Iranian Journal of Mathematical Sciences and Informatics

Vol. 13, No. 2 (2018), pp 101-109 DOI: 10.7508/ijmsi.2018.13.009

On *I*-Statistical Convergence

Shyamal Debnath*,a, Debjani Rakshit^b

^aDepartment of Mathematics, Tripura University, Agartala-799022, India.
 ^bDepartment of FST, ICFAI University, Tripura, Kamalghat, West Tripura-799210, India.

E-mail: shyamalnitamath@gmail.com E-mail: debjanirakshit88@gmail.com

ABSTRACT. In this paper we investigate the notion of I-statistical convergence and introduce I-st limit points and I-st cluster points of real number sequence and also studied some of its basic properties.

Keywords: *I*-limit point, *I*-cluster point, *I*-statistically Convergent.

2000 Mathematics subject classification: 40A35, 40D25.

1. Introduction

In 1951 Fast [6] and Steinhaus [18] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [8], Salat [19] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 13, 14, 21].

The notion of I-convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [12]. They used the notion of an ideal I of subsets of the set N to define such a concept. For an extensive view of this article we refer [4, 11, 20].

The idea of I-convergence was further extended to I-statistical convergence by Savas and Das [16]. Later on more investigation in this direction was done

Received 04 February 2016; Accepted 27 September 2017 ©2018 Academic Center for Education, Culture and Research TMU

^{*}Corresponding Author

by Savas and Das [17], Debnath and Debnath [3], Mursaleen et.al [15], Et et al. [5] and many others [9, 10, 22, 23] . In [16], Savas and Das introduced the I-statistical convergence and I- λ -statistical convergence and the relation between them. Also they studied these concept in the notion of $[V, \lambda]$ - summability method.

In the present paper we return to the view of *I*-statistical convergence as a sequential limit concept and we extend this concept in a natural way to define a *I*-statistical analogue of the set of limit points and cluster points of a real number sequence.

2. Definitions and Preliminaries

Definition 2.1. [8] If K is a subset of the positive integers N, then K_n denotes the set $\{k \in K : k \le n\}$. The natural density of K is given by $D(K) = \lim_{n \to \infty} \frac{|K_n|}{n}$.

Definition 2.2. [8] A sequence (x_n) is said to be statistically convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \ge \varepsilon\}$ has natural density zero. x_0 is called the statistical limit of the sequence (x_n) and we write st- $\lim_{n\to\infty} x_n = x_0$.

Definition 2.3. [7] If $(x_{k(j)})$ be a subsequence of a sequence $x = (x_n)$ and density of $K = \{k(j) : j \in N\}$ is zero then $(x_{k(j)})$ is called a thin subsequence. Otherwise $(x_{k(j)})$ is called a non-thin subsequence of x.

 x_0 is said to be a statistical limit point of a sequence (x_n) , if there exist a non-thin subsequence of (x_n) which conveges to x_0 .

Let Λ_x denotes the set of all statistical limit points of the sequence (x_n) .

Definition 2.4. [7] x_0 is said to be a statistical cluster point of a sequence $x = (x_n)$, provided that for each $\varepsilon > 0$ the density of the set $\{k \in N : d(x_k, x_0) < \varepsilon\}$ is not equal to 0.

Let Γ_x denotes the set of all statistical cluster points of the sequence (x_n) .

Definition 2.5. [12] Let X is a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal on X if and only if

- (i) $\emptyset \in I$;
- (ii) for each $A, B \in I$ implies $A \cup B \in I$;
- (iii) for each $A \in I$ and $B \subset A$ implies $B \in I$.

Definition 2.6. [12] Let X is a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if

- (i) $\emptyset \notin \mathcal{F}$;
- (ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
- (iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal I. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if $I \supset \{\{x\} : x \in X\}$

Definition 2.7. [12] Let $I \subset P(N)$ be a non-trivial ideal on N. A sequence (x_n) is said to be I-convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : d(x_k, x_0) \geq \varepsilon\}$ belongs to I. x_0 is called the I-limit of the sequence (x_n) and we write I- $lim_{n\to\infty}x_n = x_0$.

Definition 2.8. [12] x_0 is said to be *I*-limit point of a sequence $x = (x_n)$ provided that there is a subset $K = \{k_1 < k_2 < ...\} \subset N$ such that $K \notin I$ and $\lim x_{k_i} = x_0$.

Let $I(\Lambda_x)$ denotes the set of all *I*-limit points of the sequence x.

Definition 2.9. [12] x_0 is said to be *I*-cluster point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ the set $\{k \in N : d(x_k, x_0) < \varepsilon\} \notin I$.

Let $I(\Gamma_x)$ denotes the set of all *I*-cluster points of the sequence x.

Definition 2.10. [16] A sequence $x = (x_n)$ is said to be *I*-statistically convergent to x_0 if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in N : \frac{1}{n} | \left\{k \le n : d\left(x_n, x_0\right) \ge \varepsilon\right\} | \ge \delta\right\} \in I.$$

 x_0 is called *I*-statistical limit of the sequence (x_n) and we write, *I*-st $\lim x_n = c_0$.

Throughout the paper we consider I as an admissible ideal.

3. Main Results

Theorem 3.1. If (x_n) be a sequence such that I-st $\lim x_n = x_0$, then x_0 determined uniquely.

Proof. If possible let the sequence (x_n) be *I*-statistically convergent to two different numbers x_0 and y_0

i.e, for any $\varepsilon > 0$, $\delta > 0$ we have,

$$A_{1} = \left\{ n \in N : \frac{1}{n} | \left\{ k \leq n : d\left(x_{k}, x_{0}\right) \geq \varepsilon \right\} | < \delta \right\} \in \mathcal{F}(I)$$
 and $A_{2} = \left\{ n \in N : \frac{1}{n} | \left\{ k \leq n : d\left(x_{k}, y_{0}\right) \geq \varepsilon \right\} | < \delta \right\} \in \mathcal{F}(I)$
Therefore, $A_{1} \cap A_{2} \neq \emptyset$, since $A_{1} \cap A_{2} \in \mathcal{F}(I)$.
Let $m \in A_{1} \cap A_{2}$ and take $\varepsilon = \frac{d(x_{0}, y_{0})}{3} > 0$ so, $\frac{1}{m} | \left\{ k \leq m : d\left(x_{k}, x_{0}\right) \geq \varepsilon \right\} | < \delta$

so,
$$\frac{1}{m} | \{ k \le m : d(x_k, x_0) \ge \varepsilon \} | < \delta$$

and $\frac{1}{m} | \{ k \le m : d(x_k, y_0) \ge \varepsilon \} | < \delta$

i.e, for maximum $k \leq m$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus, we must have

 $\{k \leq m : d(x_k, x_0) < \varepsilon\} \cap \{k \leq m : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of x_0 and y_0 are disjoint.

Hence the theorem is proved.

[Downloaded from ijmsi.ir on 2025-11-04]

Theorem 3.2. For any sequence (x_n) , st-lim $x_n = x_0$ implies I-st lim $x_n = x_0$.

Proof. Let $st\text{-}limx_n = x_0$.

Then for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \le n : d(x_k, x_0) \ge \varepsilon\}$ has natural density zero.

i.e, $\lim_{n\to\infty} \frac{1}{n} | \{k \le n : d(x_k, x_0) \ge \varepsilon\} | = 0$

So for every $\varepsilon > 0$ and $\delta > 0$,

 $\{n \in N : \frac{1}{n} | \{k \le n : d(x_k, x_0) \ge \varepsilon\} | \ge \delta\}$ is a finite set and therefore belongs to I, as I is an admissible ideal.

Hence
$$I$$
-st $\lim x_n = x_0$.

But the converse is not true.

EXAMPLE 3.3. Let $I = \zeta$ be the class of $A \subset N$ that intersect a finite number of Δ_j 's where $N = \bigcup_{j=1}^{\infty} \Delta_j$ and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$.

Let $x_n = \frac{1}{n}$ and so $\lim_{n \to \infty} d(x_n, 0) = 0$. Put $\epsilon_n = d(x_n, 0)$ for $n \in \mathbb{N}$.

Now define a sequence (y_n) by $y_n = x_j$ if $n \in \triangle_j$

Let $\eta > 0$. Choose $\nu \in N$ such that $\epsilon_{\nu} < \eta$. Then

$$A(\eta) = \{n : d(y_n, 0) \ge \eta\} \subset \triangle_1 \cup \dots \cup \triangle_{\nu} \in \zeta.$$

Now,
$$\{k \le n : d(y_k, 0) \ge \eta\} \subseteq \{n \in N : d(y_n, 0) \ge \eta\}$$

i.e,
$$\frac{1}{n} | \{ k \le n : d(y_k, 0) \ge \eta \} | \le | \{ n \in N : d(y_n, 0) \ge \eta \} |$$

so for any $\delta > 0$,

$$\left\{n \in N : \frac{1}{n} | \left\{k \le n : d\left(y_k, 0\right) \ge \eta\right\} | \ge \delta\right\} \subseteq \left\{n \in N : d\left(y_n, 0\right) \ge \eta\right\} \in \zeta.$$

Therefore (y_n) is ζ -statistically convergent to 0.

But (y_n) is not a statistically convergent.

Theorem 3.4. For any sequence (x_n) , I- $lim x_n = x_0$ implies I- $st lim x_n = x_0$.

Proof. The proof is obvious. But the converse is not true.

EXAMPLE 3.5. If we take $I = I_f$ the sequence (x_n) ,

where
$$x_n = \begin{cases} 0, & n = k^2, k \in \mathbb{N} \\ 1, & otherwise \end{cases}$$

is I-statistically convergent to 1. But (x_n) is not I-convergent.

Theorem 3.6. If each subsequence of (x_n) is I-statistically convergent to ξ then (x_n) is also I-statistically convergent to ξ .

Proof. Suppose (x_n) is not *I*-statistically convergent to ξ , then there exists $\varepsilon > 0$ and $\delta > 0$ such that

 $A = \left\{ n \in \mathbb{N} : \frac{1}{n} | \left\{ k \leq n : d(x_k, \xi) \geq \varepsilon \right\} | \geq \delta \right\} \notin I$. Since I is admissible ideal so A must be an infinite set.

Let $A = \{n_1 < n_2 < ... < n_m < ...\}$. Let $y_m = x_{n_m}$ for $m \in N$. Then $(y_m)_{m \in N}$ is a subsequence of (x_n) which is not I-statistically convergent to ξ , a contradiction. Hence the theorem is proved.

But the converse is not true. We can easily show this from example 3.5.

Theorem 3.7. Let (x_n) and (y_n) be two sequences then

- (i) I-st $\lim x_n = x_0$ and $c \in R$ implies I-st $\lim cx_n = cx_0$.
- (ii) I-st $\lim x_n = x_0$ and I-st $\lim y_n = y_0$ implies I-st $\lim (x_n + y_n) = x_0 + y_0$.

Proof. (i) If c = 0, we have nothing to prove.

So we assume that $c \neq 0$.

Now,
$$\frac{1}{n} | \{ k \le n : d(cx_k, cx_0) \ge \varepsilon \} | = \frac{1}{n} | \{ k \le n : |c|d(x_k, x_0) \ge \varepsilon \} | \le \frac{1}{n} | \{ k \le n : d(x_k, x_0) \ge \frac{\varepsilon}{|c|} \} | < \delta$$

Therefore, $\left\{n \in N : \frac{1}{n} | \left\{k \leq n : d\left(cx_k, cx_0\right) \geq \varepsilon\right\} | < \delta\right\} \in \mathcal{F}(I)$.

i.e, I-st $\lim cx_n = cx_0$.

(ii) We have
$$A_1 = \left\{n \in N : \frac{1}{n} | \left\{k \le n : d\left(x_k, x_0\right) \ge \frac{\varepsilon}{2}\right\} | < \frac{\delta}{2}\right\} \in \mathcal{F}\left(I\right)$$
 and $A_2 = \left\{n \in N : \frac{1}{n} | \left\{k \le n : d\left(y_k, y_0\right) \ge \frac{\varepsilon}{2}\right\} | < \frac{\delta}{2}\right\} \in \mathcal{F}\left(I\right)$.

Since $A_1 \cap A_2 \neq \emptyset$, therefore for all $n \in A_1 \cap A_2$ we have,

$$\begin{split} &\frac{1}{n}|\left\{k \leq n: d\left(x_k + y_k, x_0 + y_0\right) \geq \varepsilon\right\}| \\ &\leq \frac{1}{n}|\left\{k \leq n: d\left(x_k, x_0\right) \geq \frac{\varepsilon}{2}\right\}| + \frac{1}{n}|\left\{k \leq n: d\left(y_k, y_0\right) \geq \frac{\varepsilon}{2}\right\}| < \delta. \\ &\text{i.e, } \left\{n \in N: \frac{1}{n}|\left\{k \leq n: d\left(x_k + y_k, x_0 + y_0\right) \geq \varepsilon\right\} < \delta\right\} \in \mathcal{F}\left(I\right). \\ &\text{Hence } I\text{-st} \lim \left(x_n + y_n\right) = \left(x_0 + y_0\right). \end{split}$$

Definition 3.8. A sequence $x = (x_n)_{n \in \mathbb{N}}$ of elements of X is said to be I^* -statistical convergent to $\xi \in X$ if and only if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(I)$, such that $st\text{-}lim\ d\ (x_{m_k}, \xi) = 0$.

Theorem 3.9. If I^* -st $\lim_{n\to\infty} x_n = \xi$ then I-st $\lim_{n\to\infty} x_n = \xi$.

Proof. Let I^* -st $\lim_{n\to\infty} x_n = \xi$. By assumption there exist a set $H \in I$ such that for $M = N \setminus H = \{m_1 < m_2 < ... < m_k < ...\}$ we have st- $\lim x_{m_k} = \xi$

i.e,
$$\lim_{n\to\infty}\frac{1}{n}|\left\{m_k\leq n:d\left(x_{m_k},\xi\right)\geq\varepsilon\right\}|=0$$

so for any $\delta > 0$, $\{n \in N : \frac{1}{n} | \{m_k \le n : d(x_{m_k}, \xi) \ge \varepsilon\} | \ge \delta\} \in I$ since I is an admissible ideal.

Now,
$$A(\varepsilon, \delta) = \left\{ n \in N : \frac{1}{n} | \left\{ k \le n : d(x_k, \xi) \ge \varepsilon \right\} | \ge \delta \right\}$$

 $\subset H \cup \left\{ n \in N : \frac{1}{n} | \left\{ m_k \le n : d(x_{m_k}, \xi) \ge \varepsilon \right\} | \ge \delta \right\} \in I$
i.e, I -st $\lim_{n \to \infty} x_n = \xi$.

But the converse may not be true.

From example 3.3. we have ζ -st $\lim_{n\to\infty} y_n = 0$.

Suppose that ζ^* -st $\lim_{n\to\infty} y_n = 0$. Then there exist a set $H \in \zeta$ such that for $M = N \setminus H = \{m_1 < m_2 < ... < m_k < ...\}$ we have st- $\lim y_{m_k} = 0$. By definition of ζ there exist a $p \in N$ such that $H \subset \triangle_1 \cup ... \cup \triangle_p$. But then $\triangle_{p+1} \subset M$, so for infinitely many $m_k \in \triangle_{p+1}$,

$$D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \ge \eta\} = 2^{-(p+1)} > 0 \text{ for } 0 < \eta < \frac{1}{p+1}$$

i.e, $D\{m_k \in \Delta_{p+1} : d(y_{m_k}, 0) \ge \eta\} \ne 0$, which is a contradicts st-lim $y_{m_k} = 0$

Hence ζ^* -st $\lim_{n\to\infty} y_n \neq 0$.

0.

Definition 3.10. An element x_0 is said to be an *I*-statistical limit point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < ...\} \subset N$ such that $M \notin I$ and $st\text{-}lim \, x_{m_k} = x_0$.

I- $S(\Lambda_x)$ denotes the set of all I-statistical limit points of the sequence (x_n) .

Theorem 3.11. If (x_n) be a sequence such that I-st $\lim x_n = x_0$ then I- $S(\Lambda_x) = \{x_0\}.$

Proof. Since (x_n) is *I*-statistically convergent to x_0 , so for each $\varepsilon > 0$ and $\delta > 0$ the set,

 $A = \left\{ n \in \mathbb{N} : \frac{1}{n} | \left\{ k \leq n : d(x_k, x_0) \geq \varepsilon \right\} | \geq \delta \right\} \in I$, where I is an admissible ideal.

Suppose I- $S(\Lambda_x)$ contains y_0 different from x_0 . i.e, $y_0 \in I$ - $S(\Lambda_x)$.

So there exist a $M \subset N$ such that $M \notin I$ and st- $lim X_{m_k} = y_0$.

Let $B = \{n \in M : \frac{1}{n} | \{k \le n : d(x_k, y_0) \ge \varepsilon\} | \ge \delta\}$. So B is a finite set and therefore $B \in I$ and so $B^c = \{n \in M : \frac{1}{n} | \{k \le n : d(x_k, y_0) \ge \varepsilon\} | < \delta\} \in \mathcal{F}(I)$.

Again let $A_1 = \{n \in M : \frac{1}{n} | \{k \le n : d(x_k, x_0) \ge \varepsilon\} | \ge \delta \}$. So $A_1 \subset A \in I$. i.e, $A_1^c \in \mathcal{F}(I)$. Therefore $A_1^c \cap B^c \ne \emptyset$, since $A_1^c \cap B^c \in \mathcal{F}(I)$

Let $p \in A_1^c \cap B^c$ and take $\varepsilon = \frac{d(x_0, y_0)}{3} > 0$

so $\frac{1}{p} | \{ k \le p : d(x_k, x_0) \ge \varepsilon \} | < \delta$

and $\frac{1}{p} | \{ k \leq p : d(x_k, y_0) \geq \varepsilon \} | < \delta$

i.e, for maximum $k \leq p$ will satisfy $d(x_k, x_0) < \varepsilon$ and $d(x_k, y_0) < \varepsilon$ for a very small $\delta > 0$.

Thus we must have.

 $\{k \leq p : d(x_k, x_0) < \varepsilon\} \cap \{k \leq p : d(x_k, y_0) < \varepsilon\} \neq \emptyset$ a contradiction, as the neighbourhood of x_0 and y_0 are disjoint.

Hence
$$I$$
- $S(\Lambda_x) = \{x_0\}.$

Definition 3.12. [15] An element x_0 is said to be an I-statistical cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{n \in N : \frac{1}{n} | \left\{k \le n : d\left(x_k, x_0\right) \ge \varepsilon\right\} | < \delta\right\} \notin I.$$

I- $S(\Gamma_x)$ denotes the set of all I-statistical cluster points of the sequence (x_n) .

Theorem 3.13. For any sequence $x = (x_n)$, I- $S(\Gamma_x)$ is closed.

Proof. Let y_0 be a limit point of the set I- $S(\Gamma_x)$ then for any $\varepsilon > 0$, I- $S(\Gamma_x) \cap B(y_0, \varepsilon) \neq 0$, where $B(y_0, \varepsilon) = \{z \in R : d(z, y_0) < \varepsilon\}$

Let $z_0 \in I$ - $S(\Gamma_x) \cap B(y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B(z_0, \varepsilon_1) \subseteq B(y_0, \varepsilon)$.

Then we have $\{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : d(x_k, y_0) \geq \varepsilon\}$ $\Rightarrow \frac{1}{n} | \{k \leq n : d(x_k, z_0) \geq \varepsilon_1\} | \geq \frac{1}{n} | \{k \leq n : d(x_k, y_0) \geq \varepsilon\} |$

Now for any $\delta > 0$,

$$\left\{n \in N : \frac{1}{n} | \left\{k \le n : d\left(x_k, z_0\right) \ge \varepsilon_1\right\} | < \delta\right\}$$

```
\subseteq \left\{n \in N: \frac{1}{n} | \left\{k \leq n: d\left(x_k, y_0\right) \geq \varepsilon\right\} | < \delta\right\} Since z_0 \in I\text{-}S(\Gamma_x) therefore, \left\{n \in N: \frac{1}{n} | \left\{k \leq n: d\left(x_k, y_0\right) \geq \varepsilon\right\} | < \delta\right\} \notin I. i.e, y_0 \in I\text{-}S(\Gamma_x). Hence the theorem is proved.
```

Theorem 3.14. For any sequence $x = (x_n)$, I- $S(\Lambda_x) \subseteq I$ - $S(\Gamma_x)$.

Proof. Let $x_0 \in I$ - $S(\Lambda_x)$. Then there exist a set $M = \{m_1 < m_2 < ...\} \notin I$ such that, st- $lim x_{m_k} = x_0 \Rightarrow lim_{k\to\infty} \frac{1}{k} |\{m_i \le k : d(x_{m_i}, x_0) \ge \varepsilon\}| = 0$.

Take $\delta > 0$, so there exist $k_0 \in N$ such that for $n > k_0$ we have,

$$\frac{1}{n} |\{m_i \le n : d(x_{m_i}, x_0) \ge \varepsilon\}| < \delta.$$

Let
$$A = \{ n \in \mathbb{N} : \frac{1}{n} | \{ m_i \le n : d(x_{m_i}, x_0) \ge \varepsilon \} | < \delta \}.$$

Also, $A \supset M/\{m_1 < m_2 < ... < m_{k_0}\}$. Since I is an admissible ideal and $M \notin I$, therefore $A \notin I$. So by definition of I-statistical cluster point $x_0 \in I$ - $S(\Gamma_x)$.

Hence the theorem is proved.

Theorem 3.15. If
$$x = (x_n)$$
 and $y = (y_n)$ be two sequences such that $\{n \in N : x_n \neq y_n\} \in I$, then (i) I - $S(\Lambda_x) = I$ - $S(\Lambda_y)$ and (ii) I - $S(\Gamma_x) = I$ - $S(\Gamma_y)$.

Proof. (i) Let $x_0 \in I$ - $S(\Lambda_x)$. So by definition there exist a set

 $K = \{k_1 < k_2 < k_3 < \cdots\}$ of N such that $K \notin I$ and st-lim $x_{k_n} = x_0$.

Since
$$\{n \in K : x_n \neq y_n\} \subset \{n \in N : x_n \neq y_n\} \in I$$
,

therefore $K' = \{n \in K : x_n = y_n\} \notin I \text{ and } K' \subseteq K.$

So we have st-lim $y_{k'_n} = x_0$.

This shows that $x_0 \in I$ - $S(\Lambda_y)$ and therefore I- $S(\Lambda_x) \subseteq I$ - $S(\Lambda_y)$.

By symmetry I- $S(\Lambda_u) \subseteq I$ - $S(\Lambda_x)$.

Hence I- $S(\Lambda_y) = I$ - $S(\Lambda_x)$.

(ii) Let $x_0 \in I$ - $S(\Gamma_x)$. So by definition for each $\varepsilon > 0$ the set, $A = \left\{ n \in N : \frac{1}{n} \middle| \left\{ k \le n : d(x_k, x_0) \ge \varepsilon \right\} \middle| < \delta \right\} \notin I$. Let $B = \left\{ n \in N : \frac{1}{n} \middle| \left\{ k \le n : d(y_k, x_0) \ge \varepsilon \right\} < \delta \right\}$. We have to prove that $S \notin I$.

Suppose $B \in I$. So, $B^c = \{n \in N : \frac{1}{n} | \{k \le n : d(y_k, x_0) \ge \varepsilon\} \ge \delta\} \in \mathcal{F}(I)$. By hypothesis the set $C = \{n \in N : x_n = y_n\} \in \mathcal{F}(I)$.

Therefore $B^c \cap C \in \mathcal{F}(I)$. Also it is clear that $B^c \cap C \subset A^c \in \mathcal{F}(I)$,

i.e, $A \in I$, which is a contradiction.

Hence $B \notin I$ and thus the result is proved.

Theorem 3.16. If g is a continuous function on X then it preserves I-statistical convergence in X.

Proof. Let I-st $\lim_{n\to\infty} x_n = \xi$.

Since g is continuous, then for each $\varepsilon_1 > 0$, there exist $\varepsilon_2 > 0$ such that if $x \in B(\xi, \varepsilon_1)$ then $g(x) \in B(g(\xi), \varepsilon_2)$.

Also we have,

```
\begin{split} C\left(\varepsilon_{1},\delta\right) &= \left\{n \in N: \frac{1}{n}|\left\{k \leq n: d\left(x_{k},\xi\right) \geq \varepsilon_{1}\right\} \mid <\delta\right\} \in \mathcal{F}\left(I\right) \\ \text{Now, } \left\{k \leq n: d\left(x_{k},\xi\right) \geq \varepsilon_{1}\right\} \supseteq \left\{k \leq n: d\left(g\left(x_{k}\right), g\left(\xi\right)\right) \geq \varepsilon_{2}\right\} \\ \text{so, } \frac{1}{n}|\left\{k \leq n: d\left(x_{k},\xi\right) \geq \varepsilon_{1}\right\} \mid \geq \frac{1}{n}|\left\{k \leq n: d\left(g\left(x_{k}\right), g\left(\xi\right)\right) \geq \varepsilon_{2}\right\} \mid \\ \text{for } \delta > 0, \left\{n \in N: \frac{1}{n}|\left\{k \leq n: d\left(x_{k},\xi\right) \geq \varepsilon_{1}\right\} \mid <\delta\right\} \\ &\subseteq \left\{n \in N: \frac{1}{n}|\left\{k \leq n: d\left(g\left(x_{k}\right), g\left(\xi\right)\right) \geq \varepsilon_{2}\right\} \mid <\delta\right\} \in \mathcal{F}\left(I\right) \\ \text{since } C\left(\varepsilon_{1},\delta\right) \in \mathcal{F}\left(I\right). \end{split}
```

Hence the theorem is proved.

ACKNOWLEDGMENTS

The authors are grateful to CSIR for their financial support to carry out this research work through the Project No. 25(0236)/14/EMR-II.

References

- R. C. Buck, The measure theoretic approach to density, Amer. J. Math., 68, (1946), 560-580.
- 2. R. C. Buck, Generalized asymptotic density, Amer. J. Math., 75, (1953), 335-346.
- S. Debnath, J. Debnath, On *I*-statistically convergent sequence spaces defined by sequences of Orlicz functions using matrix transformation, *Proyectiones J. Math.*, 33(3), (2014), 277-285.
- 4. K. Dems, On I-Cauchy sequences, Real Anal. Exchange, 30(1), (2004/2005), 123-128.
- 5. M. Et, A. Alotaibi, S. A. Mohiuddine, On (\triangle^m, I) -statistical convergence of order α , The Scientific world journal, vol 2014.
- 6. H. Fast, Sur la convergence statistique, Collog. Math., 2, (1951), 241-244.
- 7. J. A. Fridy, Statistical limit points, Proc. Amer. Math. Soc., 4, (1993), 1187-1192.
- 8. J. A. Fridy, On statistical convergence, Analysis, 5, (1985), 301-313.
- 9. M. Gurdal, H. Sari, Extremal A-statistical limit points via ideals, J. Egyptian Math. Soc., 22(1), (2014), 55-58.
- M. Gurdal, M. O. Ozgur, A generalized statistical convergence via moduli, *Electronic J. Math. Anal. & Appl.*, 3(2), (2015), 173-178.
- P. Kostyrko, M. Macaj, T. Salat, M. Sleziak, I-convergence and extremal I-limit points, Math. Slov., 4, (2005), 443-464.
- P. Kostyrko, T. Salat, W. Wilczynski, *I*-convergence, *Real Anal. Exchange*, 26(2), (2000/2001), 669-686.
- M. Mamedov, S. Pehlivan, Statistical cluster points and turnpike theorem in non convex problems, J. Math. Anal. Appl., 256, (2001), 686-693.
- D. S. Mitrinovic, J. Sandor, B. Crstici, Handbook of Number Theory, Kluwer Acad. Publ. Dordrecht-Boston-London, 1996.
- M. Mursaleen, S. Debnath, D. Rakshit, I-statistical limit superior and I-statistical limit inferior, Filomat, 31(7), (2017), 2103-2108.
- E. Savas, P. Das, A generalized statistical convergence via ideals, App. Math. Lett., 24, (2011), 826-830.
- 17. E. Savas, P. Das, On I-statistically pre-Cauchy sequences, $Taiwanese\ J.\ Math.,\ {\bf 18}(1),$ (2014), 115-126.
- H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2, (1951), 73-74.

[Downloaded from ijmsi.ir on 2025-11-04]

- T. Salat, On statistically convergent sequences of real numbers, Math. Slov., 30, (1980), 139-150.
- T. Salat, B. Tripathy, M. Ziman, On some properties of *I*-convergence, *Tatra. Mt. Math. Publ.*, 28, (2004), 279-286.
- 21. B. C. Tripathy, On statistically convergent and statistically bounded sequences, *Bull. Malaysian. Math. Soc.*, **20**, (1997), 31-33.
- 22. U. Yamanci, M. Gurdal, *I* -statistical convergence in 2-normed space, *Arab J. Math. Sci.*, **20**(1), (2014), 41-47.
- 23. U. Yamanci, M. Gurdal, I -statistically pre-Cauchy double sequences, Global J. Math. Anal., 2(4), (2014), 297-303.