

## **$L_1$ -Biharmonic Hypersurfaces in Euclidean Spaces with Three Distinct Principal Curvatures**

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**ABSTRACT.** A submanifold  $M^n$  of the Euclidean space  $\mathbb{E}^{n+m}$  is said to be biharmonic if its position map  $x : M^n \rightarrow \mathbb{E}^{n+m}$  satisfies the condition  $\Delta^2 x = 0$ , where  $\Delta$  stands for the Laplace operator. A well-known conjecture of Bang-Yen Chen says that, the only biharmonic submanifolds of Euclidean spaces are the minimal ones. In this paper, we consider a modified version of the conjecture, replacing  $\Delta$  by its extension,  $L_1$ -operator (namely,  $L_1$ -conjecture). The  $L_1$ -conjecture states that any  $L_1$ -biharmonic Euclidean hypersurface is 1-minimal. We prove that the  $L_1$ -conjecture is true for  $L_1$ -biharmonic hypersurfaces with three distinct principal curvatures and constant mean curvature of a Euclidean space of arbitrary dimension.

**Keywords:** Linearized operators  $L_1$ ,  $L_1$ -Biharmonic hypersurfaces, 1-Minimal.

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## 1. INTRODUCTION

The concept of harmonic maps plays important roles in differential geometry, computational geometry and physical theories of elastics and fluid mechanics. In applied mathematics, some partial differential equations have analytical solutions in terms of harmonic functions (see for instance [13, 14]). Sometimes, it becomes very difficult to find harmonic functions whereas biharmonic ones make help us to solve related differential equations. As a geometric example, there exists no harmonic map as  $\mathbb{T}^2 \rightarrow \mathbb{S}^2$  (whatever the metrics chosen) in the homotopy class of Brower degree  $\pm 1$  and hence, it is useful to find a biharmonic map from  $\mathbb{T}^2$  into  $\mathbb{S}^2$  ([9]). Obviously, harmonic maps are biharmonic but not vis versa. If a map biharmonic is non-harmonic, then it is said to be *proper-biharmonic*. Proper-biharmonic maps facilitate the study of pseudo-umbilical and parallel submanifolds.

A well-known conjecture of Bang-Yen Chen (in 1987) says that every biharmonic Riemannian submanifold of Euclidean  $m$ -space (of arbitrary dimension  $m$ ),  $\mathbb{E}^m$ , is minimal. Chen himself has proven his conjecture for Euclidean surfaces in 3-space,  $\mathbb{E}^3$ . The conjecture has been affirmed in some extended cases. In 1992, I. Dimitrić proved that any biharmonic hypersurface in  $\mathbb{E}^m$  (of arbitrary dimension  $m$ ) with at most two distinct principal curvatures is minimal ([7]). Also, in 1995, T. Hasanis and T. Vlachos have proven Chen conjecture on the 3-dimensional Euclidean hypersurfaces ([12]). K. Akutagawa and S. Maeta ([2]) have studied on a general version of the conjecture on complete biharmonic submanifolds of the Euclidean spaces. In 2013, B.Y. Chen and M.I. Munteanu ([6]) affirmed the conjecture for every  $\delta(2)$ -ideal or  $\delta(3)$ -ideal hypersurface of the Euclidean space of arbitrary dimension. Recently, R. Gupta ([11]) has proven that every biharmonic hypersurface with three distinct principal curvatures in  $\mathbb{E}^m$  is minimal. On the other hand, there exists a nice relation between the finite type hypersurfaces and biharmonic ones (see [7]). The theory of finite type hypersurfaces has been interested by B.Y. Chen and followed by L.J. Alias, S.M.B. Kashani and others (see [3, 5, 15]). One can see main results in the last chapter of Chen's book ([5]). In [15], S.M.B. Kashani has introduced the notion of  $L_r$ -finite type hypersurface as an extension of finite type hypersurface in the Euclidean space, followed by the first author in her doctoral thesis.

The map  $L_r$ , as a natural extension of the Laplace operator  $L_0 = \Delta$ , stands for the linearized operator of  $(r + 1)$ th mean curvature of a hypersurface  $M^n$  in  $\mathbb{E}^{n+1}$ , for  $r = 0, \dots, n - 1$  (see [3, 17]). The  $L_r$ -operator is given by  $L_r(f) = \text{tr}(P_r \circ \nabla^2 f)$  for any smooth function  $f$  on  $M^n$ , where  $P_r$  is the  $r$ -th Newton transformation associated to the second fundamental form of  $M^n$  and  $\nabla^2 f$  is the hessian of  $f$ .

It seems interesting to generalize the definition of biharmonic hypersurface by replacing  $\Delta$  by  $L_r$ . We call these hypersurfaces *L<sub>r</sub>-biharmonic*. Recently, M. Aminian and S.M.B. kashani ([4]) have stated a general version of Chen conjecture, which says that, every Euclidean hypersurface  $x : M^n \rightarrow \mathbb{E}^{n+1}$  satisfying the condition  $L_r^2 x = 0$  for some  $r$ , ( $0 \leq r \leq n - 1$ ) is *r-minimal*. They proved that the  $L_r$ -conjecture is true for Euclidean hypersurfaces with at most two principal curvatures and  $L_r$ -finite type hypersurfaces. In this paper, we prove that the  $L_1$ -conjecture is true for Euclidean hypersurfaces with three distinct principal curvature and constant mean curvature. Here is our main result:

**Theorem 1.1.** *Every L<sub>1</sub>-biharmonic hypersurface in  $\mathbb{E}^{n+1}$  with constant mean curvature and three distinct principal curvatures is 1-minimal.*

## 2. PRELIMINARIES

In this section, we recall preliminaries from [3, 10]. Let  $x : M^n \rightarrow \mathbb{E}^{n+1}$  be an isometric immersion from a Riemannian manifold  $M^n$  of dimension  $n$  into the Euclidean space  $\mathbb{E}^{n+1}$ , with a unite normal vector field (Gauss map)  $N$ . The symbols  $\nabla^0$  and  $\nabla$  stands for Levi-Civita connections on  $\mathbb{E}^{n+1}$  and  $M^n$ , respectively. The Gauss formula on  $M^n$  is given by  $\nabla_X^0 Y = \nabla_X Y + \langle SX, Y \rangle N$ , where  $S : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$  is the shape operator (or Weingarten endomorphism) of  $M^n$  defined by  $SX = -\nabla_X^0 N$ , for every tangent vector fields  $X$  and  $Y$  on  $M^n$ . As it is known, at each point  $p \in M$ ,  $S_p$  is a self-adjoint linear endomorphism on the tangent space,  $T_p M$ , and its eigenvalues  $\lambda_1(p), \dots, \lambda_n(p)$  are defined as the principal curvatures and the corresponding orthonormal vectors (local basis)  $\{e_1, \dots, e_n\}$  are called the principal directions on  $M^n$ . The characteristic polynomial of  $S$  is defined by

$$Q_S(t) = \det(tI - S) = \sum_{k=0}^n (-1)^k a_k t^{n-k},$$

where  $a_0 = 1$  and for  $k = 1, \dots, n$ ,  $a_k$  is given by

$$a_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \dots \lambda_{i_k}. \quad (2.1)$$

The *r-th mean curvature*  $H_r$  of  $M^n$  is defined by  $\binom{n}{r} H_r = a_r$  for  $1 \leq r \leq n$  and  $H_0 = 1$ . If  $H_{r+1} \equiv 0$ , the hypersurface  $M^n$  is said to be *r-minimal*. The *r-th Newton transformation* of  $M^n$  is the operator  $P_r : \mathfrak{X}(M^n) \rightarrow \mathfrak{X}(M^n)$  defined by

$$P_r = \sum_{j=0}^r (-1)^j \binom{n}{r-j} H_{r-j} S^j = \sum_{j=0}^r (-1)^j a_{r-j} S^j.$$

Equivalently,

$$P_0 = I, \quad P_r = \binom{n}{r} H_r I - S \circ P_{r-1}.$$

At each point  $p \in M$ , the restricted map  $P_r : T_p M \rightarrow T_p M$  is a self-adjoint linear operators that commutes with  $S$  and its eigenvalues with respect to the orthonormal (local) basis  $\{e_1, \dots, e_n\}$  of principal directions on  $M^n$  are given by  $P_r e_i = \mu_{i,r} e_i$  (for  $i = 1, \dots, n$ ), where

$$\mu_{i,r} = \sum_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ i_j \neq i}} \lambda_{i_1} \cdots \lambda_{i_r}.$$

We will use a helpful formula from [3] as:

$$tr(S^2 \circ P_1) = \frac{n(n-1)(n-2)}{2} (2HH_2 - H_3). \quad (2.2)$$

Associated to the Newton transformation  $P_r$ , we consider the second-order linear differential operator  $L_r : C^\infty(M) \rightarrow C^\infty(M)$  given by  $L_r(f) = tr(P_r \circ \nabla^2 f)$ , where  $\nabla^2 f : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$  denotes a self-adjoint linear operator metrically equivalent to the Hessian of  $f$ , given by  $\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle$  for every vector fields  $X$  and  $Y$  on  $M$ . In terms of the local orthonormal basis  $\{e_1, \dots, e_n\}$ ,  $L_r(f)$  is given by

$$L_r(f) = \sum_{i=1}^n \mu_{i,r} (e_i e_i f - \nabla_{e_i} e_i f). \quad (2.3)$$

### 3. $L_r$ -BIHARMONIC HYPERSURFACES IN $\mathbb{E}^{n+1}$

Let  $x : M^n \rightarrow \mathbb{E}^{n+1}$  be a connected orientable hypersurface immersed into the Euclidean space, with Gauss map  $N$ . By definition,  $M^n$  is called a  $L_r$ -biharmonic hypersurface if its position vector field satisfies the condition  $L_r^2 x = 0$ . By the equality  $L_r x = c_r H_{r+1} N$  from [3], the condition  $L_r^2 x = 0$  has another equivalent expression as  $L_r(H_{r+1} N) = 0$ . It is clear that the  $r$ -minimal hypersurface is  $L_r$ -biharmonic. By formulae in [3] page 122, we have

$$L_r^2 x = -2c_r(S \circ P_r) \nabla H_{r+1} - c_r \binom{n}{r+1} H_{r+1} \nabla H_{r+1} - c_r (tr(S^2 \circ P_r) H_{r+1} - L_r H_{r+1}) N, \quad (3.1)$$

where  $c_r = (r+1) \binom{n}{r+1}$ .

By using this formula for  $L_r^2 x$  and the identifying normal and tangent parts of the  $L_r$ -biharmonic condition  $L_r^2 x = 0$ , one obtains necessary and sufficient conditions for  $M^n$  to be  $L_r$ -biharmonic in  $\mathbb{E}^{n+1}$ , namely

$$L_r H_{r+1} = tr(S^2 \circ P_r) H_{r+1} \quad (3.2)$$

and

$$(S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2} \binom{n}{r+1} H_{r+1} \nabla H_{r+1}. \quad (3.3)$$

**3.1. Proof of theorem 1.1.** From now on, we concentrate on *L*<sub>1</sub>-biharmonic hypersurfaces  $M^n$  in a Euclidean space  $\mathbb{E}^{n+1}$  with three distinct principal curvatures and constant mean curvature  $H$ . We assume that the 2th mean curvature  $H_2$  is not constant, so there exists of an open connected subset  $\mathcal{U}$  of  $M$ , with  $\nabla H_2(p) \neq 0$  for all  $p \in \mathcal{U}$ . We shall contradict the condition  $\nabla H_2(p) \neq 0, \forall p \in \mathcal{U}$ .

We assume that  $\{e_1, e_2, \dots, e_{n-1}, e_n\}$  be a local orthonormal frame of principal directions of the shape operator,  $S$ , on  $\mathcal{U}$  such that  $Se_i = \lambda_i e_i$  ( $1 \leq i \leq n$ ). Then we have  $P_2 e_i = \mu_{i,2} e_i$ , for every  $i$ . We have

$$H_2 = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j. \quad (3.4)$$

From (3.3) (using the inductive definition of  $P_2$ ) we get

$$P_2(\nabla H_2) = \frac{3}{4}n(n-1)H_2 \nabla H_2 \text{ on } \mathcal{U}. \quad (3.5)$$

Observe from (3.5) that  $\nabla H_2$  is an eigenvector of  $P_2$  with the corresponding eigenvalue  $\frac{3}{4}n(n-1)H_2$ . Without loss of generality, we can choose  $e_1$  such that  $e_1$  is parallel to  $\nabla H_2$ . Since the shape operator  $S$  and  $P_2$  can be simultaneously diagonalized, therefore the shape operator  $S$  of  $M^n$  takes the form with respect to a suitable orthonormal frame  $\{e_1, e_2, \dots, e_{n-1}, e_n\}$

$$\begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \lambda_n \end{pmatrix}. \quad (3.6)$$

Then we have

$$\mu_{1,2} = \frac{3}{4}n(n-1)H_2. \quad (3.7)$$

We can decompose  $\nabla H_2 = \sum_{i=1}^n e_i(H_2) e_i$ . Since  $e_1$  is parallel to  $\nabla H_2$ , it follows that

$$e_1(H_2) \neq 0, e_i(H_2) = 0 \text{ for } i = 2, \dots, n. \quad (3.8)$$

We write

$$\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k, \quad i, j = 1, 2, \dots, n. \quad (3.9)$$

The compatibility conditions  $\nabla_{e_k} \langle e_i, e_i \rangle = 0$  and  $\nabla_{e_k} \langle e_i, e_j \rangle = 0$  imply respectively that

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \quad (3.10)$$

for  $i \neq j$  and  $i, j, k = 1, 2, \dots, n$ . Furthermore, it follows from the Codazzi equation that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \quad (3.11)$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \quad (3.12)$$

for distinct  $i, j, k = 1, 2, \dots, n$ .

Since  $\mu_{1,2} = \frac{3}{4}n(n-1)H_2$ , from (3.4) we have

$$H_2 = \frac{4}{n(n-1)}\lambda_1(\lambda_1 - nH), \quad (3.13)$$

therefore, we get

$$e_1(\lambda_1) \neq 0, \quad e_i(\lambda_1) = 0 \quad i = 2, \dots, n. \quad (3.14)$$

One can compute that

$$[e_i, e_j](\lambda_1) = 0, \quad i, j = 2, \dots, n,$$

which yields directly

$$\omega_{ij}^1 = \omega_{ji}^1, \quad (3.15)$$

for  $i \neq j$  and  $i, j = 2, \dots, n$ .

Now we show that  $\lambda_j \neq \lambda_1$  for  $j = 2, \dots, n$ . In fact, if  $\lambda_j = \lambda_1$  for  $j \neq 1$ , by putting  $i = 1$  in (3.11) we have that

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts the first expression of (3.14).

Since  $M^n$  has three distinct principal curvatures, we can assume that  $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1} = \lambda$  and  $\lambda_n \neq \lambda$ , hence  $\lambda_n = nH - \lambda_1 - (n-2)\lambda$ .

Consider Eqs. (3.11) and (3.12).

For  $i, j = 2, 3, \dots, n-1$ , and  $i \neq j$  in (3.11). One has

$$e_j(\lambda) = 0, \quad \text{for } j = 2, \dots, n-1. \quad (3.16)$$

For  $j = 1$  and  $i \neq 1$  in (3.11), by (3.14) we have  $\omega_{1i}^1 = 0$  ( $i \neq 1$ ). Moreover, by the first expression of (3.10) we have

$$\omega_{1i}^1 = 0, \quad i = 1, 2, \dots, n.$$

For  $i = 2, \dots, n-1, j = n$  in (3.11), by (3.16) we have

$$\omega_{ni}^n = 0, \quad i = 2, 3, \dots, n-1.$$

For  $i = 1, j = 2, 3, \dots, n$  in (3.11), we obtain

$$\omega_{n1}^n = -\frac{e_1(\lambda_1 + (n-2)\lambda)}{2\lambda_1 + (n-2)\lambda - nH}, \quad \omega_{j1}^j = \frac{e_1(\lambda)}{\lambda_1 - \lambda}, \quad j = 2, 3, \dots, n-1. \quad (3.17)$$

For  $i = n, j = 2, 3, \dots, n-1$  in (3.11), we obtain

$$\omega_{jn}^j = \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda}, \quad j = 2, 3, \dots, n-1.$$

For  $i = 1$ ,  $j \neq k$  and  $j, k = 2, 3, \dots, n-1$  in (3.12), we have

$$\omega_{k1}^j = 0, \quad j \neq k, \quad \text{and } j, k = 2, 3, \dots, n-1.$$

For  $i = n$ ,  $j \neq k$  and  $j, k = 2, 3, \dots, n-1$ , in (3.12), we get

$$\omega_{kn}^j = 0, \quad j \neq k, \quad \text{and } j, k = 2, 3, \dots, n-1.$$

For  $i = n$  and  $j = 1$ ,  $k = 2, 3, \dots, n-1$  in (3.12), and using (3.15) we get

$$\omega_{kn}^1 = \omega_{nk}^1 = 0, \quad k = 2, 3, \dots, n-1.$$

Similarly, we can also obtain

$$\omega_{1k}^n = \omega_{k1}^n = 0, \quad k = 2, 3, \dots, n-1.$$

Let us introduce two smooth functions  $\alpha$  and  $\beta$  as follows:

$$\alpha = \frac{e_1(\lambda)}{\lambda_1 - \lambda}, \quad \beta = \frac{e_1(\lambda_1 + (n-2)\lambda)}{2\lambda_1 + (n-2)\lambda - nH}. \quad (3.18)$$

We have the following:

**Lemma 3.1.** *Let  $M^n$  be an  $n$ -dimensional biharmonic hypersurface with three distinct principle curvature in Euclidean space, having the shape operator given by (3.6) with respect to suitable orthonormal farame  $\{e_1, e_2, \dots, e_{n-1}, e_n\}$ . Then we obtain*

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, \quad \nabla_{e_i} e_1 = \alpha e_i, \quad i = 2, \dots, n-1, \quad \nabla_{e_n} e_1 = -\beta e_n, \\ \nabla_{e_i} e_i &= -\alpha e_1 + \sum_{i \neq j, j=2}^{n-1} \omega_{ii}^j e_j - \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} e_n, \quad i = 2, 3, \dots, n-1, \\ \nabla_{e_i} e_j &= \sum_{i \neq j, k=2}^{n-1} \omega_{ij}^k e_k, \quad i, j = 2, 3, \dots, n-1, \\ \nabla_{e_1} e_n &= 0, \quad \nabla_{e_n} e_n = \beta e_1, \quad \nabla_{e_i} e_n = \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} e_i, \quad i = 2, 3, \dots, n-1. \end{aligned} \quad (3.19)$$

where  $\omega_{ij}^k$  satisfies (3.10) for  $i, j, k = 1, 2, \dots, n-1, n$ .

Recall the definition of the Gauss curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Using Lemma 3.1, Gauss equation and comparing the coefficients with respect to a orthonormal basis  $\{e_1, e_2, \dots, e_{n-1}, e_n\}$ , we find the following:

- $X = e_1, Y = e_2, Z = e_1$ ,

$$e_1(\alpha) + \alpha^2 = -\lambda_1 \lambda; \quad (3.20)$$

- $X = e_1, Y = e_2, Z = e_n$ ,

$$e_1 \left( \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} \right) + \alpha \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} = 0; \quad (3.21)$$

- $X = e_1, Y = e_n, Z = e_1,$

$$-e_1(\beta) + \beta^2 = -\lambda_1(nH - \lambda_1 - (n-2)\lambda); \quad (3.22)$$

- $X = e_3, Y = e_n, Z = e_1,$

$$e_n(\alpha) + (\alpha + \beta) \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} = 0; \quad (3.23)$$

- $X = e_n, Y = e_2, Z = e_n,$

$$-e_n \left( \frac{e_n(\lambda)}{nH - \lambda_1 - (n-1)\lambda} \right) + \alpha\beta - \left( \frac{e_4(\lambda)}{nH - \lambda_1 - (n-1)\lambda} \right)^2 = \lambda(nH - \lambda_1 - (n-2)\lambda). \quad (3.24)$$

Now, we consider the  $L_1$ -biharmonic equation (3.2). It follows from (2.3) and (3.19) that

$$\begin{aligned} & (\lambda_1 - nH)e_1e_1(H_2) + [(n-2)(\lambda - nH)\alpha + (\lambda_1 + (n-2)\lambda)\beta]e_1(H_2) \\ & - \frac{n(n-1)(n-2)}{2}H_2(2HH_2 - H_3) = 0. \end{aligned} \quad (3.25)$$

From (3.8) and (3.19), we obtain

$$e_i e_1(H_2) = 0, \quad i = 2, 3, \dots, n-1, n. \quad (3.26)$$

Differentiating  $\alpha$  and  $\beta$  along  $e_n$ , we get equations

$$(\lambda_1 - \lambda)e_n(\alpha) - \alpha e_n(\lambda) = e_n e_1(\lambda),$$

$$(2\lambda_1 + (n-2)\lambda - nH)e_n(\beta) + (n-2)\beta e_n(\lambda) = (n-2)e_n e_1(\lambda),$$

respectively and eliminating  $e_n e_1(\lambda)$ , we have

$$(2\lambda_1 + (n-2)\lambda - nH)e_n(\beta) = (n-2)[(\lambda_1 - \lambda)e_n(\alpha) - (\alpha + \beta)e_n(\lambda)].$$

Putting the value of  $e_n(\alpha)$  from (3.23) in the above equation, we find

$$e_n(\beta) = \frac{e_n(\lambda)(n-2)(\alpha + \beta)(n\lambda - nH)}{(2\lambda_1 + (n-2)\lambda - nH)(nH - \lambda_1 - (n-1)\lambda)}.$$

Differentiating (3.25) along  $e_n$  and using (3.26), (3.23) and  $e_n(\beta)$ , we get

$$(n-2)e_n(\lambda) \left[ \frac{(\alpha + \beta)A}{2\lambda_1 + (n-2)\lambda - nH} e_1(H_2) - H_2(nH - \lambda_1 - (n-1)\lambda)B \right] = 0. \quad (3.27)$$

where  $A := 4nH\lambda_1 - 2\lambda_1^2 - 2(n-1)\lambda\lambda_1 + 2n(n-1)H\lambda - 2n^2H^2$  and  $B := n^2H^2 + 3\lambda_1^2 + (3(n-2)^2 - 3)\lambda^2 + (2n - 4n(n-2))H\lambda - 4nH\lambda_1 + 6(n-2)\lambda\lambda_1$ . We claim that  $e_n(\lambda) = 0$ . Indeed, if  $e_n(\lambda) \neq 0$ , then

$$\frac{(\alpha + \beta)A}{2\lambda_1 + (n-2)\lambda - nH} e_1(H_2) - H_2(nH - \lambda_1 - (n-1)\lambda)B = 0, \quad (3.28)$$

Now, differentiating (3.28) along  $e_n$ , we have

$$\frac{(\alpha + \beta) [A((n-4)\lambda_1 + 2(n-2)^2\lambda + (n-2n(n-2))H) + C] e_1(H_2)}{(2\lambda_1 + (n-2)\lambda - nH)^2} + H_2 D = 0, \quad (3.29)$$

where  $C := (2n(n-1)H - 2(n-1)\lambda_1)(nH - \lambda_1 - (n-1)\lambda)(2\lambda_1 + (n-2)\lambda - nH)$  and

$$D := -(nH - \lambda_1 - (n-1)\lambda)^2[(6(n-2)^2 - 6)\lambda + (2n - 4n(n-2))H + 6(n-2)\lambda_1] + (n-1)(nH - \lambda_1 - (n-1)\lambda)B.$$

Eliminating  $e_1(H_2)$  from (3.28) and (3.29), we obtain

$$-AD(2\lambda_1 + (n-2)\lambda - nH) = (nH - \lambda_1 - (n-1)\lambda)B[A((n-4)\lambda_1 + 2(n-2)^2\lambda + (n-2n(n-2))H) + C] \quad (3.30)$$

After four times differentiating (3.30) along  $e_n$ , we get that  $nH = \lambda_1$ , which is not possible since  $\lambda_1$  is not constant. Consequently,  $e_n(\lambda) = 0$ . Therefore, (3.24) reduces to

$$\alpha\beta = \lambda(nH - \lambda_1 - (n-2)\lambda). \quad (3.31)$$

Note that (3.13) yields

$$e_1(H_2) = -\frac{4(n-2)}{n(n-1)}(2\lambda_1 - nH)e_1(\lambda) + \frac{4}{n(n-1)}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)\beta. \quad (3.32)$$

By using (3.32), (3.31), (3.22) and (3.20), we obtain

$$\begin{aligned} e_1 e_1(H_2) &= \frac{4(n-2)}{n(n-1)}\lambda_1\lambda(\lambda_1 - \lambda)(2\lambda_1 - nH) \\ &+ \frac{4}{n(n-1)}(nH - \lambda_1 - (n-2)\lambda)(2\lambda_1 - nH)((3n-2)\lambda_1\lambda + 2\lambda_1^2 - 2nH\lambda - nH\lambda_1) \\ &+ \left[ -n\alpha + 3\beta + 2\frac{(2\lambda_1 + (n-2)\lambda - nH)\beta - (n-2)(\lambda_1 - \lambda)\alpha}{2\lambda_1 - nH} \right] e_1(H_2). \end{aligned} \quad (3.33)$$

Combining (3.25) with (3.33) gives

$$(P_{1,2}\alpha + P_{2,2}\beta)e_1(H_2) = P_{3,6}, \quad (3.34)$$

where  $P_{1,2}$ ,  $P_{2,2}$  and  $P_{3,6}$  are polynomials in terms of  $\lambda$  and  $\lambda_1$  of degrees 2, 2 and 6 respectively.

Differentiating (3.34) along  $e_1$  and using (3.31), (3.22), (3.20) and (3.34), we get following relation

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}e_1(H_2), \quad (3.35)$$

where  $P_{4,8}$ ,  $P_{5,8}$  and  $P_{6,5}$  are polynomials in terms of  $\lambda$  and  $\lambda_1$  of degrees 8, 8 and 5 respectively.

Also, we have

$$e_1(H_2) = \frac{4}{n(n-1)}(2\lambda_1 - nH)(\beta(2\lambda_1 + (n-2)\lambda - nH) - (n-2)\alpha(\lambda_1 - \lambda)). \quad (3.36)$$

Combining (3.35) and (3.36), we obtain

$$\begin{aligned} & \left( P_{4,8} + \frac{4(n-2)}{n(n-1)} P_{6,5}(\lambda_1 - \lambda)(2\lambda_1 - nH) \right) \alpha \\ & + \left( P_{5,8} - \frac{4}{n(n-1)} P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH) \right) \beta = 0. \end{aligned} \quad (3.37)$$

On the other hand, combining (3.36) with (3.34) and using (3.31), we find

$$P_{2,2}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)\beta^2 - P_{1,2}(n-2)(\lambda_1 - \lambda)(2\lambda_1 - nH)\alpha^2 = L, \quad (3.38)$$

where  $L$  is given by

$$\begin{aligned} L &= \lambda(nH - \lambda_1 - (n-2)\lambda)(2\lambda_1 - nH) \\ & (P_{2,2}(n-2)(\lambda_1 - \lambda) - P_{1,2}(2\lambda_1 + (n-2)\lambda - nH)) \\ & + \frac{n(n-1)}{4} P_{3,6}. \end{aligned} \quad (3.39)$$

Using (3.37) and (3.31), we get

$$\alpha^2 = \frac{\frac{4}{n(n-1)} P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH) - P_{5,8}}{P_{4,8} + \frac{4(n-2)}{n(n-1)} P_{6,5}(\lambda_1 - \lambda)(2\lambda_1 - nH)} \lambda(nH - \lambda_1 - (n-2)\lambda),$$

$$\beta^2 = \frac{-\frac{4(n-2)}{n(n-1)} P_{6,5}(\lambda_1 - \lambda)(2\lambda_1 - nH) - P_{4,8}}{P_{5,8} - \frac{4}{n(n-1)} P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)} \lambda(nH - \lambda_1 - (n-2)\lambda). \quad (3.40)$$

Eliminating  $\alpha^2$  and  $\beta^2$  from (3.38), we obtain

$$\begin{aligned} & \lambda(nH - \lambda_1 - (n-2)\lambda)(2\lambda_1 - nH) \\ & [P_{1,2}(n-2)(\lambda_1 - \lambda)(P_{5,8} - \frac{4}{n(n-1)} P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH))^2 \\ & - P_{2,2}(2\lambda_1 + (n-2)\lambda - nH)(P_{4,8} + \frac{4(n-2)}{n(n-1)} P_{6,5}(\lambda_1 - \lambda)(2\lambda_1 - nH))]^2 \\ & = L(P_{5,8} - \frac{4}{n(n-1)} P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH)) \\ & (P_{4,8} + \frac{4(n-2)}{n(n-1)} P_{6,5}(\lambda_1 - \lambda)(2\lambda_1 - nH)), \end{aligned} \quad (3.41)$$

which is a polynomial equation of degree 22 in terms of  $\lambda$  and  $\lambda_1$ .

Now consider an integral curve of  $e_1$  passing through  $p = \gamma(t_0)$  as  $\gamma(t)$ ,  $t \in I$ .

Since  $e_i(\lambda_1) = e_i(\lambda) = 0$  for  $i = 2, \dots, n$  and  $e_1(\lambda_1), e_1(\lambda) \neq 0$ , we can assume  $t = t(\lambda)$  and  $\lambda_1 = \lambda_1(\lambda)$  in some neighborhood of  $\lambda_0 = \lambda(t_0)$ . Using (3.37), we have

$$\begin{aligned} \frac{d\lambda_1}{d\lambda} &= \frac{d\lambda_1}{dt} \frac{dt}{d\lambda} = \frac{e_1(\lambda_1)}{e_1(\lambda)} \\ &= \frac{(2\lambda_1 + (n-2)\lambda - nH)\beta - (n-2)(\lambda_1 - \lambda)\alpha}{(\lambda_1 - \lambda)\alpha} \\ &= \frac{\left(P_{4,8} + \frac{4(n-2)}{n(n-1)}P_{6,5}(\lambda_1 - \lambda)(2\lambda_1 - nH)\right)(2\lambda_1 + (n-2)\lambda - nH)}{\left(\frac{4}{n(n-1)}P_{6,5}(2\lambda_1 + (n-2)\lambda - nH)(2\lambda_1 - nH) - P_{5,8}\right)(\lambda_1 - \lambda)} - (n-2) \end{aligned} \quad (3.42)$$

Differentiating (3.41) with respect to  $\lambda$  and substituting  $\frac{d\lambda_1}{d\lambda}$  from (3.42), we get

$$f(\lambda_1, \lambda) = 0, \quad (3.43)$$

another algebraic equation of degree 30 in terms of  $\lambda_1$  and  $\lambda$ .

We rewrite (3.41) and (3.43) respectively in the following forms

$$\sum_{i=0}^{22} f_i(\lambda_1) \lambda^i, \quad \sum_{i=0}^{30} g_i(\lambda_1) \lambda^i, \quad (3.44)$$

where  $f_i(\lambda_1)$  and  $g_j(\lambda_1)$  are polynomial functions of  $\lambda_1$ . We eliminate  $\lambda^{30}$  between these two polynomials of (3.44) by multiplying  $g_{30}\lambda^8$  and  $f_{22}$  respectively on the first and second equations of (3.44), we obtain a new polynomial equation in  $\lambda$  of degree 29. Combining this equation with the first equation of (3.44), we successively obtain a polynomial equation in  $\lambda$  of degree 28. In a similar way, by using the first equation of (3.44) and its consequences we are able to gradually eliminate  $\lambda$ . At last, we obtain a non-trivial algebraic polynomial equation in  $\lambda_1$  with constant coefficients. Therefore, we conclude that the real function  $\lambda_1$  must be a constant, which is a contradiction. Hence  $H_2$  is constant on  $M^n$ . If  $H_2 \neq 0$ , by using (3.2) and (2.2) we obtain that  $H_3$  is constant. Therefore all the mean curvatures  $H_r$  are constant functions, this is equivalent to  $M^n$  is isoparametric. An isoparametric hypersurface of Euclidean space can have at most two distinct principal curvatures ([18]), which is a contradiction. So  $H_2 = 0$ .

In conclusion, we get Theorem 1.1.

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