Atomic Systems in 2-inner Product Spaces

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ABSTRACT. In this paper, the concept of a family of local atoms in a 2-inner product space is introduced and then this concept is generalized to an atomic system for an operator. Next a characterization of atomic systems is proved. This characterization lead us to obtain a new frame which is a generalization of frames in 2-inner product spaces.

1. Introduction and Preliminaries

Frames in Hilbert spaces were introduced by Duffin and Schaffer [9] in the context of nonharmonic Fourier series in 1952. In 1986, frames were brought to life by Daubechies *et al.* [7]. Now frames play an important role not only in the theoretics but also in many kinds of applications, and have been widely applied in signal processing [13], sampling [10, 11], coding and communications [19], filter bank theory [2], system modeling [8], and so on.

Atomic systems for bounded linear operators on Hilbert spaces have been introduced by L. Găvruţa in [15] as a generalization of families of local atoms

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[12]. A sequence $\{f_j\}_{j\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called an *atomic system* for a bounded linear operator K on $\mathcal{H}$ if

i) the series $\sum_{j\in\mathbb{N}} c_j f_j$ converges for all $c = (c_j) \in l^2 := \{\{b_j\}_{j\in\mathbb{N}} : \sum_{j\in\mathbb{N}} |b_j|^2 < \infty\}$;

ii) there exists $C > 0$ such that for every $f \in \mathcal{H}$ there exists $a_f = (a_j) \in l^2$ such that $\|a_f\|_{l^2} \leq C\|f\|$ and $Kf = \sum_{j\in\mathbb{N}} a_j f_j$.

It is proved that this concept is equivalent to K-frames, where K is a bounded linear operator on separable Hilbert space $\mathcal{H}$ [15].

A sequence $\{f_j\}_{j\in\mathbb{N}}$ is said to be a K-*frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{j\in\mathbb{N}} |\langle f, f_j \rangle|^2 \leq B\|f\|^2, \ \forall f \in \mathcal{H}.$$

We refer to [20] for more results on these concepts. In addition, the authors generalized these concepts and gave some new results in Hilbert modules [5] and Banach spaces [6]. Note that frames in Hilbert spaces are just a particular case of K-frames, when K is the identity operator on these Hilbert spaces.

The concepts of 2-inner product spaces and 2-normed spaces have been studied by many authors [3, 4, 14, 16, 17, 18]. In the sequel, we introduce 2-inner product and 2-normed spaces.

Definition 1.1. Suppose that X is a vector space of dimension grater than 1 over the field $\mathbb{F}$ (*either* $\mathbb{R}$ *or* $\mathbb{C}$). If there exists a mapping $\langle ., .|. \rangle : X \times X \times X \to \mathbb{F}$ with the properties

1. $\langle f, f|h \rangle \geq 0$ and $\langle f, f|h \rangle = 0$ if and only if f and h are linearly dependent;
2. $\langle f, f|h \rangle = \langle h, h|f \rangle$;
3. $\langle g, f|h \rangle = \overline{\langle f, g|h \rangle}$;
4. $\langle \alpha f, g|h \rangle = \alpha \langle f, g|h \rangle$ for $\alpha \in \mathbb{F}$;
5. $\langle f_1 + f_2, g|h \rangle = \langle f_1, g|h \rangle + \langle f_2, g|h \rangle$,

then the pair $(X, \langle ., .|. \rangle)$ is called a 2-inner product space. The map $\langle ., .|. \rangle$ is said to be a 2-inner product on X.

Some basic properties of 2-inner product $\langle ., .|. \rangle$ cane be immediately obtained as follows (see [3, 4]).

- $\langle 0, g|h \rangle = \langle f, 0|h \rangle = \langle f, g|0 \rangle = 0$;
- $\langle f, \alpha g|h \rangle = \overline{\alpha} \langle f, g|h \rangle$;
- $\langle f, g|\alpha h \rangle = |\alpha|^2 \langle f, g|h \rangle$;

for all $f, g, h \in X$ and $\alpha \in \mathbb{F}$.

One of the most important properties of 2-inner product is the Cauchy-Schwarz inequality

$$|\langle f, g|h \rangle|^2 \leq \langle f, f|h \rangle \langle g, g|h \rangle, \quad f, g, h \in X.$$

For a given 2-inner product space $(X, \langle ., . | . \rangle)$ we can define a function $\| ., . \|$ on $X \times X$ by

$$\|f, h\| = \langle f, f | h \rangle^{\frac{1}{2}} \tag{1.1}$$

for all $f, h \in X$.

The above mentioned function satisfies the following conditions:

    a. $\|f, h\| \geq 0$ and $\|f, h\| = 0$ if and only if f and h are linearly dependent;
    b. $\|f, h\| = \|h, f\|$;
    c. $\|\alpha f, h\| = |\alpha| \|f, h\|, \alpha \in \mathbb{F}$;
    d. $\|f_1 + f_2, h\| \leq \|f_1, h\| + \|f_2, h\|$.

A 2-norm on a vector space X is a function $\| ., . \|$ defined on $X \times X$ satisfying the conditions (a) to (d) and $(X, \| ., . \|)$ is called a linear 2-normed space. Whenever a 2-inner product space $(X, \langle ., . | . \rangle)$ is given, we consider it as a linear 2-normed space $(X, \| ., . \|)$ via the 2-norm defined by (1.1).

Let X be a 2-inner product space. A sequence $\{f_j\}$ is called convergent if there exists $f \in X$ such that $\lim_{j \to \infty} \|f_j - f, h\| = 0$, for all $h \in X$. Similarly, we can define a Cauchy sequence in X. Also, X is said to be a 2-Hilbert space if it is complete (see [18]).

Now we are ready to state the concept of a 2-frame which was introduced in [1]. A sequence $\{f_j\}$ in a 2-Hilbert space $(X, \langle ., . | . \rangle)$ is called a 2-frame associated to $h \in X$ if there exist $A, B > 0$ such that

$$A\|f, h\|^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f, h\|^2, \forall f \in X. \tag{1.2}$$

If the right side of (1.2) holds, then $\{f_j\}$ is called a 2-Bessel sequence.

In this paper, we shall introduce 2-atomic systems as a generalization of families of local 2-atoms. A characterization of 2-atomic systems is given. This leads us to obtain a generalization of 2-frame.

2. Main Results

In this section we are going to define the concept of a family of local 2-atoms. Next we will generalize this concept to a 2-atomic system for a linear operator and then a generalization of 2-frames will be studied.

In the sequel we assumed that $(X, \langle ., . | . \rangle)$ is a 2-Hilbert space, $h \in X$ and $\langle h \rangle$ is the subspace generated by h.

Definition 2.1. Let $\{f_j\}$ be a 2-Bessel sequence in a 2-inner product space X, $h \in X$ and Y be a closed subspace of X. We say that $\{f_j\}$ is a family of local 2-atoms for Y associated to h if there exists a sequence of bilinear functionals $\{c_j\}$ on $X \times \langle h \rangle$ such that
$i)$ $\sum_j |c_j(f, h)|^2 \leq C\|f, h\|^2$, for some $C > 0$;
$ii)$ $f = \sum_j c_j(f, h)f_j$,
for all $f \in Y$.

Note that a map $c_j : X \times \langle h \rangle \to \mathbb{F}$ is called a bilinear functional if the following conditions hold for every $f, g \in X$ and $\alpha \in \mathbb{F}$.

(i) $c_j(\alpha f + g, h) = \alpha c_j(f, h) + c_j(g, h)$;

(ii) $c_j(f, \alpha h) = \alpha c_j(f, h)$.

In the following proposition, it is proved that every family of local 2-atoms is indeed a 2-frame sequence.

Proposition 2.2. *Suppose that $\{f_j\}$ is a family of local 2-atoms for Y, a closed subspace of 2-inner product space X, then $\{f_j\}$ is a 2-frame for Y associated to h.*

Proof. It is enough to show that $\{f_j\}$ has a lower bound. Since $\{f_j\}$ is a family of local 2-atoms, there exists a sequence of bilinear functionals $\{c_j\}$ such that $\sum_j |c_j(f, h)|^2 \le C\|f, h\|^2, f \in Y$, for some $C > 0$.

$$
\begin{aligned}
\|f, h\|^4 &= (\langle f, f | h \rangle)^2 \\
&= (\langle f, \sum_j c_j(f, h) f_j | h \rangle)^2 \\
&= (\sum_j \overline{c_j(f, h)} \langle f, f_j | h \rangle)^2 \\
&\le \sum_j |c_j(f, h)|^2 \sum_j |\langle f, f_j | h \rangle|^2 \\
&\le C\|f, h\|^2 \sum_j |\langle f, f_j | h \rangle|^2,
\end{aligned}
$$

it means that $\frac{1}{C}\|f, h\|^2 \le \sum_j |\langle f, f_j | h \rangle|^2$. $\qquad\qquad\square$

Assume that $(X, \langle .,.,;. \rangle)$ is a 2-Hilbert space and $h \in X$. The algebraic complement of $\langle h \rangle$ in X is denoted by M_h, i.e. $\langle h \rangle \oplus M_h = X$.

One may see that

$$\langle f, g \rangle_h = \langle f, g | h \rangle, \ f, g \in X.$$

defines a semi-inner product on X (see [1]). This semi-inner product induces the following inner product on the quotient space $\frac{X}{\langle h \rangle}$ denoted by M_h as follows:

$$\langle f + \langle h \rangle, g + \langle h \rangle \rangle_h = \langle f, g \rangle_h, \ f, g \in X.$$

So M_h with respect to $\|f\|_h := \sqrt{\langle f, f \rangle_h}, f \in M_h$, is a normed space. The completion of the inner product space M_h is denoted by X_h.

With these notations, one can rewrite (1.2) as follows:

$$A\|f\|_h^2 \le \sum_j |\langle f, f_j \rangle_h|^2 \le B\|f\|_h^2, \ \forall f \in X_h.$$

Now we are going to generalize the concept of a family of local 2-atoms.

Definition 2.3. Let X be a 2-inner product space and fix $h \in X$. Let K_h be a bounded linear operator on the Hilbert space X_h. A sequence $\{f_j\} \subseteq X$ is

called a 2-atomic system for K_h associated to h if

i) $\{f_j\}$ is a 2-Bessel sequence;

ii) for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$, where $\|a_f\|_{\ell^2} \leq C\|f, h\|_X$ and C is a positive constant.

Note that the convergence of the series $\sum_j a_j f_j$ is in the topology of X. Also if $\{f_j\} \subseteq X_h$ then the convergence of the series $\sum_j a_j f_j$ is in the topology of X implies its convergence in X_h.

A characterization of a 2-atomic system corresponding to $h \in X$ is given as follows which lead us to obtain a generalization of 2-frame.

Theorem 2.4. *Let K_h be a bounded linear operator on X_h. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:*

(i) $\{f_j\}$ *is a 2-atomic system for K_h;*

(ii) *there exist $A, B > 0$ such that*

$$A\|K_h^* f\|_h^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h;$$

(iii) $\{f_j\} \subseteq X_h$ *is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that*

$$K_h f = \sum_j \langle f, g_j | h \rangle f_j, f \in X_h;$$

(iv) $\{f_j\} \subseteq X_h$ *is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that*

$$K_h^* f = \sum_j \langle f, f_j | h \rangle g_j, f \in X_h;$$

(v) $\{Q_h f_j\}$ *is a 2-atomic system for the bounded linear operator $Q_h K_h$, where Q_h is an injective operator on X_h.*

Proof. $i \to ii$) For every $f \in X_h$ we have

$$
\begin{aligned}
\|K_h^* f\|^2 &= \|K_h^* f, h\|^2 \\
&= \sup\{|\langle K_h^* f, g | h \rangle|^2 : g \in X_h, \|g, h\| = 1\} \\
&= \sup\{|\langle f, K_h g | h \rangle|^2 : g \in X_h, \|g, h\| = 1\}.
\end{aligned}
$$

By definition of a 2-atomic system for K_h, there exists $C > 0$ such that $K_h g = \sum_j b_j f_j$ with $\|b_g\|_{\ell^2} = \|\{b_j\}\|_{\ell^2} \leq C\|g, h\|$ and so

$$
\begin{aligned}
\|K_h^* f\|^2 &= \sup\{|\langle f, \sum_j b_j f_j | h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\
&= \sup\{|\sum_j \overline{b_j}\langle f, f_j | h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\
&\leq \sup\{\sum_j |b_j|^2 \sum_j |\langle f, f_j | h\rangle|^2 : g \in X_h, \|g, h\| = 1\} \\
&\leq C^2\|g, h\|^2 \sum_j |\langle f, f_j | h\rangle|^2 \\
&= C^2 \sum_j |\langle f, f_j | h\rangle|^2.
\end{aligned}
$$

It means that $\frac{1}{C^2}\|K_h^* f\|^2 \leq \sum_j |\langle f, f_j | h\rangle|^2$.

$ii \to iii$) Similar to Theorem 3 of [15], there exists a 2-Bessel sequence $\{g_j\} \in X_h$ such that

$$
K_h f = \sum_j \langle f, g_j\rangle_h f_j = \sum_j \langle f, g_j | h\rangle f_j.
$$

$iii \to iv$) For $f, g \in X_h$ we have

$$
\begin{aligned}
\langle K_h f, g\rangle_h &= \langle \sum_j \langle f, g_j | h\rangle f_j, g\rangle_h \\
&= \sum_j \langle f, g_j | h\rangle \langle f_j, g | h\rangle \\
&= \sum_j \langle f, g_j\rangle_h \langle f_j, g\rangle_h \\
&= \langle f, \sum_j \langle g, f_j | h\rangle g_j\rangle_h,
\end{aligned}
$$

that is $K_h^* f = \sum_j \langle f, f_j | h\rangle g_j$.

$iv \to iii$) It is similar to $iii \to iv$ so we omit it.

$i \to v$) Since $\{f_j\}$ is a 2-atomic system for K_h, for any $f \in X_h$ there exists $a_f = \{a_j\} \in \ell^2$ such that $K_h f = \sum_j a_j f_j$ so $Q_h K_h f = \sum_j a_j Q_h f_j$, i.e. $\{Q_h f_j\}$ is a 2-atomic system for $Q_h K_h$.

$v \to i$) Since $\{Q_h f_j\}$ is a 2-atomic system for $Q_h K_h$, for any $f \in X_h$ there exists $\{b_j\} \in \ell^2$ such that $Q_h K_h f = \sum_j b_j Q_h f_j$ so $Q_h(K_h f - \sum_j b_j f_j) = 0$. Due to injectivity of Q_h, $K_h f = \sum_j b_j f_j$. $\qquad\square$

As a result of Theorem 2.4 the following definition is given.

Definition 2.5. Let K_h be a bounded linear operator on X_h. A sequence $\{f_j\}$ in X is called 2-K-frame if there exist $A, B >$ such that

$$A\|K_h^* f\|_h^2 \leq \sum_j |\langle f, f_j | h \rangle|^2 \leq B\|f\|_h^2, \forall f \in X_h.$$

Trivially a 2-frame, which was defined in [1], is a special case of 2-K-frames with $K_h = I$.
A consequence of Theorem 2.4 is given as follows.

Theorem 2.6. *Let P_{Y_h} be the orthogonal projection on Y_h as a closed subspace of X_h. Then for a sequence $\{f_j\} \subseteq X_h$ the following statements are equivalent:*

  (i) *$\{f_j\}$ is a family of local 2-atoms for Y_h;*
 (ii) *$\{f_j\}$ is a 2-atomic system for P_{Y_h};*
(iii) *$\{f_j\}$ is a 2-P_{Y_h}-frame;*
(iv) *$\{f_j\}$ is a 2-Bessel sequence and there exists a 2-Bessel sequence $\{g_j\}$ such that*

$$P_{Y_h} f = \sum_j \langle f, g_j | h \rangle f_j = \sum_j \langle f, f_j | h \rangle g_j, f \in X_h;$$

 (v) *$\{Q_h f_j\}$ is a 2-atomic system for bounded linear operator $Q_h P_{Y_h}$, where Q_h is an injective operator on X_h.*

Proof. $i \to ii$ is obvious.
$ii \longleftrightarrow iii$, $iii \longleftrightarrow iv$, $iv \longleftrightarrow ii$ and $v \longleftrightarrow i$ hold from Theorem 2.4.
$iv \to i$) Since $P_{Y_h} f = \sum_j \langle f, g_j | h \rangle f_j$, it is enough to put $c_j(f, h) = \langle f, g_j | h \rangle$ because it is linear and

$$\sum_j |c_j(f, h)|^2 = \sum_j |\langle f, g_j | h \rangle|^2 \leq D\|f, h\|^2,$$

where D is the upper 2-frame bound of $\{g_j\}$. $\qquad\qquad\square$

EXAMPLE 2.7. Let $n \in \mathbb{N}$ be odd and consider $X = \mathbb{R}^n$ with the following standard two inner product

$$\langle x, y | z \rangle = \det \begin{pmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{pmatrix}$$

where $\langle ., . \rangle$ is the inner product of $\mathbb{R}^n$. Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{R}^n$ and $h = e_n$. Trivially in this case $X_h = \mathbb{R}^{n-1}$ and one can see that its induced inner product is the standard inner product of $\mathbb{R}^{n-1}$. Now define the operator K_h on X_h by

$$K_h(e_{2i}) = e_i, i = 1, 2, ..., \frac{n-1}{2} \text{ and otherwise } K_h e_i = e_i, i \leq n-1.$$

Then one can see that $e_1, e_1, e_3, e_2, ..., e_{\frac{n-1}{2}}, e_{n-1}$ is a 2-K_h-frame.

Acknowledgments

The authors wish to thank the anonymous reviewers for their specific and useful comments.

References

1. A. A. Arefijamaal, Gh. Sadeghi, Frames in 2-inner product spaces, *Iran. J. Math. Sci. Inform.*, **8**(2), (2013), 123–130.
2. H. Bolcskei, F. Hlawatsch, H. G. Feichtinger, Frame-theoretic analyssis of oversampled filter banks, *IEEE Trans. Signal Process.*, **46**, (1998), 3256–3268.
3. Y. J. Cho, P. C. S. Lin, S. S. Kim, A. Misiak, Theorey of 2-inner Product Spaces, Nova Science Publishers, Inc., New York, 2001.
4. Y. J. Cho, M. Matic, J. E. Pecaric, On Gram's deteminant in 2-inner product spaces, *J. Korean Math. Soc.*, **38**(4), (2001), 1125–1156.
5. B. Dastourian, M. Janfada, ∗-frames for operators on Hilbert modules, *Wavelets and Linear algebras.*, **3**, (2016), 27–43.
6. B. Dastourian, M. Janfada, Frames for operators in Banach spaces via semi-inner products, *Int. J. Wavelets Multiresult. Inf. Process.*, **14**(3), (2016), 1650011 (17 pages).
7. I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, *J. Math. Phys.*, **27**, (1986), 1271–1283.
8. N. E. Dudey Ward, J. R. Partington, A construction of rational wavelets and frames in Hardy-Sobolev space with applications to system modelling, *SIAM J. Control Optim.* **36**, (1998), 654–679.
9. J. Duffin, A.C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72**, (1952), 341–366.
10. Y. C. Eldar, Sampling with arbitrary sampling and reconstruction spaces and oblique dual frame vectors. *J. Fourier. Anal. Appl.* **9**(1), (2003), 77–96.
11. Y. C. Eldar, T. Werther, General framework for consistent sampling in Hilbert spaces, *Int. J. Wavelets Multiresult. Inf. Process.* **3**(3), (2005), 347–359.
12. H.G. Feichtinger, T. Werther, Atomic systems for subspaces, in: *L. Zayed (Ed.)*, *Proceedings SampTA*, (2001), *Orlando, FL*, (2001), 163–165.
13. P. J. S. G. Ferreira, Mathematics for multimedia signal processing II: Discrete finite frames and signal reconstruction, In: *Byrnes, J.S. (ed.) Signal processing for multimedia*, IOS Press, Amsterdam (1999), 35–54.
14. S. Gahler, Linear 2-normierte Raume, *Math. Nachr.*, **28**, (1965), 1–43.
15. L. Găvruţa, Frames for operators, *Appl. Comput. Harmon. Anal.*, *32*, (2012), 139–144.
16. S. M. Gozali, H. Gunawan, On b-orthogonality in 2-normed spaces, *J. Indones. Math. Soc.* **16**(2), (2010), 127–132.
17. H. Gunawan, Inner products on n-inner product spaces, *Soochow J. Math.* **28**(4), (2002), 389–398.
18. Z. Lewandowska, Bounded 2-linear operators on 2-normed sets, *Glas. Mat. Ser. III*, **39**(59), (2004), 301–312.
19. T. Strohmer, R. Jr. Heath, Grassmanian frames with applications to coding and communications, *Appl. Comput. Harmon. Anal.*, **14**, (2003), 257–275.
20. X. Xiao, Y. Zhu, L. Găvruţa, Some Properties of K-Frames in Hilbert Spaces, *Results. Math.*, **63**(3-4), (2013), 1243–1255.