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## New Jensen and Ostrowski Type Inequalities for General Lebesgue Integral with Applications

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ABSTRACT. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for  $f$ -divergence measure are provided as well.

**Keywords:** Ostrowski's inequality, Jensen's inequality,  $f$ -Divergence measures.

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### 1. INTRODUCTION

Let  $(\Omega, \mathcal{A}, \mu)$  be a measurable space consisting of a set  $\Omega$ , a  $\sigma$ -algebra  $\mathcal{A}$  of parts of  $\Omega$  and a countably additive and positive measure  $\mu$  on  $\mathcal{A}$  with values in  $\mathbb{R} \cup \{\infty\}$ . Assume, for simplicity, that  $\int_{\Omega} d\mu = 1$ . Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R}, f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(t)| d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel  $\int_{\Omega} w d\mu$  instead of  $\int_{\Omega} w(t) d\mu(t)$ .

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:

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**Theorem 1.1.** Let  $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable convex function on  $(m, M)$  and  $f : \Omega \rightarrow [m, M]$  so that  $\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$ . Then we have the inequality:

$$\begin{aligned} 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left( \int_{\Omega} f d\mu \right) \\ &\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu. \end{aligned} \quad (1.1)$$

In the case of discrete measure, we have:

**Corollary 1.2.** Let  $\Phi : [m, M] \rightarrow \mathbb{R}$  be a differentiable convex function on  $(m, M)$ . If  $x_i \in [m, M]$  and  $w_i \geq 0$  ( $i = 1, \dots, n$ ) with  $W_n := \sum_{i=1}^n w_i = 1$ , then one has the counterpart of Jensen's weighted discrete inequality:

$$\begin{aligned} 0 &\leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi \left( \sum_{i=1}^n w_i x_i \right) \\ &\leq \sum_{i=1}^n w_i \Phi'(x_i) x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|. \end{aligned} \quad (1.2)$$

*Remark 1.3.* We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

If  $f, g : \Omega \rightarrow \mathbb{R}$  are  $\mu$ -measurable functions and  $f, g, fg \in L(\Omega, \mu)$ , then we may consider the Čebyšev functional

$$T(f, g) := \int_{\Omega} fg d\mu - \int_{\Omega} f d\mu \int_{\Omega} g d\mu. \quad (1.3)$$

The following result is known in the literature as the Grüss inequality

$$|T(f, g)| \leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta), \quad (1.4)$$

provided

$$-\infty < \gamma \leq f(t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(t) \leq \Delta < \infty \quad (1.5)$$

for  $\mu$ -a.e.  $t \in \Omega$ .

The constant  $\frac{1}{4}$  is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that  $-\infty < \gamma \leq f(t) \leq \Gamma < \infty$  for  $\mu$ -a.e.  $t \in \Omega$ , then by the Grüss inequality for  $g = f$  and by the Schwarz's integral inequality, we have

$$\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \left[ \int_{\Omega} f^2 d\mu - \left( \int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} (\Gamma - \gamma). \quad (1.6)$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$\begin{aligned}
 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left( \int_{\Omega} f d\mu \right) \\
 &\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\
 &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \\
 &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \left[ \int_{\Omega} f^2 d\mu - \left( \int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \\
 &\leq \frac{1}{4} [\Phi'(M) - \Phi'(m)] (M - m),
 \end{aligned} \tag{1.7}$$

provided that  $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable convex function on  $(m, M)$  and  $f : \Omega \rightarrow [m, M]$  so that  $\Phi \circ f, f, \Phi' \circ f, f \cdot (\Phi' \circ f) \in L(\Omega, \mu)$ , with  $\int_{\Omega} d\mu = 1$ .

The following reverse of the Jensen's inequality also holds [33]:

**Theorem 1.4.** *Let  $\Phi : I \rightarrow \mathbb{R}$  be a continuous convex function on the interval of real numbers  $I$  and  $m, M \in \mathbb{R}$ ,  $m < M$  with  $[m, M] \subset \overset{\circ}{I}$ , where  $\overset{\circ}{I}$  is the interior of  $I$ . If  $f : \Omega \rightarrow \mathbb{R}$  is  $\mu$ -measurable, satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that  $f, \Phi \circ f \in L(\Omega, \mu)$ , then

$$\begin{aligned}
 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left( \int_{\Omega} f d\mu \right) \\
 &\leq \left( M - \int_{\Omega} f d\mu \right) \left( \int_{\Omega} f d\mu - m \right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\
 &\leq \frac{1}{4} (M - m) [\Phi'_-(M) - \Phi'_+(m)],
 \end{aligned} \tag{1.8}$$

where  $\Phi'_-$  is the left and  $\Phi'_+$  is the right derivative of the convex function  $\Phi$ .

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean  $\frac{1}{b-a} \int_a^b \Phi(t) dt$  and the value  $\Phi(x)$ ,  $x \in [a, b]$ .

For various results related to Ostrowski's inequality see [6]-[9], [15]-[41], [43] and the references therein.

**Theorem 1.5.** *Let  $\Phi : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $\Phi' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|\Phi'\|_{\infty} := \sup_{t \in (a, b)} |\Phi'(t)| <$*

$\infty$ . Then

$$\left| \Phi(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|\Phi'\|_\infty (b-a), \quad (1.9)$$

for all  $x \in [a, b]$  and the constant  $\frac{1}{4}$  is the best possible.

Now, for  $\gamma, \Gamma \in \mathbb{C}$  and  $[a, b]$  an interval of real numbers, define the sets of complex-valued functions [34]

$$\begin{aligned} &\bar{U}_{[a,b]}(\gamma, \Gamma) \\ &:= \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re} \left[ (\Gamma - f(t)) \overline{(f(t) - \gamma)} \right] \geq 0 \text{ for almost every } t \in [a, b] \right\} \end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a, b] \right\}.$$

The following representation result may be stated [34].

**Proposition 1.6.** For any  $\gamma, \Gamma \in \mathbb{C}$ ,  $\gamma \neq \Gamma$ , we have that  $\bar{U}_{[a,b]}(\gamma, \Gamma)$  and  $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$  are nonempty, convex and closed sets and

$$\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma). \quad (1.10)$$

On making use of the complex numbers field properties we can also state that:

**Corollary 1.7.** For any  $\gamma, \Gamma \in \mathbb{C}$ ,  $\gamma \neq \Gamma$ , we have that

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) = \{ f : [a, b] \rightarrow \mathbb{C} \mid &(\operatorname{Re} \Gamma - \operatorname{Re} f(t)) (\operatorname{Re} f(t) - \operatorname{Re} \gamma) \\ &+ (\operatorname{Im} \Gamma - \operatorname{Im} f(t)) (\operatorname{Im} f(t) - \operatorname{Im} \gamma) \geq 0 \text{ for a.e. } t \in [a, b] \}. \end{aligned} \quad (1.11)$$

Now, if we assume that  $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$  and  $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$ , then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a, b] \rightarrow \mathbb{C} \mid &\operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ &\text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a, b] \}. \end{aligned} \quad (1.12)$$

One can easily observe that  $\bar{S}_{[a,b]}(\gamma, \Gamma)$  is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \quad (1.13)$$

The following result holds [34]:

**Theorem 1.8.** Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , the interior of  $I$ . For some  $\gamma, \Gamma \in \mathbb{C}$ ,  $\gamma \neq \Gamma$ , assume that  $\Phi' \in \bar{U}_{[a,b]}(\gamma, \Gamma)$  ( $= \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ ). If  $g : \Omega \rightarrow [a, b]$  is Lebesgue  $\mu$ -measurable on  $\Omega$  and such that  $\Phi \circ g, g \in L(\Omega, \mu)$ , then we have the inequality

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} |g - x| d\mu \quad (1.14)$$

for any  $x \in [a, b]$ .

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \frac{\gamma + \Gamma}{2} \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{4} (b-a) |\Gamma - \gamma| \end{aligned} \quad (1.15)$$

and

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \\ & \leq \frac{1}{2} |\Gamma - \gamma| \left( \int_{\Omega} g^2 d\mu - \left( \int_{\Omega} g d\mu \right)^2 \right)^{1/2} \\ & \leq \frac{1}{4} (b-a) |\Gamma - \gamma|. \end{aligned} \quad (1.16)$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left( \int_{\Omega} g d\mu - x \right) \right|, \quad x \in [a, b],$$

under various assumptions on the absolutely continuous function  $\Phi$ , which in the particular case of  $x = \int_{\Omega} g d\mu$  provides some results connected with Jensen's inequality while in the case  $\lambda = 0$  provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

## 2. SOME IDENTITIES

The following result holds [34]:

**Lemma 2.1.** *Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , the interior of  $I$ . If  $g : \Omega \rightarrow [a, b]$  is Lebesgue  $\mu$ -measurable on  $\Omega$  and such that  $\Phi \circ g, g \in L(\Omega, \mu)$ , then we have the equality*

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left( \int_{\Omega} g d\mu - x \right) \\ & = \int_{\Omega} \left[ (g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] d\mu \end{aligned} \quad (2.1)$$

for any  $\lambda \in \mathbb{C}$  and  $x \in [a, b]$ .

In particular, we have

$$\int_{\Omega} \Phi \circ g d\mu - \Phi(x) = \int_{\Omega} \left[ (g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu, \quad (2.2)$$

for any  $x \in [a, b]$ .

*Remark 2.2.* With the assumptions of Lemma 2.1 we have

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) \\ &= \int_{\Omega} \left[ \left( g - \frac{a+b}{2} \right) \int_0^1 \Phi' \left( (1-s) \frac{a+b}{2} + sg \right) ds \right] d\mu. \end{aligned} \quad (2.3)$$

**Corollary 2.3.** *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \\ &= \int_{\Omega} \left[ \left( g - \int_{\Omega} g d\mu \right) \int_0^1 \Phi' \left( (1-s) \int_{\Omega} g d\mu + sg \right) ds \right] d\mu. \end{aligned} \quad (2.4)$$

*Proof.* We observe that since  $g : \Omega \rightarrow [a, b]$  and  $\int_{\Omega} d\mu = 1$  then  $\int_{\Omega} g d\mu \in [a, b]$  and by taking  $x = \int_{\Omega} g d\mu$  in (2.2) we get (2.4).  $\square$

**Corollary 2.4.** *With the assumptions of Lemma 2.1 we have*

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \frac{1}{b-a} \int_a^b \Phi(x) dx - \lambda \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \\ &= \int_{\Omega} \left\{ \frac{1}{b-a} \int_a^b \left[ (g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] dx \right\} d\mu. \end{aligned} \quad (2.5)$$

*Proof.* Follows by integrating the identity (2.1) over  $x \in [a, b]$ , dividing by  $b-a > 0$  and using Fubini's theorem.  $\square$

**Corollary 2.5.** *Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , the interior of  $I$ . If  $g, h : \Omega \rightarrow [a, b]$  are Lebesgue  $\mu$ -measurable on  $\Omega$  and such that  $\Phi \circ g, \Phi \circ h, g, h \in L(\Omega, \mu)$ , then we have the equality*

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu - \lambda \left( \int_{\Omega} g d\mu - \int_{\Omega} h d\mu \right) \\ &= \int_{\Omega} \int_{\Omega} \left[ (g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) ds \right] \\ & \quad \times d\mu(t) d\mu(\tau) \end{aligned} \quad (2.6)$$

for any  $\lambda \in \mathbb{C}$  and  $x \in [a, b]$ .

In particular, we have

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu \\ &= \int_{\Omega} \int_{\Omega} \left[ (g(t) - h(\tau)) \int_0^1 \Phi'((1-s)h(\tau) + sg(t)) ds \right] d\mu(t) d\mu(\tau), \end{aligned} \quad (2.7)$$

for any  $x \in [a, b]$ .

*Remark 2.6.* The above inequality (2.6) can be extended for two measures as follows

$$\begin{aligned} & \int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left( \int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right) \\ &= \int_{\Omega_1} \int_{\Omega_2} \left[ (g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) ds \right] \\ & \times d\mu_1(t) d\mu_2(\tau), \end{aligned} \tag{2.8}$$

for any  $\lambda \in \mathbb{C}$  and  $x \in [a, b]$  and provided that  $\Phi \circ g, g \in L(\Omega_1, \mu_1)$  while  $\Phi \circ h, h \in L(\Omega_2, \mu_2)$ .

*Remark 2.7.* If  $w \geq 0$   $\mu$ -almost everywhere ( $\mu$ -a.e.) on  $\Omega$  with  $\int_{\Omega} w d\mu > 0$ , then by replacing  $d\mu$  with  $\frac{w d\mu}{\int_{\Omega} w d\mu}$  in (2.1) we have the weighted equality

$$\begin{aligned} & \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w (\Phi \circ g) d\mu - \Phi(x) - \lambda \left( \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w g d\mu - x \right) \\ &= \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w \cdot \left[ (g - x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] d\mu \end{aligned} \tag{2.9}$$

for any  $\lambda \in \mathbb{C}$  and  $x \in [a, b]$ , provided  $\Phi \circ g, g \in L_w(\Omega, \mu)$  where

$$L_w(\Omega, \mu) := \left\{ g \mid \int_{\Omega} w |g| d\mu < \infty \right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

### 3. INEQUALITIES FOR DERIVATIVES OF BOUNDED VARIATION

The following result holds:

**Theorem 3.1.** *Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \mathring{I}$ , the interior of  $I$  and with the property that the derivative  $\Phi'$  is of bounded variation on  $[a, b]$ . If  $g : \Omega \rightarrow [a, b]$  is Lebesgue  $\mu$ -measurable on  $\Omega$  and such that  $\Phi \circ g, g \in L(\Omega, \mu)$ , then we have*

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - x \right) \right| \\ & \leq \frac{1}{2} \bigvee_a^b(\Phi') \int_{\Omega} |g - x| d\mu \end{aligned} \tag{3.1}$$

for any  $x \in [a, b]$ .

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{2} \bigvee_a^b(\Phi') \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{2} (b-a) \bigvee_a^b(\Phi') \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \int_{\Omega} g d\mu \right) \right| &\leq \frac{1}{2} \bigvee_a^b(\Phi') \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu & (3.3) \\ &\leq \frac{1}{2} \bigvee_a^b(\Phi') \left( \int_{\Omega} g^2 d\mu - \left( \int_{\Omega} g d\mu \right)^2 \right)^{1/2} \\ &\leq \frac{1}{4} (b-a) \bigvee_a^b(\Phi'). \end{aligned}$$

*Proof.* From the identity (2.1) we have

$$\begin{aligned} &\int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - x \right) & (3.4) \\ &= \int_{\Omega} \left[ (g-x) \int_0^1 \left( \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) ds \right] d\mu \end{aligned}$$

for any  $x \in [a, b]$ .

Taking the modulus in (3.4) we get

$$\begin{aligned} &\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \int_{\Omega} g d\mu - x \right) \right| & (3.5) \\ &\leq \int_{\Omega} \left| (g-x) \int_0^1 \left( \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) \right| ds d\mu \\ &\leq \int_{\Omega} |g-x| \int_0^1 \left| \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds d\mu \end{aligned}$$

for any  $x \in [a, b]$ .

Since  $\Phi'$  is of bounded variation on  $[a, b]$ , then for any  $s \in [0, 1]$ ,  $x \in [a, b]$  and  $t \in \Omega$  we have

$$\begin{aligned} &\left| \Phi'((1-s)x + sg(t)) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| \\ &= \frac{1}{2} |\Phi'((1-s)x + sg(t)) - \Phi'(a) + \Phi'((1-s)x + sg(t)) - \Phi'(b)| \\ &\leq \frac{1}{2} [|\Phi'((1-s)x + sg(t)) - \Phi'(a)| + |\Phi'(b) - \Phi'((1-s)x + sg(t))|] \\ &\leq \frac{1}{2} \bigvee_a^b(\Phi'). \end{aligned}$$

Then we have

$$\begin{aligned} &\int_{\Omega} |g-x| \int_0^1 \left| \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds d\mu & (3.6) \\ &\leq \frac{1}{2} \bigvee_a^b(\Phi') \int_{\Omega} |g-x| d\mu \end{aligned}$$



for any  $x \in [a, b]$ .

Making use of (3.5) and (3.6) we deduce the desired result (3.1).  $\square$

*Remark 3.2.* Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , the interior of  $I$  and with the property that the derivative  $\Phi'$  is of bounded variation on  $[a, b]$ . If  $x_i \in [m, M]$  and  $w_i \geq 0$  ( $i = 1, \dots, n$ ) with  $W_n := \sum_{i=1}^n w_i = 1$ , then one has the weighted discrete inequality:

$$\begin{aligned} & \left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^n w_i x_i - x \right) \right| \\ & \leq \frac{1}{2} \bigvee_a^b(\Phi') \sum_{i=1}^n w_i |x_i - x| \end{aligned} \quad (3.7)$$

for any  $x \in [a, b]$ .

In particular, we have

$$\begin{aligned} & \left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left( \sum_{i=1}^n w_i x_i - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{2} \bigvee_a^b(\Phi') \sum_{i=1}^n w_i \left| x_i - \frac{a+b}{2} \right| \leq \frac{1}{4} (b-a) \bigvee_a^b(\Phi') \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} & \left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \right| \leq \frac{1}{2} \bigvee_a^b(\Phi') \sum_{i=1}^n w_i \left| x_i - \sum_{i=1}^n w_i x_i \right| \\ & \leq \frac{1}{2} \bigvee_a^b(\Phi') \left( \sum_{j=1}^n w_j x_j^2 - \left( \sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2} \\ & \leq \frac{1}{4} (b-a) \bigvee_a^b(\Phi'). \end{aligned} \quad (3.9)$$

#### 4. INEQUALITIES FOR LIPSCHITZIAN DERIVATIVES

The following result holds:

**Theorem 4.1.** Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , the interior of  $I$  and with the property that the derivative  $\Phi'$  is Lipschitzian with the constant  $K > 0$  on  $[a, b]$ . If  $g : \Omega \rightarrow [a, b]$  is Lebesgue  $\mu$ -measurable on  $\Omega$  and such that  $\Phi \circ g, g \in L(\Omega, \mu)$ , then we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left( \int_{\Omega} g d\mu - x \right) \right| \\ & \leq \frac{1}{2} K \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right] \end{aligned} \quad (4.1)$$

for any  $x \in [a, b]$ , where  $\sigma_\mu(g)$  is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left( \int_\Omega \left( g - \int_\Omega g d\mu \right)^2 d\mu \right)^{1/2} = \left( \int_\Omega g^2 d\mu - \left( \int_\Omega g d\mu \right)^2 \right)^{1/2}.$$

In particular, we have

$$\begin{aligned} & \left| \int_\Omega \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \frac{a+b}{2} \right) \left( \int_\Omega g d\mu - \frac{a+b}{2} \right) \right| \quad (4.2) \\ & \leq \frac{1}{2} K \left[ \sigma_\mu^2(g) + \left( \int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \end{aligned}$$

and

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi \left( \int_\Omega g d\mu \right) \right| \leq \frac{1}{2} K \sigma_\mu^2(g) \leq \frac{1}{8} K (b-a)^2. \quad (4.3)$$

*Proof.* From the identity (2.1) we have for  $\lambda = \Phi'(x)$  that

$$\begin{aligned} & \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left( \int_\Omega g d\mu - x \right) \quad (4.4) \\ & = \int_\Omega \left[ (g-x) \int_0^1 (\Phi'((1-s)x + sg) - \Phi'(x)) ds \right] d\mu \end{aligned}$$

for any  $x \in [a, b]$ .

Taking the modulus in (4.4) we get

$$\begin{aligned} & \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left( \int_\Omega g d\mu - x \right) \right| \quad (4.5) \\ & \leq \int_\Omega |g-x| \left| \int_0^1 (\Phi'((1-s)x + sg) - \Phi'(x)) ds \right| d\mu \\ & \leq \int_\Omega \left[ |g-x| \int_0^1 |\Phi'((1-s)x + sg) - \Phi'(x)| ds \right] d\mu \\ & \leq K \int_\Omega \left[ |g-x| \int_0^1 s |g-x| ds \right] d\mu = \frac{1}{2} K \int_\Omega (g-x)^2 d\mu \end{aligned}$$

for any  $x \in [a, b]$ .

However,

$$\begin{aligned} & \int_{\Omega} (g - x)^2 d\mu \\ &= \int_{\Omega} \left( g - \int_{\Omega} g d\mu + \int_{\Omega} g d\mu - x \right)^2 d\mu \\ &= \int_{\Omega} \left( g - \int_{\Omega} g d\mu \right)^2 d\mu + 2 \int_{\Omega} \left( g - \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) d\mu \\ &+ \int_{\Omega} \left( \int_{\Omega} g d\mu - x \right)^2 d\mu \\ &= \int_{\Omega} \left( g - \int_{\Omega} g d\mu \right)^2 d\mu + \left( \int_{\Omega} g d\mu - x \right)^2 \end{aligned}$$

for any  $x \in [a, b]$ , and by (4.5) we get the desired result (4.1). □

**Corollary 4.2.** *Let  $\Phi : I \rightarrow \mathbb{C}$  be a twice differentiable functions on  $[a, b] \subset \overset{\circ}{I}$  with  $\|\Phi''\|_{[a,b],\infty} := \text{ess sup}_{t \in [a,b]} |\Phi''(t)| < \infty$ . Then the inequalities (4.1)-(4.3) hold for  $K = \|\Phi''\|_{[a,b],\infty}$ .*

*Remark 4.3.* Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$  and with the property that the derivative  $\Phi'$  is Lipschitzian with the constant  $K > 0$  on  $[a, b]$ . If  $x_i \in [m, M]$  and  $w_i \geq 0$  ( $i = 1, \dots, n$ ) with  $W_n := \sum_{i=1}^n w_i = 1$ , then one has the weighted discrete inequality:

$$\begin{aligned} & \left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) - \Phi'(x) \left( \sum_{i=1}^n w_i x_i - x \right) \right| \tag{4.6} \\ & \leq \frac{1}{2} K \left[ \sigma_w^2(\mathbf{x}) + \left( \sum_{i=1}^n w_i x_i - x \right)^2 \right] \end{aligned}$$

for any  $x \in [a, b]$ , where

$$\sigma_w(\mathbf{x}) := \left( \sum_{i=1}^n w_i \left( x_i - \sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2} = \left( \sum_{i=1}^n w_i x_i^2 - \left( \sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2}.$$

The following lemma may be stated:

**Lemma 4.4.** *Let  $u : [a, b] \rightarrow \mathbb{R}$  and  $l, L \in \mathbb{R}$  with  $L > l$ . The following statements are equivalent:*

- (i) *The function  $u - \frac{l+L}{2} \cdot e$ , where  $e(t) = t$ ,  $t \in [a, b]$  is  $\frac{1}{2}(L - l)$ -Lipschitzian;*
- (ii) *We have the inequalities*

$$l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each } t, s \in [a, b] \quad \text{with } t \neq s; \tag{4.7}$$

- (iii) *We have the inequalities*

$$l(t - s) \leq u(t) - u(s) \leq L(t - s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s. \tag{4.8}$$

Following [53], we can introduce the definition of  $(l, L)$ -Lipschitzian functions:

**Definition 4.5.** The function  $u : [a, b] \rightarrow \mathbb{R}$  which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be  $(l, L)$ -Lipschitzian on  $[a, b]$ .

If  $L > 0$  and  $l = -L$ , then  $(-L, L)$ -Lipschitzian means  $L$ -Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of  $(l, L)$ -Lipschitzian functions.

**Proposition 4.6.** Let  $u : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $-\infty < l = \inf_{t \in [a, b]} u'(t)$  and  $\sup_{t \in [a, b]} u'(t) = L < \infty$ , then  $u$  is  $(l, L)$ -Lipschitzian on  $[a, b]$ .

The following result holds.

**Corollary 4.7.** Let  $\Phi : I \rightarrow \mathbb{R}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , with the property that the derivative  $\Phi'$  is  $(l, L)$ -Lipschitzian on  $[a, b]$ , where  $l, L \in \mathbb{R}$  with  $L > l$ . If  $g : \Omega \rightarrow [a, b]$  is Lebesgue  $\mu$ -measurable on  $\Omega$  and such that  $\Phi \circ g, g \in L(\Omega, \mu)$ , then we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left( \int_{\Omega} g d\mu - x \right) \right. \\ & \quad \left. - \frac{1}{4} (L + l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right] \right| \\ & \leq \frac{1}{4} (L - l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} g d\mu - x \right)^2 \right] \end{aligned} \quad (4.9)$$

for any  $x \in [a, b]$ .

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right) \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right. \\ & \quad \left. - \frac{1}{4} (L + l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right)^2 \right] \right| \\ & \leq \frac{1}{4} (L - l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right)^2 \right] \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} \left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\int_{\Omega} g d\mu\right) - \frac{1}{4} (L + l) \sigma_{\mu}^2(g) \right| & \leq \frac{1}{4} (L - l) \sigma_{\mu}^2(g) \\ & \leq \frac{1}{16} (L - l) (b - a)^2. \end{aligned} \quad (4.11)$$

*Proof.* Consider the auxiliary function  $\Psi : [a, b] \rightarrow \mathbb{R}$  given by

$$\Psi(x) = \Phi(x) - \frac{1}{4}(L+l)x^2.$$

We observe that  $\Psi$  is differentiable and

$$\Psi'(x) = \Phi'(x) - \frac{1}{2}(L+l)x.$$

Since  $\Phi'$  is  $(l, L)$ -Lipschitzian on  $[a, b]$  it follows that  $\Psi'$  is Lipschitzian with the constant  $\frac{1}{2}(L-l)$ , so we can apply Theorem 4.1 for  $\Psi$ , i.e. we have the inequality

$$\begin{aligned} & \left| \int_{\Omega} \Psi \circ gd\mu - \Psi(x) - \Psi'(x) \left( \int_{\Omega} gd\mu - x \right) \right| \\ & \leq \frac{1}{4}(L-l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} gd\mu - x \right)^2 \right]. \end{aligned} \quad (4.12)$$

However

$$\begin{aligned} & \int_{\Omega} \Psi \circ gd\mu - \Psi(x) - \Psi'(x) \left( \int_{\Omega} gd\mu - x \right) \\ & = \int_{\Omega} \Phi \circ gd\mu - \Phi(x) - \Phi'(x) \left( \int_{\Omega} gd\mu - x \right) \\ & \quad - \frac{1}{4}(L+l) \left[ \int_{\Omega} g^2 d\mu - x^2 - 2x \left( \int_{\Omega} gd\mu - x \right) \right] \\ & = \int_{\Omega} \Phi \circ gd\mu - \Phi(x) - \Phi'(x) \left( \int_{\Omega} gd\mu - x \right) \\ & \quad - \frac{1}{4}(L+l) \left[ \sigma_{\mu}^2(g) + \left( \int_{\Omega} gd\mu - x \right)^2 \right] \end{aligned}$$

and by (4.12) we get the desired result (4.9).  $\square$

*Remark 4.8.* We observe that if the function  $\Phi$  is twice differentiable on  $\overset{\circ}{I}$  and for  $[a, b] \subset \overset{\circ}{I}$  we have

$$-\infty < l \leq \Phi''(x) \leq L < \infty \text{ for any } x \in [a, b],$$

then  $\Phi'$  is  $(l, L)$ -Lipschitzian on  $[a, b]$  and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

**Theorem 4.9.** *Let  $\Phi : I \rightarrow \mathbb{C}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , the interior of  $I$  and with the property that the derivative  $\Phi'$  is Lipschitzian with the constant  $K > 0$  on  $[a, b]$ . If  $g : \Omega \rightarrow [a, b]$  is Lebesgue  $\mu$ -measurable on*

$\Omega$  and such that  $\Phi \circ g, g \in L(\Omega, \mu)$ , then we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right| \quad (4.13) \\ & \leq \frac{1}{2} K \left[ \left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} |g - x| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\ & \leq \frac{1}{2} K \left[ \left| x - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} |g - x| d\mu \end{aligned}$$

for any  $x \in [a, b]$ , where

$$\left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} := \operatorname{ess\,sup}_{t \in \Omega} \left| g(t) - \int_{\Omega} g d\mu \right| < \infty.$$

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \quad (4.14) \\ & \leq \frac{1}{2} K \left[ \left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \right. \\ & \quad \left. + \int_{\Omega} \left| g - \frac{a+b}{2} \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\ & \leq \frac{1}{2} K \left[ \left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu. \end{aligned}$$

*Proof.* From the identity (2.1) we have for  $\lambda = \Phi' \left( \int_{\Omega} g d\mu \right)$  that

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \quad (4.15) \\ & = \int_{\Omega} \left[ (g-x) \int_0^1 \left( \Phi'((1-s)x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right] d\mu \end{aligned}$$

for any  $x \in [a, b]$ .

Taking the modulus in (4.15) we get

$$\begin{aligned}
 & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right| \quad (4.16) \\
 & \leq \int_{\Omega} |g - x| \left| \int_0^1 \left( \Phi'((1-s)x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) \right) ds \right| d\mu \\
 & \leq \int_{\Omega} \left[ |g - x| \int_0^1 \left| \Phi'((1-s)x + sg) - \Phi' \left( \int_{\Omega} g d\mu \right) \right| ds \right] d\mu \\
 & \leq K \int_{\Omega} \left[ |g - x| \int_0^1 \left| (1-s)x + sg - \int_{\Omega} g d\mu \right| ds \right] d\mu \\
 & = K \int_{\Omega} \left[ |g - x| \int_0^1 \left| (1-s)x + sg - (1-s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right| ds \right] d\mu \\
 & := B.
 \end{aligned}$$

Using the triangle inequality we have for any  $t \in \Omega$

$$\begin{aligned}
 & \int_0^1 \left| (1-s)x + sg(t) - (1-s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right| ds \\
 & \leq \int_0^1 (1-s) \left| x - \int_{\Omega} g d\mu \right| ds + \int_0^1 s \left| g(t) - \int_{\Omega} g d\mu \right| ds \\
 & = \frac{1}{2} \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g(t) - \int_{\Omega} g d\mu \right| \right]
 \end{aligned}$$

and then

$$\begin{aligned}
 B & \leq \frac{1}{2} K \int_{\Omega} |g - x| \left[ \left| x - \int_{\Omega} g d\mu \right| + \left| g(t) - \int_{\Omega} g d\mu \right| \right] d\mu \quad (4.17) \\
 & = \frac{1}{2} K \left[ \int_{\Omega} |g - x| \left| x - \int_{\Omega} g d\mu \right| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right].
 \end{aligned}$$

Making use of (4.16) and (4.17) we deduce the desired result (4.13).  $\square$

**Corollary 4.10.** *Let  $\Phi : I \rightarrow \mathbb{R}$  be an absolutely continuous functions on  $[a, b] \subset \overset{\circ}{I}$ , with the property that the derivative  $\Phi'$  is  $(l, L)$ -Lipschitzian on  $[a, b]$ , where  $l, L \in \mathbb{R}$  with  $L > l$ . If  $g : \Omega \rightarrow [a, b]$  is Lebesgue  $\mu$ -measurable on  $\Omega$  and such that  $\Phi \circ g, g \in L(\Omega, \mu)$ , then we have*

$$\begin{aligned}
 & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - x \right) \right. \quad (4.18) \\
 & \quad \left. - \frac{1}{4} (L + l) \left[ \sigma_{\mu}^2(g) - \left( x - \int_{\Omega} g d\mu \right)^2 \right] \right| \\
 & \leq \frac{1}{4} (L - l) \left[ \int_{\Omega} |g - x| \left| x - \int_{\Omega} g d\mu \right| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\
 & \leq \frac{1}{4} (L - l) \left[ \int_{\Omega} |g - x| \left| x - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} |g - x| d\mu
 \end{aligned}$$

for any  $x \in [a, b]$ .

In particular, we have

$$\begin{aligned}
 & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left( \frac{a+b}{2} \right) - \Phi' \left( \int_{\Omega} g d\mu \right) \left( \int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right. \\
 & \quad \left. - \frac{1}{4} (L+l) \left[ \sigma_{\mu}^2(g) - \left( \frac{a+b}{2} - \int_{\Omega} g d\mu \right)^2 \right] \right| \\
 & \leq \frac{1}{4} (L-l) \left[ \left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \right. \\
 & \quad \left. + \int_{\Omega} \left| g - \frac{a+b}{2} \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\
 & \leq \frac{1}{4} (L-l) \left[ \left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu.
 \end{aligned} \tag{4.19}$$

## 5. APPLICATIONS FOR $f$ -DIVERGENCE

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár  $f$ -divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set  $\Omega$  and the  $\sigma$ -finite measure  $\mu$  are given. Consider the set of all probability densities on  $\mu$  to be  $\mathcal{P} := \{p|p: \Omega \rightarrow \mathbb{R}, p(t) \geq 0, \int_{\Omega} p(t) d\mu(t) = 1\}$ . The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

$$D_{KL}(p, q) := \int_{\Omega} p(t) \ln \left[ \frac{p(t)}{q(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}, \tag{5.1}$$

where  $\ln$  is to base  $e$ .

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance*  $D_v$ , *Hellinger distance*  $D_H$  [45],  $\chi^2$ -*divergence*  $D_{\chi^2}$ ,  $\alpha$ -*divergence*  $D_{\alpha}$ , *Bhattacharyya distance*  $D_B$  [3], *Harmonic distance*  $D_{H\alpha}$ , *Jeffrey's distance*  $D_J$  [47],



triangular discrimination  $D_\Delta$  [65], etc... They are defined as follows:

$$D_v(p, q) := \int_{\Omega} |p(t) - q(t)| d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.2)$$

$$D_H(p, q) := \int_{\Omega} \left| \sqrt{p(t)} - \sqrt{q(t)} \right| d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.3)$$

$$D_{\chi^2}(p, q) := \int_{\Omega} p(t) \left[ \left( \frac{q(t)}{p(t)} \right)^2 - 1 \right] d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.4)$$

$$D_\alpha(p, q) := \frac{4}{1 - \alpha^2} \left[ 1 - \int_{\Omega} [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t) \right], \quad p, q \in \mathcal{P}; \quad (5.5)$$

$$D_B(p, q) := \int_{\Omega} \sqrt{p(t)q(t)} d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.6)$$

$$D_{Ha}(p, q) := \int_{\Omega} \frac{2p(t)q(t)}{p(t) + q(t)} d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.7)$$

$$D_J(p, q) := \int_{\Omega} [p(t) - q(t)] \ln \left[ \frac{p(t)}{q(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.8)$$

$$D_\Delta(p, q) := \int_{\Omega} \frac{[p(t) - q(t)]^2}{p(t) + q(t)} d\mu(t), \quad p, q \in \mathcal{P}. \quad (5.9)$$

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár  $f$ -divergence is defined as follows [13]

$$I_f(p, q) := \int_{\Omega} p(t) f \left[ \frac{q(t)}{p(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}, \quad (5.10)$$

where  $f$  is convex on  $(0, \infty)$ . It is assumed that  $f(u)$  is zero and strictly convex at  $u = 1$ . By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár  $f$ -divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár  $f$ -divergence see [13], [14] and [68].

The following result holds:

**Proposition 5.1.** *Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a twice differentiable convex function with the property that  $f(1) = 0$  and there exists the constants  $\gamma, \Gamma$  so that*

$$-\infty < \gamma \leq f(t) \leq \Gamma < \infty.$$

*Assume that  $p, q \in \mathcal{P}$  and there exists the constants  $0 < r < 1 < R < \infty$  such that*

$$r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega. \quad (5.11)$$

If  $x \in [r, R]$ , then we have the inequality

$$\begin{aligned} & \left| I_f(p, q) - f(x) - f'(x)(1-x) - \frac{1}{4}(L+l) \left[ D_{\chi^2}(p, q) + (1-x)^2 \right] \right| \quad (5.12) \\ & \leq \frac{1}{4}(L-l) \left[ D_{\chi^2}(p, q) + (1-x)^2 \right]. \end{aligned}$$

In particular, we have

$$\begin{aligned} & \left| I_f(p, q) - f\left(\frac{r+R}{2}\right) - f'\left(\frac{r+R}{2}\right) \left(1 - \frac{r+R}{2}\right) \right. \quad (5.13) \\ & \quad \left. - \frac{1}{4}(L+l) \left[ D_{\chi^2}(p, q) + \left(1 - \frac{r+R}{2}\right)^2 \right] \right| \\ & \leq \frac{1}{4}(L-l) \left[ D_{\chi^2}(p, q) + \left(1 - \frac{r+R}{2}\right)^2 \right] \end{aligned}$$

and

$$\left| I_f(p, q) - \frac{1}{4}(L+l) D_{\chi^2}(p, q) \right| \leq \frac{1}{4}(L-l) D_{\chi^2}(p, q). \quad (5.14)$$

*Proof.* From (4.9) we have

$$\begin{aligned} & \left| \int_{\Omega} p(t) f\left(\frac{q(t)}{p(t)}\right) d\mu(t) - f(x) - f'(x)(1-x) \right. \\ & \quad \left. - \frac{1}{4}(L+l) \left[ \int_{\Omega} p(t) \left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2 \right] \right| \\ & \leq \frac{1}{4}(L-l) \left[ \int_{\Omega} p(t) \left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2 \right] \end{aligned}$$

for any  $x \in [r, R]$ , which is equivalent to (5.12). □

Utilising Corollary 4.10 we can state the following result as well:

**Proposition 5.2.** *With the assumptions in Proposition 5.1, we have*

$$\begin{aligned} & \left| I_f(p, q) - f(x) - f'(1)(1-x) - \frac{1}{4}(L+l) \left[ D_{\chi^2}(p, q) - (1-x)^2 \right] \right| \quad (5.15) \\ & \leq \frac{1}{4}(L-l) \left[ |x-1| \int_{\Omega} |q-xp| d\mu + \int_{\Omega} |q-xp| \left| \frac{q}{p} - 1 \right| d\mu \right] \\ & \leq \frac{1}{4}(L-l) \left[ |x-1| + \left\| \frac{q}{p} - 1 \right\|_{\Omega, \infty} \right] \int_{\Omega} |q-xp| d\mu \end{aligned}$$

for any  $x \in [r, R]$ .

If we consider the convex function  $f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(t) = t \ln t$  then

$$\begin{aligned} I_f(p, q) & := \int_{\Omega} p(t) \frac{q(t)}{p(t)} \ln \left[ \frac{q(t)}{p(t)} \right] d\mu(t) = \int_{\Omega} q(t) \ln \left[ \frac{q(t)}{p(t)} \right] d\mu(t) \\ & = D_{KL}(q, p). \end{aligned}$$

We have  $f'(t) = \ln t + 1$  and  $f''(t) = \frac{1}{t}$  and then we can choose  $l = \frac{1}{R}$  and  $L = \frac{1}{r}$ . Applying the inequality (5.14) we get

$$\left| D_{KL}(q, p) - \left( \frac{R+r}{4rR} \right) D_{\chi^2}(p, q) \right| \leq \frac{R-r}{4rR} D_{\chi^2}(p, q). \quad (5.16)$$

If we consider the convex function  $f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(t) = -\ln t$  then

$$\begin{aligned} I_f(p, q) &:= - \int_{\Omega} p(t) \ln \left[ \frac{q(t)}{p(t)} \right] d\mu(t) = \int_{\Omega} p(t) \ln \left[ \frac{p(t)}{q(t)} \right] d\mu(t) \\ &= D_{KL}(p, q). \end{aligned}$$

We have  $f'(t) = -\frac{1}{t}$  and  $f''(t) = \frac{1}{t^2}$  and then we can choose  $l = \frac{1}{R^2}$  and  $L = \frac{1}{r^2}$ . Applying the inequality (5.14) we get

$$\left| D_{KL}(p, q) - \frac{R^2 + r^2}{4R^2r^2} D_{\chi^2}(p, q) \right| \leq \frac{R^2 - r^2}{4R^2r^2} D_{\chi^2}(p, q). \quad (5.17)$$

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