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Isoclinic Classification of Some Pairs (G, G') of p-Groups

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Abstract. The equivalence relation isoclinism partitions the class of all pairs of groups into families. In this paper, a complete classification of the set of all pairs (G, G') is established whenever, p is a prime number greater than 3 and G is a p-group of order at most p^5. Moreover, the classification of pairs (H, H') for extra special p-groups H is also given.

Keywords: Pairs of groups, Isoclinism, Classification of p-groups.

2000 Mathematics subject classification: 20D15, 20E99, 20D60.

1. Introduction

The problem of classification of groups or objects in an algebraic concept or finding the resemblance of some groups of objects and study their common properties has been the interest of many researchers. For example, the resemblance or similarity among three objects of a set is discussed in [7]. F. Ayatollah Zadeh Shirazi and A. Hosseini [1] tried to introduce $(\alpha, \beta)-$ linear connectivity concept as a tool to classify the class of all linear connected topological spaces. This article deals with the classification of some pairs of groups.

By a pair of groups (G, N) we simply mean a group G with normal subgroup N. Pairs of groups as a tool for simultaneous study of a group and its subgroup, have been verified by some authors during these recent two decades.

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For instance, Ellis [2, 3] introduced the notions of capability and Schur multiplier for a pair of groups and turned out some of their properties. Also, a work of Moghaddam *et al.* [8] demonstrated some more properties of the Schur multiplier of a pair, as well as a recent work of Pourmirzaei *et al.* [9] has given both a criterion for characterizing the capability of a pair and also a complete classification of finitely generated abelian capable pairs. Hassanzadeh *et al.* [5] established a new notion of nilpotency for a pair which is lain between the usual notion of nilpotency for a group and its subgroup. This new concept will be defined in such a way that the nilpotency of G implies that of (G, N) and the nilpotency of (G, N) forces N to be nilpotent. On the other hand, Salemkar *et al.* [10] introduced the notion of isoclinism on pairs of groups and derived some of its properties.

Now, it seems that the accomplished investigations on pairs of groups provide a suitable grounds for thinking of the classification of pairs of groups. The inspirational paper in this area, is the work of P. Hall [4] on the classification of some p-groups. In this paper we intend to present a complete classification of pairs (G, G') with respect to the isoclinism of pairs, when G is a p-group of order at most p^5.

The process of classifying of these pairs when G is a p-group of order at most p^4 will be done using the following factors; the order of G', the notion of nilpotency class for G and the structure of central factor group of G. Whereas, these tools will not be sufficient for classifying the set of all pairs (G, G') when G is a p-group of order p^5. Therefore, we need some more information for such p-groups. The works of P. Hall [4] and James [6] are two valuable references for attaining the desired information. Accordingly, our strategy is as follows. We first do a screening in the set of all pairs (G, G') according to choosing the isoclinic groups G from the families Φ_i $(1 \leq i \leq 10)$. Indeed, we get one set for each family Φ_i. Then we will verify whether their derived pairs in each created set are isoclinic. This is the second step of screening. Finally, we will compare the pairs from different sets to obtain disjoint families. This will complete our screening and yields the required classification.

But, for recognizing the isoclinism families of these pairs, we need some technical lemmas and theorems in which we try to adopt some properties of such pairs, depending on the structure of the group itself. Moreover, since the derived subgroup of each extra special p-group is cyclic and has order p, one can easily obtain a complete classification of all pairs (G, G') for extra special p-groups G.

2. Preliminaries

The notion of isoclinism between two groups which was first introduced by P. Hall [4] later extended by Salemkar *et al.* [10] for pairs of groups as follows.

Definition 2.1. Let (G_1, N_1) and (G_2, N_2) be two pairs of groups and $Z(G_i, N_i) = Z(G_i) \cap N_i$ be the center of pair (G_i, N_i). Then the pair of mappings (α, β) is an isoclinism between the pairs (G_1, N_1) and (G_2, N_2), if

$$\alpha : \frac{G_1}{Z(G_1, N_1)} \longrightarrow \frac{G_2}{Z(G_2, N_2)} \quad \textit{and} \quad \beta : [N_1, G_1] \longrightarrow [N_2, G_2],$$

are both isomorphisms, such that $\alpha(N_1/Z(G_1, N_1)) = N_2/Z(G_2, N_2)$ and for all $n_1 \in N_1$ and $g_1 \in G_1$ we have $\beta([n_1, g_1]) = [n_2, g_2]$, where $g_2 \in \alpha(\overline{g}_1)$ and $n_2 \in \alpha(\overline{n}_1)$. In such a situation, we say that (G_1, N_1) is isoclinic to (G_2, N_2) and write $(G_1, N_1) \sim (G_2, N_2)$.

Clearly, an isoclinism between two pairs is an equivalence relation among all pairs of groups. This new concept motivates us to verify the classification of pairs (G, N) with respect to this notion. Before doing this, one should note that if N is chosen to be equal to G, then the notion of isoclinism between pairs coincides with the usual notion of isoclinism between groups, and if N is the trivial group, the isoclinism of pairs will be the usual isomorphism of groups. On the other hand, if N is chosen to be the center of G, then the notion of isoclinism of groups implies the isoclinism of pairs. This means that the classification of all pairs $(G, Z(G))$ is weaker than the classification of the groups G with respect to the notion of isoclinism of groups. In other words, each class of isoclinism of groups lies in one of the isoclinism class of pairs $(G, Z(G))$, but the inverse is not true. Also, one can easily see that if N is not equal to the center of G, then the notions of isoclinism of pairs and isoclinism of groups are not comparable with each other. The present paper follows the problem when N is chosen to be the derived subgroup. Also, since our classification is inspired P. Hall's classification, therefore the chosen group G assumed to be a p-group of order at most p^5 $(p > 3)$.

One should recall that the main factor which was used by P.Hall in his classification is the structure of central factor group. Accordingly, he classified the groups of order p^3, p^4 and p^5 into the families $\{\Phi_1, \Phi_2\}, \{\Phi_1, \Phi_2, \Phi_3\}$ and $\{\Phi_1, \ldots, \Phi_{10}\}$, respectively, in which Φ_1 contains abelian groups and the other Φ_i's depend on some structural properties. We shall use these notions and notations throughout the paper.

Now, let G_1 and G_2 be two isoclinic groups. Then the isoclinism between (G_1, G_1') and (G_2, G_2') implies that the orders of G_1 and G_2 must be equal. In other words, if G_1 and G_2 are two isoclinic groups of different orders, then their derived pairs are not isoclinic. With this in mind, we classify the set of all pairs (G, G') by considering the order of G and isoclinism families, $\Phi_1, \Phi_2, \ldots, \Phi_{10}$. Since the order and isoclinism family are two main factors for this classification, then we nominate the following notations for the isoclinism families of pairs (G, G').

Notation 1. *Let l be a natural number and G be a group of order p^n. The symbol $\Psi(A, n_l)$ denotes the l-th isoclinism family of pairs (G, G'), in which G belongs to the set A. Moreover, $\Phi_{(r,s)}$ denotes the union of all groups in families Φ_r and Φ_s.*

More information about the structure of families $\Psi(A, n_l)$ will subsequently be introduced. But before embarking on the classification, we provide some fundamental statements.

Theorem 2.1. *Let G and H be two abelian groups. Then*

$$(G, G') \sim (H, H') \quad \text{if and only if} \quad G \cong H.$$

Therefore, the classification of pairs (G, G') for abelian group G leads only to the isomorphism. Thus, for abelian p-group G of order p^n, the number of isoclinism classes of pairs (G, G') is just equal to the number of partitions of n.

Theorem 2.2. *Let G and H be two groups of nilpotency class 2. Then*

$$(G, G') \sim (H, H') \quad \text{if and only if} \quad \frac{G}{G'} \cong \frac{H}{H'}.$$

Proof. Let K be a nilpotent group of class 2. Since the derived subgroup of K is contained in the center of K, one can immediately conclude the result by Definition 2.1. □

One of the interesting results of Theorem 2.2 is as follows.

Corollary 2.2.1. *Let G and H be two extra special p-groups. Then two pairs (G, G') and (H, H') are isoclinic if and only if G and H have the same order.*

A complete classification of pairs (G, G') for extra special p-groups G is given by Corollary 2.2.1. Using this fact, if the order of G is equal to p^{2k+1}, then for each value of $k = 1, 2, \cdots$ there is just one such family.

P. Hall [4] stated that in every isoclinism family there exist groups S with the property that $Z(S) \subseteq S'$. These groups are called stem groups.

Lemma 2.1. *Let Φ be an isoclinism family of groups, in which every group is a stem group. Then for any two groups G_1 and G_2 in this family, we have $(G_1, G_1') \sim (G_2, G_2')$.*

Proof. Let G_1 and G_2 belong to Φ and (α, β) be an isoclinism between them. It can be seen that $(\alpha, \beta|_{\gamma_3(G)})$ is an isoclinism between (G_1, G_1') and (G_2, G_2'). □

3. The Classification of Pairs (G, G'), for Groups G of Order at Most p^5 $(p > 3)$

3.1. $|G| = p^3$**.** The classification of pairs (G, G') for p-groups G of order p^3 is straightforward. Because, for abelian group G we have three disjoint families according to the partitions of 3. Besides, every non-abelian group of order p^3

is an extra special p-group and hence, the constructed pairs of such groups are isoclinic, by Corollary 2.2.1. Note that we need to determine the structure of $G/Z(G, G')$ for recognizing the distinction of the families.

Theorem 3.1. *Let p be a prime number greater than 3 and G be a group of order p^3. Then, the set of all pairs (G, G') is partitioned into four isoclinism families $\Psi(\Phi_1, 3_1), \Psi(\Phi_1, 3_2), \Psi(\Phi_1, 3_3)$ and $\Psi(\Phi_2, 3_4)$. The structure of $G/Z(G, G')$ in these families will be $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p, \oplus_1^3 \mathbb{Z}_p$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$, respectively.*

3.2. $|G| = p^4$**.** P. Hall [4] pointed that two groups of order p^4 are isoclinic if and only if their nilpotency class are the same. Now, by Theorem 2.1, for abelian p-groups G of order p^4, there are five disjoint families for pairs (G, G'), which are named $\Psi(\Phi_1, 4_1)$, $\Psi(\Phi_1, 4_2)$, $\Psi(\Phi_1, 4_3)$, $\Psi(\Phi_1, 4_4)$ and $\Psi(\Phi_1, 4_5)$. The structure of $G/Z(G, G')$ in these families is $\mathbb{Z}_{p^4}$, $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $\oplus_1^4 \mathbb{Z}_p$, respectively.

Lemma 3.1. *Let G be a p-group of order p^4 and nilpotency class 2. Then, the set of all pairs (G, G') is partitioned into families $\Psi(\Phi_2, 4_6)$ and $\Psi(\Phi_2, 4_7)$. The structure of the factor group $G/Z(G, G')$ in these families will be $\oplus_1^3 \mathbb{Z}_p$ and $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$, respectively.*

Proof. By Theorem 2.2, it is enough to obtain the non-isomorphic structures for the factor group G/G'. Since G is a non-abelian group, then its derived factor group may have structures $\oplus_1^3 \mathbb{Z}_p$ or $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. Now, the result will follow if we show that there exist groups of order p^4, which have the stated structures for their derived factors. Using James' list [6], one can verify that the groups $\Phi_2(211)b$ and $\Phi_2(31)$, are as desirable. □

Lemma 3.2. *Let $\Phi_2(1^3)$ be the extra special p-group of order p^3 and exponent p. If G is a p-group of order p^4 and nilpotency class 3, then the set of all pairs (G, G') will lie in the family $\Psi(\Phi_3, 4_8)$, in which the structure of its factor group $G/Z(G, G')$ is $\Phi_2(1^3)$.*

Proof. It is an easy matter to check that every group of order p^4 and nilpotency class 3 is a stem group. In addition, these groups are isoclinic. Hence by the same argument as Lemma 2.1, the result follows. □

The following theorem will enable us to take full advantage of the results so far obtained.

Theorem 3.2. *Let p be a prime number greater than 3 and G be a group of order p^4. Then, the set of all pairs (G, G') is partitioned into eight isoclinism families $\Psi(\Phi_1, 4_1), \Psi(\Phi_1, 4_2), \ldots, \Psi(\Phi_2, 4_7)$ and $\Psi(\Phi_3, 4_8)$.*

3.3. $|G| = p^5$**.** P. Hall [4] proved that the number of isoclinism families for groups of order p^5 is eight when $p = 2$, and ten for $p > 2$. He also introduced ten families $\Phi_1, \Phi_2, \ldots, \Phi_{10}$, for groups of order p^5 $(p > 3)$. In these families, all the members are characterized by some properties. We used these properties to classify the pairs (G, G') of p-groups of order p^5 $(p > 3)$. For this, we explain these properties and some useful information in each family. The family Φ_1 is the set of all abelian groups, Φ_2 contains every group which its central factor group is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and its derived subgroup is cyclic of order p. Every group in Φ_3 has non-abelian central factor group of order p^3 and exponent p, and also the order of its derived subgroup is p^2. Central factors of groups in families Φ_4 and Φ_5 are elementary abelian p-groups of orders p^3 and p^4, respectively. Meantime, for each group in these families we know that its center and its derived subgroup are coincide. But the order of the derived subgroup in such families is p^2 and p, respectively. For groups with non-abelian central factor group of order p^3 and exponent p, anyone can see that the intersection of the derived and center subgroups of such groups has order p^2. This family is called Φ_6. The other families Φ_7, Φ_8, Φ_9 and Φ_{10} have non-abelian central factor group of order p^4 and their central factors have the structures $\Phi_2(1^3) \times \mathbb{Z}_p, \Phi_2(22), \Phi_3(1^4)$ and $\Phi_3(1^4)$, respectively (for the exact structures of $\Phi_2(22)$ and $\Phi_3(1^4)$ see [6]). Although groups in families Φ_9 and Φ_{10} have the same central factor group and even have the same derived subgroup of order p^3, these groups are not isoclinic. Actually, groups in the family Φ_9 have an abelian subgroup of index p, while those in the family Φ_{10} do not have. This fact implies that the groups of order p^5, where their central factor groups are isomorphic to $\Phi_3(1^4)$, have been classified into families Φ_9 and Φ_{10}.

Now, we can follow our purpose. First, let G be an abelian p-group of order p^5. There are seven disjoint families for pairs (G, G'), using Theorem 2.1, which are named $\Psi(\Phi_1, 5_1),\ \Psi(\Phi_1, 5_2), \ldots,$ and $\Psi(\Phi_1, 5_7)$. The structure of $G/Z(G, G')$ in these families is $\mathbb{Z}_{p^5},\ \mathbb{Z}_{p^4} \oplus \mathbb{Z}_p, \ldots,$ and $\oplus_1^5 \mathbb{Z}_p$.

Lemma 3.3. *Let G be a group of order p^5 in family Φ_2. Then the set of all pairs (G, G') is partitioned into four families $\Psi(\Phi_2, 5_8), \Psi(\Phi_2, 5_9), \Psi(\Phi_2, 5_{10})$ and $\Psi(\Phi_2, 5_{11})$. The structure of the factor group $G/Z(G, G')$ in these families will be $\oplus_1^4 \mathbb{Z}_p,\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p,\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$, respectively.*

Proof. Since the nilpotency class of G is 2, then using Theorem 2.2, it is sufficient to obtain the non-isomorphic derived factors of G. Since G/G' is an abelian group of order p^4, then it may have one of structures $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p,\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2},\ \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\oplus_1^4 \mathbb{Z}_p$. Using James' list [6], one can check that all of these structures can be found in this family. For example, the groups $\Phi_2(41),\ \Phi_2(221)d,\ \Phi_2(2111)c$ and $\Phi_2(1^5)$ have the above derived factors respectively. This completes the proof. □

Lemma 3.4. *Let G be a group of order p^5 in family Φ_3. Then the set of all pairs (G, G') is partitioned into families $\Psi(\Phi_3, 5_{12})$ and $\Psi(\Phi_3, 5_{13})$. The factor group $G/Z(G, G')$ in these families will be $\Phi_2(1^4)$ and $\Phi_2(211)c$, respectively.*

Proof. Note that in the family Φ_3 we have $\gamma_3(G) = Z(G, G') \cong \mathbb{Z}_p$. Then, the factor group $G/Z(G, G')$ is a group of order p^4 and nilpotency class 2. Although, the structure of this factor group can be isomorphic to one of the groups, $\Phi_2(1^4) = \Phi_2(1^3) \times Z_p$, $\Phi_2(211)a$, $\Phi_2(31)$, $\Phi_2(22)$, $\Phi_2(211)b$ or $\Phi_2(211)c$, but by considering the presentations of G, given by James [6], one can obtain only two groups $\Phi_2(1^4)$ and $\Phi_2(211)c$ for the stated factor group. Now, it is easy to see that all the pairs (G, G'), having the same factor group $G/Z(G, G')$, are isoclinic. Therefore, the result holds.

□

Lemma 3.5. *Let G and H be two isoclinic groups in family Φ_i, for $4 \leq i \leq 10$. Then $(G, G') \sim (H, H')$.*

Proof. It is not difficult to show that every group in these families is a stem group. Now, the result follows from Lemma 2.1. □

According to Lemma 3.5, the obtained families are nominated $\Psi(\Phi_4, 5_{14})$, $\Psi(\Phi_5, 5_8)$, $\Psi(\Phi_6, 5_{15})$, $\Psi(\Phi_7, 5_{12})$, $\Psi(\Phi_8, 5_{16})$, $\Psi(\Phi_9, 5_{17})$ and $\Psi(\Phi_{10}, 5_{18})$, respectively.

The remainder of this section will be devoted to show that which of the families $\Psi(\Phi_k, 5_j)$' s are disjoint.

By Definition 2.1, it is enough to compare these six families, $\Psi(\Phi_2, 5_8)$, $\Psi(\Phi_5, 5_8)$, $\Psi(\Phi_3, 5_{12})$, $\Psi(\Phi_7, 5_{12})$, $\Psi(\Phi_9, 5_{17})$ and $\Psi(\Phi_{10}, 5_{18})$ together.

Certainly, using Theorem 2.2, the pairs in families $\Psi(\Phi_2, 5_8)$ and $\Psi(\Phi_5, 5_8)$ are isoclinic. So these two families are coincide. We nominate the symbol $\Psi(\Phi_{(2,5)}, 5_8)$ for the family containing all of pairs of two families $\Psi(\Phi_2, 5_8)$ and $\Psi(\Phi_5, 5_8)$. Moreover, one can see that the pairs in families $\Psi(\Phi_3, 5_{12})$ and $\Psi(\Phi_7, 5_{12})$ are also isoclinic. For this, it is enough to show that a pair in the family $\Psi(\Phi_3, 5_{12})$ is isoclinic to one of pairs in family $\Psi(\Phi_7, 5_{12})$. Let $G_1 = \Phi_3(2111)a$ and $G_2 = \Phi_7(2111)a$, where these groups defined in [6] as follows.

$$G_1 = \left\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \;\middle|\; \begin{array}{c} [\alpha_j, \alpha] = \alpha_{j+1} \ (j = 1, 2) \\ \alpha^p = \alpha_3, \ \alpha_i^p = 1 \ (i = 1, 2, 3, 4) \\ [a, b] = 1 \\ \text{(for all other commutators of generators } a \text{ and } b) \end{array} \right\rangle,$$

and

$$G_2 = \left\langle \beta, \beta_1, \beta_2, \beta_3, \beta_4 \;\middle|\; \begin{array}{c} [\beta_j, \beta] = \beta_{j+1} \ (j = 1, 2) \\ [\beta_1, \beta_4] = \beta^p = \beta_3, \ \beta_i^p = 1 \ (i = 1, 2, 3, 4) \\ [a, b] = 1 \\ \text{(for all other commutators of generators } a \text{ and } b) \end{array} \right\rangle.$$

Let

$$\begin{array}{llclcllcl} \alpha_0: & \frac{G_1}{G_1'\cap Z(G_1)} & \longrightarrow & \frac{G_2}{G_2'\cap Z(G_2)} & \quad and \quad & \beta_0: & \gamma_3(G_1) & \longrightarrow & \gamma_3(G_2) \\ & \overline{\alpha} & \longmapsto & \overline{\beta} & & & \alpha_3 & \longmapsto & \beta_3. \\ & \overline{\alpha_i} & \longmapsto & \overline{\beta_i} & & & & & \end{array}$$

It is easy to see that (α_0, β_0) is an isoclinism between (G_1, G_1') and (G_2, G_2'). So, the families $\Psi(\Phi_3, 5_{12})$ and $\Psi(\Phi_7, 5_{12})$ are coincide. We nominate the symbol $\Psi(\Phi_{(3,7)}, 5_{12})$ for the family containing all the pairs in these two families.

Lemma 3.6. *Let (α, β) be an isoclinism between (G_1, N_1) and (G_2, N_2). If $Z(G_1, N_1) \subseteq H_1 \subseteq G_1$ and $\alpha(H_1/Z(G_1, N_1)) = H_2/Z(G_2, N_2)$, then $(H_1, H_1 \cap N_1) \sim (H_2, H_2 \cap N_2)$.*

Proof. Consider $\bar{\alpha} : \frac{H_1}{Z(H_1, H_1\cap N_1)} \longrightarrow \frac{H_2}{Z(H_2, H_2\cap N_2)}$ with $\bar{\alpha}(\bar{h_1}) = \bar{h_2}$, in which $h_2 \in \alpha(h_1 Z(G_1, N_1))$, and $\bar{\beta} : [H_1 \cap N_1, H_1] \longrightarrow [H_2 \cap N_2, H_2]$ with $\bar{\beta}(x) = \beta(x)$. It is easy to see that, the pair $(\bar{\alpha}, \bar{\beta})$ is an isoclinism between two pairs $(H_1, H_1 \cap N_1)$ and $(H_2, H_2 \cap N_2)$. □

Theorem 3.3. *The pairs in the families $\Psi(\Phi_9, 5_{17})$ and $\Psi(\Phi_{10}, 5_{18})$ are not isoclinic.*

Proof. Assume, by way of contradiction, that the pair (α, β) is an isoclinism between two pairs (G_1, G_1') and (G_2, G_2') in the families $\Psi(\Phi_9, 5_{17})$ and $\Psi(\Phi_{10}, 5_{18})$, respectively. Therefore, $G_1 \in \Phi_9$ and $G_2 \in \Phi_{10}$. Now, let L_1 be an abelian subgroup of index p in G_1 and L_2 be the image in G_2 of $L_1/Z(G_1, G_1')$, or equivalently $\alpha(L_1/Z(G_1, G_1')) = L_2/Z(G_2, G_2')$. Hence, the order of L_2 will be p^4. Since $Z(G_1, G_1')$ is contained in L_1, by Lemma 3.6, we have $(L_1, G_1') \sim (L_2, L_2 \cap G_2')$. Thus $L_1/G_1' \cong L_2/(Z(L_2) \cap G_2')$. But the order of G_1' is p^3, then L_1/G_1' is a cyclic group of order p. Therefore, $L_2/(Z(L_2) \cap G_2') \cong \mathbb{Z}_p$, and hence L_2 is an abelian subgroup of index p in G_2. This contradiction completes the proof. □

Now, we sum up our conclusions about the obtained families as follows.

Theorem 3.4. *Let p be a prime number greater than 3. If G is a p-group of order p^5, then the set of all pairs (G, G') is partitioned into eighteen families, $\Psi(\Phi_1, 5_1), \Psi(\Phi_1, 5_2), \ldots, \Psi(\Phi_9, 5_{17})$ and $\Psi(\Phi_{10}, 5_{18})$.*

Moreover, since groups of order at most p^2 are abelian, then the set of all derived pairs of these groups are classified into four distinct isoclinism families $\Psi(\Phi_1, 0_1), \Psi(\Phi_1, 1_1), \Psi(\Phi_1, 2_1)$ and $\Psi(\Phi_1, 2_2)$.

Finally, by comparing the families $\Psi(A, n_l)$ (for $0 \leq n \leq 5,\ 1 \leq l \leq 18$) and whatever established in Theorems 3.1, 3.2 and 3.4, one can deduce that the set of all pairs (G, G'), for p-groups G of order at most p^5, is partitioned into twenty six introduced isoclinism families. More precisely, it is easy to see that, only the following families are isoclinic together and others are distinct. $\Psi(\Phi_1, 3_2) \sim \Psi(\Phi_2, 4_7)$, $\Psi(\Phi_1, 3_3) \sim \Psi(\Phi_2, 4_6) \sim \Psi(\Phi_4, 5_{14})$, $\Psi(\Phi_1, 2_2) \sim \Psi(\Phi_2, 3_4)$, $\Psi(\Phi_1, 4_2) \sim \Psi(\Phi_2, 5_{11})$, $\Psi(\Phi_1, 4_3) \sim \Psi(\Phi_2, 5_{10})$, $\Psi(\Phi_1, 4_4) \sim \Psi(\Phi_2, 5_9)$, and $\Psi(\Phi_1, 4_5) \sim \Psi(\Phi_{(2,5)}, 5_8)$. Therefore we will have twenty six isoclinism families whenever G is a p-group of order at most p^5. Ttable 1 provides some structural properties of these families.

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$Isoclinism family$	$G/Z(G,G')$	$[G',G]$	$G'/Z(G,G')$
θ_1	1	1	1
θ_2	$\mathbb{Z}_p$	1	1
θ_3	$\mathbb{Z}_{p^2}$	1	1
θ_4	$\mathbb{Z}_{p^3}$	1	1
θ_5	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$	1	1
θ_6	$\oplus_1^3 \mathbb{Z}_p$	1	1
θ_7	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	1	1
θ_8	$\mathbb{Z}_{p^4}$	1	1
θ_9	$\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p$	1	1
θ_{10}	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$	1	1
θ_{11}	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$	1	1
θ_{12}	$\oplus_1^4 \mathbb{Z}_p$	1	1
θ_{13}	$\Phi_2(1^3)$	$\mathbb{Z}_p$	$\mathbb{Z}_p$
θ_{14}	$\mathbb{Z}_{p^5}$	1	1
θ_{15}	$\mathbb{Z}_{p^4} \oplus \mathbb{Z}_p$	1	1
θ_{16}	$\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$	1	1
θ_{17}	$\mathbb{Z}_{p^3} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$	1	1
θ_{18}	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$	1	1
θ_{19}	$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$	1	1
θ_{20}	$\oplus_1^5 \mathbb{Z}_p$	1	1
θ_{21}	$\Phi_2(1^4)$	$\mathbb{Z}_p$	$\mathbb{Z}_p$
θ_{22}	$\Phi_2(211)c$	$\mathbb{Z}_p$	$\mathbb{Z}_p$
θ_{23}	$\Phi_2(1^3)$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	$\mathbb{Z}_p$
θ_{24}	$\Phi_2(22)$	$\mathbb{Z}_p$	$\mathbb{Z}_p$
θ_{25}	$\Phi_3(1^4)$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$
θ_{26}	$\Phi_3(1^4)$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$	$\mathbb{Z}_p \oplus \mathbb{Z}_p$

TABLE 1