to \{-1,1\}$

$$f(v_{ij}) = \begin{cases} -1 & \text{if } 1 \le i \le \lfloor \frac{p}{2} \rfloor - 1 \text{ and } 1 \le j \le \lceil \frac{p^{n-1}}{2} \rceil, \\ -1 & \text{if } \lfloor \frac{p}{2} \rfloor \le i \le p \text{ and } 1 \le j \le \lfloor \frac{p^{n-1}}{2} \rfloor, \\ 1 & \text{otherwise.} \end{cases}$$

Let 
$$v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor - 1} A_i$$
. So  $|N(v) \cap V_f^-| = \frac{1}{2} (p^n - p^{n-1} - 4)$ . So  $f[v] = f(v) + 4 \ge 3$ . If  $v \in \bigcup_{i=\lfloor \frac{p}{2} \rfloor}^p A_i$ , then  $|N(v) \cap V_f^-| = \frac{1}{2} (p^n - p^{n-1} - 2)$ . So  $f[v] = f(v) + 2 \ge 1$ . Hence,  $f$  is a signed dominating function.

$$f(v) + 4 \ge 3$$
. If  $v \in \bigcup_{i=|\frac{p}{2}|}^{p} A_i$ , then  $|N(v) \cap V_f^-| = \frac{1}{2}(p^n - p^{n-1} - 2)$ .

So  $f[v] = f(v) + 2 \ge 1$ . Hence, f is a signed dominating function. Since  $|V_f^-| = \frac{1}{2}(p^n - 3)$ ,  $\omega(f) = 3$ . So  $\gamma_S(\Gamma) \leq 3$ . On the contrary, suppose  $\gamma_S(\Gamma) < 3$ . So there is a  $\gamma_S$ -function g such that  $\omega(g) < 3$ . So  $|V_g^-| > \frac{1}{2}(p^n - 3)$ . Let  $|V_g^-| = \frac{1}{2}(p^n - 1)$ . If  $A_i \cap V_g^- = \emptyset$  for some  $1 \le i \le p$ , then  $g[v] = 1 - p^{n-1}$  for every  $v \in A_i$ . Hence,  $A_i \cap V_g^- \ne \emptyset$  for every  $1 \le i \le p$ . If  $|A_i \cap V_g^-| \ge \lceil \frac{p^{n-1}}{2} \rceil$  for every  $1 \le i \le p$ , then  $|V_g^-| \ge \frac{1}{2}(p^n + p)$ . This is impossible. So there is  $j\in\{1,2,\ldots,p\}$  such that  $|A_j\cap V_g^-|\leq \lfloor \frac{p^{n-1}}{2}\rfloor$ . Let  $u\in A_j\cap V_g^-$ . So  $g[u]=deg(u)+1-2|N(u)\cap V_g^-|<0$ . This is contradiction. Therefore  $\gamma_{\scriptscriptstyle S}(\Gamma) = 3.$ 

Corollary 2.7. For every integer n, there is a  $(p^n - p^{n-1})$ -regular graph  $\Gamma$ with  $p^n$  vertices such that  $\gamma_{\scriptscriptstyle S}(\Gamma)=3$ .

#### 3. Cayley Graphs of Order pq

In this section p and q are distinct prime numbers where p < q. Let B(1, pq)be a generator of  $\mathbb{Z}_{pq}$ . For  $1 \leq i \leq p$  and  $1 \leq j \leq q$ , set

$$A_i = \{i + kp : 0 \le k \le q - 1\}$$

and

$$B_j = \{j + k'q : 0 \le k' \le p - 1\}.$$

With these notations in mind we will prove the following results.

**Lemma 3.1.** Let  $\mathbb{Z}_{pq} = \langle S \rangle$  and S = B(1, pq). Then following statuents hold.

- i.  $V(Cay(\mathbb{Z}_{pq}:S)) = \bigcup_{i=1}^{p} A_i \text{ and } Cay(\mathbb{Z}_{pq}:S) \text{ is a p-partite graph.}$ ii.  $V(Cay(\mathbb{Z}_{pq}:S)) = \bigcup_{i=1}^{p} B_j \text{ and } Cay(\mathbb{Z}_{pq}:S) \text{ is a q-partite graph.}$
- iii. Let  $1 \leq i \leq p$ . For any  $x \in A_i$  there is some  $1 \leq j \leq q$  such that
- iv.  $|A_i \cap B_j| = 1$  for every i, j.

i. Let  $s \in V(Cay(\mathbb{Z}_{pq} : S))$ . If  $p \mid s$ , then  $s \in A_p$ . Otherwise,  $s \in A_i$  where s = kp + i for some  $1 \le k \le (p-1)$ . Thus  $V(Cay(\mathbb{Z}_{pq} : \mathbb{Z}_{pq} : \mathbb{Z}_{pq}))$ 

$$S(S) = \bigcup_{i=1}^{p} A_i$$
. Since  $1 \le i \ne j \le p$ ,  $A_i \cap A_j = \emptyset$ . We show that the

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induced subgraph on  $A_i$  is empty. Let  $l + t \in E(Cay(\mathbb{Z}_{pq} : S))$ . If  $l, t \in A_s$  for some  $1 \le s \le p$ , then l = s + kp, t = s + k'p. So  $p \mid (l - t)$ . This is impossible.

- ii. The proof is likewise part i.
- iii. Let  $1 \leq i \leq p$  and let  $x \in A_i$ . If  $x \leq q$ , then  $x \in B_x$ . If not, x = i + kp > q such that  $1 \le k \le q - 1$ . Hence,  $x \equiv t \pmod{q}$  where  $1 \le t \le q$ , and so  $x \in B_t$ .
- iv. By Case iii and since  $|A_i| = q$  and also for every  $j \neq j'$ ,  $B_i \cap B_{i'} = \emptyset$ , the result reaches.

**Theorem 3.2.** [6] For any graph  $\Gamma$ ,  $\left\lceil \frac{n}{1+\Delta(\Gamma)} \right\rceil \leq \gamma(\Gamma) \leq n-\Delta(\Gamma)$  where  $\Delta(\Gamma)$ is the maximum degree of  $\Gamma$ .

**Theorem 3.3.** Let  $\mathbb{Z}_{pq} = \langle S \rangle$  and S = B(1, pq). Then the following is hold.

$$\gamma(Cay(\mathbb{Z}_{pq}:S)) = \begin{cases} 2 & p=2; \\ 3 & p>2. \end{cases}$$

*Proof.* Let p=2. By Lemma 3.1,  $D=\{i,i+q\}$  is a dominating set. Since  $Cay(\mathbb{Z}_{pq}:S)$  is a (q-1)-regular graph, by Theorem 3.2,  $\gamma(Cay(\mathbb{Z}_{pq}:S)) \geq 2$ . Thus  $\gamma(Cay(\mathbb{Z}_{pq}:S))=2$ .

Let p > 2. We define  $D = \{1, 2, s\}$  where  $s \in A_1 \setminus N(2)$ . Since 1, 2 are adjacent,  $N(1) \cup N(2) = V(Cay(\mathbb{Z}_{pq} : S)) \setminus D$ . Thus D is a dominating set. As a consequence,  $\gamma(Cay(\mathbb{Z}_{pq}:S)) \leq 2$ . It is enough to show that  $\gamma(Cay(\mathbb{Z}_{pq}:S))\neq 2$ . Let  $D'=\{x,y\}$ . We show that D' is not a dominating set. If  $x, y \in A_i$  for some  $1 \le i \le p$ , then for every  $z \in A_i \setminus D'$ ,  $z \notin N(D')$ . If not,  $x \in A_i$  and  $y \in A_j$  for some  $1 \le i \ne j \le p$ . If x, y are adjacent, then there is  $x' \in A_i \setminus \{x\}$  such that  $x' \notin N(y)$ . Thus D' is not dominating set. If x and y are not adjacent, then there is  $z \in A_l$ ,  $l \neq i, j$ , such that the induced subgraph on  $\{x, y, z\}$  is empty. Hence, D' is not a dominating set and the proof is completed.

**Theorem 3.4.** Let  $\mathbb{Z}_{pq} = \langle S \rangle$  where  $p \in \{2, 3, 5\}$  and S = B(1, pq). Then

$$\gamma_{\scriptscriptstyle S}(Cay(\mathbb{Z}_{pq}:S)) = p.$$

*Proof.* Let  $A = \{1, 1+p, \dots, 1+(\lfloor \frac{q}{2} \rfloor - 1)p\}$  and  $B = \{i+tq: i \in A \text{ and } 1 \leq 1\}$  $t \leq p-1$ . We define  $f: V(Cay(\mathbb{Z}_{pq}:S)) \to \{-1,1\}$  such that

$$f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise.} \end{cases}$$

Let  $v \in V(Cay(\mathbb{Z}_{pq}:S))$ . If f(v) = -1, then

$$f[v] = -1 + (p-1)(q-1) - 2\left(\left(\lfloor \frac{q}{2} \rfloor - 1\right)(p-1)\right) = 2p - 3.$$

Otherwise,

$$f[v] = 1 + (p-1)(q-1) - 2\left\lfloor \frac{q}{2} \right\rfloor (p-1) = 1.$$

Hence, f is a dominating function. Also

$$\omega(f) = pq - 2\left(|A| + |B|\right) = pq - 2\left(\left\lfloor\frac{q}{2}\right\rfloor + (p-1)\left\lfloor\frac{q}{2}\right\rfloor\right) = p.$$

It is enough to show that f has the minimal wait. Let, to the contrary, g be a dominating function and  $\omega(g) < \omega(f)$ . So  $|V_g^-| > |V_f^-|$ . Without lose of generality, suppose that  $|V_g^-| = p \lfloor \frac{q}{2} \rfloor + 1$ . Let  $A_i^- = A_i \cap V_g^-$ ,  $A_i^+ = A_i \setminus A_i^-$  and  $B_j^- = B_j \cap V_g^-$ . We will reach the contradiction by three steps.

Step 1. For every  $1 \le i \le p, A_i^- \ne \emptyset$ .

On the contrary, let  $A_s^- = \emptyset$  for some  $1 \le s \le p$ . Let  $u \in A_s$ . Then by Lemma 3.1,  $u \in A_s \cap B_t$  for some  $1 \le t \le q$ . So

$$g[u] = (p-1)(q-1) + 1 - 2(|V_q^-| - |B_t^-|) \ge 1.$$

Thus  $|B_t^-| \ge \lceil \frac{q}{2} \rceil$ . Hence,  $|V_g^-| \ge |A_s| \lceil \frac{q}{2} \rceil$ . This imolies  $q + (q-p) \lfloor \frac{q}{2} \rfloor < 1$ . This is a contradiction. Hence,  $A_s^- \ne \emptyset$ .

Similar argument applies for  $B_j$ . Therefore,  $B_j^- \neq \emptyset$  for every  $1 \leq j \leq q$ .

Step 2. For every  $1 \le i \le p$ ,  $|A_i^-| \ge \lfloor \frac{q}{2} \rfloor$ .

On the contrary, Let  $|A_l^-| < \lfloor \frac{q}{2} \rfloor$  for some  $1 \le l \le p$ . Without lose of generality suppose that  $|A_l^-| = \lfloor \frac{q}{2} \rfloor - 1$ . Let  $v \in A_l$ . By Lemma 3.1,  $v \in A_l \cap B_k$  for some  $1 \le k \le q$ . If g(v) = -1, then  $g[v] = (p-1)(q-1)-1-2(|V_g^-|-|A_l^-|-|B_k^-|+2) \ge 1$ . Then  $|B_k^- \setminus \{v\}| \ge 4$ . If g(v) = 1, then  $|B_k^- \setminus \{v\}| \ge 2$ . Hence,  $|V_g^-| \ge 4|A_l^-|+|A_l^-|+2|A_l^+|$ . As a consequence p > 8. This is impossible.

Therefore, for every  $1 \leq i \leq p$ ,  $|A_i^-| \geq \lfloor \frac{q}{2} \rfloor$  and since  $|V_g^-| = p \lfloor \frac{q}{2} \rfloor + 1$ , we may suppose that  $|A_1^-| = \lceil \frac{q}{2} \rceil$  and  $|A_i^-| = \lfloor \frac{q}{2} \rfloor$  for  $2 \leq i \leq p$ .

Step 3. For every  $1 \le j \le q$ ,  $|B_j^-| \ge \lceil \frac{p}{2} \rceil$ .

On the contrary, let  $|B_h^-| < \lceil \frac{p}{2} \rceil$  for some  $1 \le h \le q$ . Suppose that  $|B_h^-| = \lfloor \frac{p}{2} \rfloor$ . By Lemma 3.1,  $B_h \cap A_i \ne \emptyset$  for any  $1 \le i \le p$ . Let  $z \in B_h^- \cap A_i$ . Thus

$$\begin{split} g[z] &= -1 + (p-1)(q-1) - 2\left(|V_g^-| - |A_i^-| - |B_h^-| + 2\right) \\ &\leq -1 + (p-1)(q-1) - 2\left(p\left\lfloor\frac{q}{2}\right\rfloor + 1 - \left\lceil\frac{q}{2}\right\rceil - \lfloor\frac{p}{2}\rfloor + 2\right) \\ &\leq p - 6 \end{split}$$

Since  $p \in \{2, 3, 5\}$ ,  $g[z] \le -1$ . This is a contradiction.

By Step 3,  $|V_g^-| \ge q\lceil \frac{p}{2}\rceil$ . Hence,  $p\lfloor \frac{q}{2}\rfloor + 1 \ge q\lceil \frac{p}{2}\rceil$ . So  $p+q \le 2$ . This is impossible. Therefore  $\gamma_S(Cay(G:S)) = \omega(f) = p$ .

**Theorem 3.5.** Let  $\mathbb{Z}_{pq} = \langle S \rangle$  where  $p \geq 7$  and S = B(1, pq). Then

$$\gamma_{s}(Cay(\mathbb{Z}_{pq}:S))=5.$$

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Proof. We define  $f:V(Cay(\mathbb{Z}_{pq}:S)) \to \{-1,1\}$  such that f(i)=-1 if and only if  $i\in\{1,2,\ldots,\frac{pq-5}{2}\}$ . It is easily seen that  $\lfloor\frac{q}{2}\rfloor\leq |A_i^-|\leq \lceil\frac{q}{2}\rceil$  for every  $1\leq i\leq p$ . Also  $\lfloor\frac{p}{2}\rfloor\leq |B_j^-|\leq \lceil\frac{p}{2}\rceil$  for any  $1\leq j\leq q$ . Let  $v\in A_t\cap B_s$  such that  $1\leq t\leq p$  and  $1\leq s\leq q$ . In the worst situation,  $|A_t^-|=\lfloor\frac{q}{2}\rfloor$  and  $|B_s^-|=\lfloor\frac{p}{2}\rfloor$ . In this case  $1\leq f[v]\leq 5$ . Hence, f is a signed dominating function. Also  $\omega(f)=pq-2|V_f^-|=5$ . Thus  $\gamma_S(Cay(\mathbb{Z}_{pq}:S))\leq 5$ . What is left is to show that if g is a  $\gamma_S$ -function, then  $\omega(g)\geq 5$ . On the contrary, suppose that g be a  $\gamma_S$ -function and  $\omega(g)<\omega(f)$ . Hence,  $|V_g^-|<|V_f^-|$ . There is no loss of generality in assuming  $|V_g^-|=\frac{pq-3}{2}$ . Let  $A_i^-=A_i\cap V_g^-$  and  $B_j^-=B_j\cap V_g^-$ . In order to reach the contradiction we use two following steps:

Step 1.  $A_i^- \neq \emptyset$  for every  $1 \leq i \leq p$ .

On the contrary, suppose that for some  $1 \leq m \leq p$ ,  $A_m^- = \emptyset$ . Let  $w \in A_m$ . So there is  $1 \leq \ell \leq q$  such that  $w \in A_m \cap B_\ell$ . Hence,  $g[w] = (p-1)(q-1)+1-2(|V_g^-|-|B_\ell^-|) \geq 1$ . Thus  $|B_\ell^-| \geq \frac{p+q-4}{2}$ . So  $|V_g^-| \geq q(\frac{p+q-4}{2})$ . Hence,  $pq-3 \geq q(pq-4)$ . This makes a contradiction. By similar argument we have  $B_j^- \neq \emptyset$  for every  $1 \leq j \leq q$ .

Step 2. For every  $1 \le i \le p$ ,  $|A_i^-| \ge \lfloor \frac{q}{2} \rfloor$ . On the contrary, let  $|A_l^-| = \lfloor \frac{q}{2} \rfloor - 1$ . Let  $v \in A_l$ . There is  $1 \le l' \le q$  such that  $v \in A_l \cap B_{l'}$ . If g(v) = -1, then  $g[v] = (p-1)(q-1) + 1 - 2(|V_g^-| - |A_l^-| - |B_{l'}^-| + 2) \ge 1$ . Hence,  $|B_{l'}^- \setminus \{v\}| \ge \lceil \frac{p}{2} \rceil$ . If g(v) = 1, then  $|B_{l'}^-| \ge \lfloor \frac{p}{2} \rfloor$ . Therefore,  $|V_g^-| \ge |A_l^-| (\lceil \frac{p}{2} \rceil + 1) + |A_l^+| \lfloor \frac{p}{2} \rfloor$ . This implies that  $q \le 3$ . This is a contradiction.

Likewise Step 2,  $|B_j^-| \ge \lfloor \frac{p}{2} \rfloor$  for every  $1 \le j \le q$ . Since  $|V_g^-| = \frac{pq-3}{2}$ , there is  $1 \le k \le p$  such that  $|A_k^-| = \lfloor \frac{q}{2} \rfloor$ . On the other hand, suppose that for  $1 \le t \le q$ ,  $|B_{l_r}^-| = \lfloor \frac{p}{2} \rfloor$ . Let  $u \in A_k^- \cap B_s^-$ . If  $s \in \{l_1, \dots, l_t\}$ , then

$$g[u] = -1 + (p-1)(q-1) - 2\left(|V_g^-| - |A_k^-| - |B_s^-| + 2\right)$$

$$= -1 + (p-1)(q-1) - 2\left(\frac{pq-3}{2} - \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{p}{2} \right\rfloor + 2\right)$$

$$= -3$$

This is a contradiction by g is a signed dominating function. Hence, s is not in  $\{l_1, \dots, l_t\}$ . Since  $|A_k^-| = \lfloor \frac{q}{2} \rfloor$ ,  $q - t \geq \lfloor \frac{q}{2} \rfloor$  and so  $t \leq \lceil \frac{q}{2} \rceil$ . As a consequence,

$$|V_g^-| \geq t \lfloor \frac{p}{2} \rfloor + (q-t) \lceil \frac{p}{2} \rceil \geq \lceil \frac{q}{2} \rceil \lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor \lceil \frac{p}{2} \rceil.$$

Since  $|V_q^-| = \frac{pq-3}{2}$ , this makes a contradiction. Therefore,

$$\gamma_s(Cay(\mathbb{Z}_{pq}:S))=5.$$

**Corollary 3.6.** For any k-regular graph  $\Gamma$  on n vertices  $\gamma_S(\Gamma) \geq \frac{n}{k+1}$ . Hence,  $\gamma_S(\Gamma) \geq 1$ . It is easy to check that  $\gamma_S(\Gamma) = 1$  if and only if  $\Gamma$  is a complete

graph and n is odd. Furthermore, for any prime numbers p < q, there is a (p-1)(q-1)-regular graph  $\Gamma$  with pq vertices such that  $\gamma_s(\Gamma) \in \{2,3,5\}$ .

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