

New Jensen and Ostrowski Type Inequalities for General Lebesgue Integral with Applications

S. S. Dragomir^{a,b}

^aMathematics, College of Engineering & Science, Victoria University,
 PO Box 14428 Melbourne City, MC 8001, Australia.

^bSchool of Computational & Applied Mathematics, University of the
 Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.

E-mail: sever.dragomir@vu.edu.au

ABSTRACT. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for f -divergence measure are provided as well.

Keywords: Ostrowski's inequality, Jensen's inequality, f -Divergence measures.

2000 Mathematics subject classification: 26D15, 94A17.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω , a σ -algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_{\Omega} d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R}, f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(t)| d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w d\mu$ instead of $\int_{\Omega} w(t) d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:

Received 01 March 2015; Accepted 02 May 2016
 ©2016 Academic Center for Education, Culture and Research TMU

Theorem 1.1. Let $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \rightarrow [m, M]$ so that $\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:

$$\begin{aligned} 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left(\int_{\Omega} f d\mu \right) \\ &\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu. \end{aligned} \quad (1.1)$$

In the case of discrete measure, we have:

Corollary 1.2. Let $\Phi : [m, M] \rightarrow \mathbb{R}$ be a differentiable convex function on (m, M) . If $x_i \in [m, M]$ and $w_i \geq 0$ ($i = 1, \dots, n$) with $W_n := \sum_{i=1}^n w_i = 1$, then one has the counterpart of Jensen's weighted discrete inequality:

$$\begin{aligned} 0 &\leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi \left(\sum_{i=1}^n w_i x_i \right) \\ &\leq \sum_{i=1}^n w_i \Phi'(x_i) x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|. \end{aligned} \quad (1.2)$$

Remark 1.3. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [36].

If $f, g : \Omega \rightarrow \mathbb{R}$ are μ -measurable functions and $f, g, fg \in L(\Omega, \mu)$, then we may consider the Čebyšev functional

$$T(f, g) := \int_{\Omega} f g d\mu - \int_{\Omega} f d\mu \int_{\Omega} g d\mu. \quad (1.3)$$

The following result is known in the literature as the Grüss inequality

$$|T(f, g)| \leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta), \quad (1.4)$$

provided

$$-\infty < \gamma \leq f(t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(t) \leq \Delta < \infty \quad (1.5)$$

for μ -a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f(t) \leq \Gamma < \infty$ for μ -a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} (\Gamma - \gamma). \quad (1.6)$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$\begin{aligned}
 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left(\int_{\Omega} f d\mu \right) \\
 &\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\
 &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \\
 &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \\
 &\leq \frac{1}{4} [\Phi'(M) - \Phi'(m)] (M - m),
 \end{aligned} \tag{1.7}$$

provided that $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \rightarrow [m, M]$ so that $\Phi \circ f, f, \Phi' \circ f, f \cdot (\Phi' \circ f) \in L(\Omega, \mu)$, with $\int_{\Omega} d\mu = 1$.

The following reverse of the Jensen's inequality also holds [33]:

Theorem 1.4. *Let $\Phi : I \rightarrow \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I . If $f : \Omega \rightarrow \mathbb{R}$ is μ -measurable, satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $f, \Phi \circ f \in L(\Omega, \mu)$, then

$$\begin{aligned}
 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left(\int_{\Omega} f d\mu \right) \\
 &\leq \left(M - \int_{\Omega} f d\mu \right) \left(\int_{\Omega} f d\mu - m \right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\
 &\leq \frac{1}{4} (M - m) [\Phi'_-(M) - \Phi'_+(m)],
 \end{aligned} \tag{1.8}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ .

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [55], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a} \int_a^b \Phi(t) dt$ and the value $\Phi(x)$, $x \in [a, b]$.

For various results related to Ostrowski's inequality see [6]-[9], [15]-[41], [43] and the references therein.

Theorem 1.5. *Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $\Phi' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|\Phi'\|_{\infty} := \sup_{t \in (a, b)} |\Phi'(t)| < \infty$*

∞ . Then

$$\left| \Phi(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|\Phi'\|_{\infty} (b-a), \quad (1.9)$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions [34]

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) \\ := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re} \left[(\Gamma - f(t)) \left(\overline{f(t)} - \bar{\gamma} \right) \right] \geq 0 \text{ for almost every } t \in [a, b] \right\} \end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a, b] \right\}.$$

The following representation result may be stated [34].

Proposition 1.6. For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma, \Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ are nonempty, convex and closed sets and

$$\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma). \quad (1.10)$$

On making use of the complex numbers field properties we can also state that:

Corollary 1.7. For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) = \{ f : [a, b] \rightarrow \mathbb{C} \mid (\operatorname{Re} \Gamma - \operatorname{Re} f(t)) (\operatorname{Re} f(t) - \operatorname{Re} \gamma) \\ + (\operatorname{Im} \Gamma - \operatorname{Im} f(t)) (\operatorname{Im} f(t) - \operatorname{Im} \gamma) \geq 0 \text{ for a.e. } t \in [a, b] \}. \end{aligned} \quad (1.11)$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a, b] \}. \end{aligned} \quad (1.12)$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \quad (1.13)$$

The following result holds [34]:

Theorem 1.8. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, the interior of I . For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $\Phi' \in \bar{U}_{[a,b]}(\gamma, \Gamma) (= \bar{\Delta}_{[a,b]}(\gamma, \Gamma))$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have the inequality

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\gamma + \Gamma}{2} \left(\int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} |g - x| d\mu \quad (1.14)$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\frac{a+b}{2} \right) - \frac{\gamma + \Gamma}{2} \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{4} (b-a) |\Gamma - \gamma| \end{aligned} \quad (1.15)$$

and

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\int_{\Omega} g d\mu \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \\ & \leq \frac{1}{2} |\Gamma - \gamma| \left(\int_{\Omega} g^2 d\mu - \left(\int_{\Omega} g d\mu \right)^2 \right)^{1/2} \\ & \leq \frac{1}{4} (b-a) |\Gamma - \gamma|. \end{aligned} \quad (1.16)$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_{\Omega} g d\mu - x \right) \right|, \quad x \in [a, b],$$

under various assumptions on the absolutely continuous function Φ , which in the particular case of $x = \int_{\Omega} g d\mu$ provides some results connected with Jensen's inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. SOME IDENTITIES

The following result holds [34]:

Lemma 2.1. *Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, the interior of I . If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have the equality*

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_{\Omega} g d\mu - x \right) \\ & = \int_{\Omega} \left[(g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] d\mu \end{aligned} \quad (2.1)$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$\int_{\Omega} \Phi \circ g d\mu - \Phi(x) = \int_{\Omega} \left[(g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu, \quad (2.2)$$

for any $x \in [a, b]$.

Remark 2.2. With the assumptions of Lemma 2.1 we have

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\frac{a+b}{2} \right) \\ &= \int_{\Omega} \left[\left(g - \frac{a+b}{2} \right) \int_0^1 \Phi' \left((1-s) \frac{a+b}{2} + sg \right) ds \right] d\mu. \end{aligned} \quad (2.3)$$

Corollary 2.3. With the assumptions of Lemma 2.1 we have

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\int_{\Omega} g d\mu \right) \\ &= \int_{\Omega} \left[\left(g - \int_{\Omega} g d\mu \right) \int_0^1 \Phi' \left((1-s) \int_{\Omega} g d\mu + sg \right) ds \right] d\mu. \end{aligned} \quad (2.4)$$

Proof. We observe that since $g : \Omega \rightarrow [a, b]$ and $\int_{\Omega} d\mu = 1$ then $\int_{\Omega} g d\mu \in [a, b]$ and by taking $x = \int_{\Omega} g d\mu$ in (2.2) we get (2.4). $\square$

Corollary 2.4. With the assumptions of Lemma 2.1 we have

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \frac{1}{b-a} \int_a^b \Phi(x) dx - \lambda \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \\ &= \int_{\Omega} \left\{ \frac{1}{b-a} \int_a^b \left[(g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] dx \right\} d\mu. \end{aligned} \quad (2.5)$$

Proof. Follows by integrating the identity (2.1) over $x \in [a, b]$, dividing by $b-a > 0$ and using Fubini's theorem. $\square$

Corollary 2.5. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I . If $g, h : \Omega \rightarrow [a, b]$ are Lebesgue μ -measurable on Ω and such that $\Phi \circ g, \Phi \circ h, g, h \in L(\Omega, \mu)$, then we have the equality

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu - \lambda \left(\int_{\Omega} g d\mu - \int_{\Omega} h d\mu \right) \\ &= \int_{\Omega} \int_{\Omega} \left[(g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) ds \right] \\ &\quad \times d\mu(t) d\mu(\tau) \end{aligned} \quad (2.6)$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu \\ &= \int_{\Omega} \int_{\Omega} \left[(g(t) - h(\tau)) \int_0^1 \Phi'((1-s)h(\tau) + sg(t)) ds \right] d\mu(t) d\mu(\tau), \end{aligned} \quad (2.7)$$

for any $x \in [a, b]$.

Remark 2.6. The above inequality (2.6) can be extended for two measures as follows

$$\begin{aligned} & \int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left(\int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right) \\ &= \int_{\Omega_1} \int_{\Omega_2} \left[(g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) ds \right] \\ & \quad \times d\mu_1(t) d\mu_2(\tau), \end{aligned} \quad (2.8)$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$ and provided that $\Phi \circ g, g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

Remark 2.7. If $w \geq 0$ μ -almost everywhere (μ -a.e.) on Ω with $\int_{\Omega} w d\mu > 0$, then by replacing $d\mu$ with $\frac{w d\mu}{\int_{\Omega} w d\mu}$ in (2.1) we have the weighted equality

$$\begin{aligned} & \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w (\Phi \circ g) d\mu - \Phi(x) - \lambda \left(\frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w g d\mu - x \right) \\ &= \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w \cdot \left[(g - x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] d\mu \end{aligned} \quad (2.9)$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$, provided $\Phi \circ g, g \in L_w(\Omega, \mu)$ where

$$L_w(\Omega, \mu) := \left\{ g \mid \int_{\Omega} w |g| d\mu < \infty \right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

3. INEQUALITIES FOR DERIVATIVES OF BOUNDED VARIATION

The following result holds:

Theorem 3.1. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a, b]$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_{\Omega} g d\mu - x \right) \right| \\ & \leq \frac{1}{2} \bigvee_a^b(\Phi') \int_{\Omega} |g - x| d\mu \end{aligned} \quad (3.1)$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{2} \bigvee_a^b(\Phi') \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{2} (b-a) \bigvee_a^b(\Phi') \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\int_{\Omega} g d\mu \right) \right| &\leq \frac{1}{2} \bigvee_a^b (\Phi') \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \\ &\leq \frac{1}{2} \bigvee_a^b (\Phi') \left(\int_{\Omega} g^2 d\mu - \left(\int_{\Omega} g d\mu \right)^2 \right)^{1/2} \\ &\leq \frac{1}{4} (b-a) \bigvee_a^b (\Phi'). \end{aligned} \quad (3.3)$$

Proof. From the identity (2.1) we have

$$\begin{aligned} &\int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_{\Omega} g d\mu - x \right) \\ &= \int_{\Omega} \left[(g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) ds \right] d\mu \end{aligned} \quad (3.4)$$

for any $x \in [a, b]$.

Taking the modulus in (3.4) we get

$$\begin{aligned} &\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\int_{\Omega} g d\mu - x \right) \right| \\ &\leq \int_{\Omega} \left| (g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right) \right| ds d\mu \\ &\leq \int_{\Omega} |g-x| \int_0^1 \left| \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds d\mu \end{aligned} \quad (3.5)$$

for any $x \in [a, b]$.

Since Φ' is of bounded variation on $[a, b]$, then for any $s \in [0, 1]$, $x \in [a, b]$ and $t \in \Omega$ we have

$$\begin{aligned} &\left| \Phi'((1-s)x + sg(t)) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| \\ &= \frac{1}{2} |\Phi'((1-s)x + sg(t)) - \Phi'(a) + \Phi'((1-s)x + sg(t)) - \Phi'(b)| \\ &\leq \frac{1}{2} [|\Phi'((1-s)x + sg(t)) - \Phi'(a)| + |\Phi'(b) - \Phi'((1-s)x + sg(t))|] \\ &\leq \frac{1}{2} \bigvee_a^b (\Phi'). \end{aligned}$$

Then we have

$$\begin{aligned} &\int_{\Omega} |g-x| \int_0^1 \left| \Phi'((1-s)x + sg) - \frac{\Phi'(a) + \Phi'(b)}{2} \right| ds d\mu \\ &\leq \frac{1}{2} \bigvee_a^b (\Phi') \int_{\Omega} |g-x| d\mu \end{aligned} \quad (3.6)$$

for any $x \in [a, b]$.

Making use of (3.5) and (3.6) we deduce the desired result (3.1). $\square$

Remark 3.2. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a, b]$. If $x_i \in [m, M]$ and $w_i \geq 0$ ($i = 1, \dots, n$) with $W_n := \sum_{i=1}^n w_i = 1$, then one has the weighted discrete inequality:

$$\left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\sum_{i=1}^n w_i x_i - x \right) \right| \quad (3.7)$$

$$\leq \frac{1}{2} \bigvee_a^b(\Phi') \sum_{i=1}^n w_i |x_i - x|$$

for any $x \in [a, b]$.

In particular, we have

$$\left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'(a) + \Phi'(b)}{2} \left(\sum_{i=1}^n w_i x_i - \frac{a+b}{2} \right) \right| \quad (3.8)$$

$$\leq \frac{1}{2} \bigvee_a^b(\Phi') \sum_{i=1}^n w_i \left| x_i - \frac{a+b}{2} \right| \leq \frac{1}{4} (b-a) \bigvee_a^b(\Phi')$$

and

$$\left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \right| \leq \frac{1}{2} \bigvee_a^b(\Phi') \sum_{i=1}^n w_i \left| x_i - \sum_{i=1}^n w_i x_i \right| \quad (3.9)$$

$$\leq \frac{1}{2} \bigvee_a^b(\Phi') \left(\sum_{j=1}^n w_j x_j^2 - \left(\sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2}$$

$$\leq \frac{1}{4} (b-a) \bigvee_a^b(\Phi').$$

4. INEQUALITIES FOR LIPSCHITZIAN DERIVATIVES

The following result holds:

Theorem 4.1. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have

$$\left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_{\Omega} g d\mu - x \right) \right| \quad (4.1)$$

$$\leq \frac{1}{2} K \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - x \right)^2 \right]$$

for any $x \in [a, b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu \right)^2 d\mu \right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu \right)^2 \right)^{1/2}.$$

In particular, we have

$$\begin{aligned} & \left| \int_\Omega \Phi \circ g d\mu - \Phi \left(\frac{a+b}{2} \right) - \Phi' \left(\frac{a+b}{2} \right) \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{2} K \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \end{aligned} \quad (4.2)$$

and

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi \left(\int_\Omega g d\mu \right) \right| \leq \frac{1}{2} K \sigma_\mu^2(g) \leq \frac{1}{8} K (b-a)^2. \quad (4.3)$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$\begin{aligned} & \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_\Omega g d\mu - x \right) \\ & = \int_\Omega \left[(g-x) \int_0^1 (\Phi'((1-s)x + sg) - \Phi'(x)) ds \right] d\mu \end{aligned} \quad (4.4)$$

for any $x \in [a, b]$.

Taking the modulus in (4.4) we get

$$\begin{aligned} & \left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_\Omega g d\mu - x \right) \right| \\ & \leq \int_\Omega |g-x| \left| \int_0^1 (\Phi'((1-s)x + sg) - \Phi'(x)) ds \right| d\mu \\ & \leq \int_\Omega \left[|g-x| \int_0^1 |\Phi'((1-s)x + sg) - \Phi'(x)| ds \right] d\mu \\ & \leq K \int_\Omega \left[|g-x| \int_0^1 s |g-x| ds \right] d\mu = \frac{1}{2} K \int_\Omega (g-x)^2 d\mu \end{aligned} \quad (4.5)$$

for any $x \in [a, b]$.

However,

$$\begin{aligned}
 & \int_{\Omega} (g - x)^2 d\mu \\
 &= \int_{\Omega} \left(g - \int_{\Omega} g d\mu + \int_{\Omega} g d\mu - x \right)^2 d\mu \\
 &= \int_{\Omega} \left(g - \int_{\Omega} g d\mu \right)^2 d\mu + 2 \int_{\Omega} \left(g - \int_{\Omega} g d\mu \right) \left(\int_{\Omega} g d\mu - x \right) d\mu \\
 &+ \int_{\Omega} \left(\int_{\Omega} g d\mu - x \right)^2 d\mu \\
 &= \int_{\Omega} \left(g - \int_{\Omega} g d\mu \right)^2 d\mu + \left(\int_{\Omega} g d\mu - x \right)^2
 \end{aligned}$$

for any $x \in [a, b]$, and by (4.5) we get the desired result (4.1). $\square$

Corollary 4.2. Let $\Phi : I \rightarrow \mathbb{C}$ be a twice differentiable functions on $[a, b] \subset \overset{\circ}{I}$ with $\|\Phi''\|_{[a, b], \infty} := \text{ess sup}_{t \in [a, b]} |\Phi''(t)| < \infty$. Then the inequalities (4.1)-(4.3) hold for $K = \|\Phi''\|_{[a, b], \infty}$.

Remark 4.3. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$ and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $x_i \in [m, M]$ and $w_i \geq 0$ ($i = 1, \dots, n$) with $W_n := \sum_{i=1}^n w_i = 1$, then one has the weighted discrete inequality:

$$\begin{aligned}
 & \left| \sum_{i=1}^n w_i \Phi(x_i) - \Phi(x) - \Phi'(x) \left(\sum_{i=1}^n w_i x_i - x \right) \right| \\
 & \leq \frac{1}{2} K \left[\sigma_w^2(\mathbf{x}) + \left(\sum_{i=1}^n w_i x_i - x \right)^2 \right]
 \end{aligned} \tag{4.6}$$

for any $x \in [a, b]$, where

$$\sigma_w(\mathbf{x}) := \left(\sum_{i=1}^n w_i \left(x_i - \sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2} = \left(\sum_{i=1}^n w_i x_i^2 - \left(\sum_{k=1}^n w_k x_k \right)^2 \right)^{1/2}.$$

The following lemma may be stated:

Lemma 4.4. Let $u : [a, b] \rightarrow \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:

- (i) The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a, b]$ is $\frac{1}{2}(L - l)$ -Lipschitzian;
- (ii) We have the inequalities

$$l \leq \frac{u(t) - u(s)}{t - s} \leq L \quad \text{for each } t, s \in [a, b] \quad \text{with } t \neq s; \tag{4.7}$$

- (iii) We have the inequalities

$$l(t - s) \leq u(t) - u(s) \leq L(t - s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s. \tag{4.8}$$

Following [53], we can introduce the definition of (l, L) -Lipschitzian functions:

Definition 4.5. The function $u : [a, b] \rightarrow \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 4.4 is said to be (l, L) -Lipschitzian on $[a, b]$.

If $L > 0$ and $l = -L$, then $(-L, L)$ -Lipschitzian means L -Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l, L) -Lipschitzian functions.

Proposition 4.6. Let $u : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $-\infty < l = \inf_{t \in [a, b]} u'(t)$ and $\sup_{t \in [a, b]} u'(t) = L < \infty$, then u is (l, L) -Lipschitzian on $[a, b]$.

The following result holds.

Corollary 4.7. Let $\Phi : I \rightarrow \mathbb{R}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, with the property that the derivative Φ' is (l, L) -Lipschitzian on $[a, b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_{\Omega} g d\mu - x \right) \right. \\ & \quad \left. - \frac{1}{4} (L + l) \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - x \right)^2 \right] \right| \\ & \leq \frac{1}{4} (L - l) \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - x \right)^2 \right] \end{aligned} \quad (4.9)$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right) \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right. \\ & \quad \left. - \frac{1}{4} (L + l) \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right)^2 \right] \right| \\ & \leq \frac{1}{4} (L - l) \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right)^2 \right] \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\int_{\Omega} g d\mu\right) - \frac{1}{4} (L + l) \sigma_{\mu}^2(g) \right| \leq \frac{1}{4} (L - l) \sigma_{\mu}^2(g) \\ & \leq \frac{1}{16} (L - l) (b - a)^2. \end{aligned} \quad (4.11)$$

Proof. Consider the auxiliary function $\Psi : [a, b] \rightarrow \mathbb{R}$ given by

$$\Psi(x) = \Phi(x) - \frac{1}{4}(L+l)x^2.$$

We observe that Ψ is differentiable and

$$\Psi'(x) = \Phi'(x) - \frac{1}{2}(L+l)x.$$

Since Φ' is (l, L) -Lipschitzian on $[a, b]$ it follows that Ψ' is Lipschitzian with the constant $\frac{1}{2}(L-l)$, so we can apply Theorem 4.1 for Ψ , i.e. we have the inequality

$$\begin{aligned} & \left| \int_{\Omega} \Psi \circ g d\mu - \Psi(x) - \Psi'(x) \left(\int_{\Omega} g d\mu - x \right) \right| \\ & \leq \frac{1}{4}(L-l) \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - x \right)^2 \right]. \end{aligned} \quad (4.12)$$

However

$$\begin{aligned} & \int_{\Omega} \Psi \circ g d\mu - \Psi(x) - \Psi'(x) \left(\int_{\Omega} g d\mu - x \right) \\ & = \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_{\Omega} g d\mu - x \right) \\ & \quad - \frac{1}{4}(L+l) \left[\int_{\Omega} g^2 d\mu - x^2 - 2x \left(\int_{\Omega} g d\mu - x \right) \right] \\ & = \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_{\Omega} g d\mu - x \right) \\ & \quad - \frac{1}{4}(L+l) \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - x \right)^2 \right] \end{aligned}$$

and by (4.12) we get the desired result (4.9). $\square$

Remark 4.8. We observe that if the function Φ is twice differentiable on $\overset{\circ}{I}$ and for $[a, b] \subset \overset{\circ}{I}$ we have

$$-\infty < l \leq \Phi''(x) \leq L < \infty \text{ for any } x \in [a, b],$$

then Φ' is (l, L) -Lipschitzian on $[a, b]$ and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

Theorem 4.9. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a, b]$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on

Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left(\int_{\Omega} g d\mu \right) \left(\int_{\Omega} g d\mu - x \right) \right| \\ & \leq \frac{1}{2} K \left[\left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} |g - x| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\ & \leq \frac{1}{2} K \left[\left| x - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} |g - x| d\mu \end{aligned} \quad (4.13)$$

for any $x \in [a, b]$, where

$$\left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} := \operatorname{ess\,sup}_{t \in \Omega} \left| g(t) - \int_{\Omega} g d\mu \right| < \infty.$$

In particular, we have

$$\begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\frac{a+b}{2} \right) - \Phi' \left(\int_{\Omega} g d\mu \right) \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{2} K \left[\left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \right. \\ & \quad \left. + \int_{\Omega} \left| g - \frac{a+b}{2} \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\ & \leq \frac{1}{2} K \left[\left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu. \end{aligned} \quad (4.14)$$

Proof. From the identity (2.1) we have for $\lambda = \Phi' \left(\int_{\Omega} g d\mu \right)$ that

$$\begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left(\int_{\Omega} g d\mu \right) \left(\int_{\Omega} g d\mu - x \right) \\ & = \int_{\Omega} \left[(g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \Phi' \left(\int_{\Omega} g d\mu \right) \right) ds \right] d\mu \end{aligned} \quad (4.15)$$

for any $x \in [a, b]$.

Taking the modulus in (4.15) we get

$$\begin{aligned}
 & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left(\int_{\Omega} g d\mu \right) \left(\int_{\Omega} g d\mu - x \right) \right| \\
 & \leq \int_{\Omega} |g - x| \left| \int_0^1 \left(\Phi'((1-s)x + sg) - \Phi' \left(\int_{\Omega} g d\mu \right) \right) ds \right| d\mu \\
 & \leq \int_{\Omega} \left[|g - x| \int_0^1 \left| \Phi'((1-s)x + sg) - \Phi' \left(\int_{\Omega} g d\mu \right) \right| ds \right] d\mu \\
 & \leq K \int_{\Omega} \left[|g - x| \int_0^1 \left| (1-s)x + sg - \int_{\Omega} g d\mu \right| ds \right] d\mu \\
 & = K \int_{\Omega} \left[|g - x| \int_0^1 \left| (1-s)x + sg - (1-s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right| ds \right] d\mu \\
 & := B.
 \end{aligned} \tag{4.16}$$

Using the triangle inequality we have for any $t \in \Omega$

$$\begin{aligned}
 & \int_0^1 \left| (1-s)x + sg(t) - (1-s) \int_{\Omega} g d\mu - s \int_{\Omega} g d\mu \right| ds \\
 & \leq \int_0^1 (1-s) \left| x - \int_{\Omega} g d\mu \right| ds + \int_0^1 s \left| g(t) - \int_{\Omega} g d\mu \right| ds \\
 & = \frac{1}{2} \left[\left| x - \int_{\Omega} g d\mu \right| + \left| g(t) - \int_{\Omega} g d\mu \right| \right]
 \end{aligned}$$

and then

$$\begin{aligned}
 B & \leq \frac{1}{2} K \int_{\Omega} |g - x| \left[\left| x - \int_{\Omega} g d\mu \right| + \left| g(t) - \int_{\Omega} g d\mu \right| \right] d\mu \\
 & = \frac{1}{2} K \left[\int_{\Omega} |g - x| \left| x - \int_{\Omega} g d\mu \right| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right].
 \end{aligned} \tag{4.17}$$

Making use of (4.16) and (4.17) we deduce the desired result (4.13). $\square$

Corollary 4.10. *Let $\Phi : I \rightarrow \mathbb{R}$ be an absolutely continuous functions on $[a, b] \subset I$, with the property that the derivative Φ' is (l, L) -Lipschitzian on $[a, b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have*

$$\begin{aligned}
 & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \Phi' \left(\int_{\Omega} g d\mu \right) \left(\int_{\Omega} g d\mu - x \right) \right. \\
 & \quad \left. - \frac{1}{4} (L + l) \left[\sigma_{\mu}^2(g) - \left(x - \int_{\Omega} g d\mu \right)^2 \right] \right| \\
 & \leq \frac{1}{4} (L - l) \left[\left| x - \int_{\Omega} g d\mu \right| \int_{\Omega} |g - x| d\mu + \int_{\Omega} |g - x| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\
 & \leq \frac{1}{4} (L - l) \left[\left| x - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} |g - x| d\mu
 \end{aligned} \tag{4.18}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned}
 & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\frac{a+b}{2} \right) - \Phi' \left(\int_{\Omega} g d\mu \right) \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right. \\
 & \quad \left. - \frac{1}{4} (L+l) \left[\sigma_{\mu}^2(g) - \left(\frac{a+b}{2} - \int_{\Omega} g d\mu \right)^2 \right] \right| \\
 & \leq \frac{1}{4} (L-l) \left[\left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \right. \\
 & \quad \left. + \int_{\Omega} \left| g - \frac{a+b}{2} \right| \left| g - \int_{\Omega} g d\mu \right| d\mu \right] \\
 & \leq \frac{1}{4} (L-l) \left[\left| \frac{a+b}{2} - \int_{\Omega} g d\mu \right| + \left\| g - \int_{\Omega} g d\mu \right\|_{\Omega, \infty} \right] \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu.
 \end{aligned} \tag{4.19}$$

5. APPLICATIONS FOR f -DIVERGENCE

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [47], Kullback and Leibler [52], Rényi [58], Havrda and Charvat [44], Kapur [50], Sharma and Mittal [62], Burbea and Rao [5], Rao [57], Lin [53], Csiszár [12], Ali and Silvey [1], Vajda [68], Shioya and Da-te [63] and others (see for example [54] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [57], genetics [54], finance, economics, and political science [60], [66], [67], biology [56], the analysis of contingency tables [42], approximation of probability distributions [11], [51], signal processing [48], [49] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár f -divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ -finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \{p|p : \Omega \rightarrow \mathbb{R}, p(t) \geq 0, \int_{\Omega} p(t) d\mu(t) = 1\}$. The Kullback-Leibler divergence [52] is well known among the information divergences. It is defined as:

$$D_{KL}(p, q) := \int_{\Omega} p(t) \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}, \tag{5.1}$$

where $\ln$ is to base e .

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v , *Hellinger distance* D_H [45], χ^2 -divergence D_{χ^2} , α -divergence D_{α} , *Bhattacharyya distance* D_B [3], *Harmonic distance* D_{H_a} , *Jeffrey's distance* D_J [47],

triangular discrimination D_{Δ} [65], etc... They are defined as follows:

$$D_v(p, q) := \int_{\Omega} |p(t) - q(t)| d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.2)$$

$$D_H(p, q) := \int_{\Omega} \left| \sqrt{p(t)} - \sqrt{q(t)} \right| d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.3)$$

$$D_{\chi^2}(p, q) := \int_{\Omega} p(t) \left[\left(\frac{q(t)}{p(t)} \right)^2 - 1 \right] d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.4)$$

$$D_{\alpha}(p, q) := \frac{4}{1 - \alpha^2} \left[1 - \int_{\Omega} [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t) \right], \quad p, q \in \mathcal{P}; \quad (5.5)$$

$$D_B(p, q) := \int_{\Omega} \sqrt{p(t)q(t)} d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.6)$$

$$D_{Ha}(p, q) := \int_{\Omega} \frac{2p(t)q(t)}{p(t) + q(t)} d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.7)$$

$$D_J(p, q) := \int_{\Omega} [p(t) - q(t)] \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}; \quad (5.8)$$

$$D_{\Delta}(p, q) := \int_{\Omega} \frac{[p(t) - q(t)]^2}{p(t) + q(t)} d\mu(t), \quad p, q \in \mathcal{P}. \quad (5.9)$$

For other divergence measures, see the paper [50] by Kapur or the book on line [64] by Taneja.

Csiszár f -divergence is defined as follows [13]

$$I_f(p, q) := \int_{\Omega} p(t) f \left[\frac{q(t)}{p(t)} \right] d\mu(t), \quad p, q \in \mathcal{P}, \quad (5.10)$$

where f is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár f -divergence. There are also many others which are not in this class (see for example [64]). For the basic properties of Csiszár f -divergence see [13], [14] and [68].

The following result holds:

Proposition 5.1. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable convex function with the property that $f(1) = 0$ and there exists the constants γ, Γ so that*

$$-\infty < \gamma \leq f(t) \leq \Gamma < \infty.$$

Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that

$$r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega. \quad (5.11)$$

If $x \in [r, R]$, then we have the inequality

$$\begin{aligned} & \left| I_f(p, q) - f(x) - f'(x)(1-x) - \frac{1}{4}(L+l) \left[D_{\chi^2}(p, q) + (1-x)^2 \right] \right| \\ & \leq \frac{1}{4}(L-l) \left[D_{\chi^2}(p, q) + (1-x)^2 \right]. \end{aligned} \quad (5.12)$$

In particular, we have

$$\begin{aligned} & \left| I_f(p, q) - f\left(\frac{r+R}{2}\right) - f'\left(\frac{r+R}{2}\right) \left(1 - \frac{r+R}{2}\right) \right. \\ & \quad \left. - \frac{1}{4}(L+l) \left[D_{\chi^2}(p, q) + \left(1 - \frac{r+R}{2}\right)^2 \right] \right| \\ & \leq \frac{1}{4}(L-l) \left[D_{\chi^2}(p, q) + \left(1 - \frac{r+R}{2}\right)^2 \right] \end{aligned} \quad (5.13)$$

and

$$\left| I_f(p, q) - \frac{1}{4}(L+l) D_{\chi^2}(p, q) \right| \leq \frac{1}{4}(L-l) D_{\chi^2}(p, q). \quad (5.14)$$

Proof. From (4.9) we have

$$\begin{aligned} & \left| \int_{\Omega} p(t) f\left(\frac{q(t)}{p(t)}\right) d\mu(t) - f(x) - f'(x)(1-x) \right. \\ & \quad \left. - \frac{1}{4}(L+l) \left[\int_{\Omega} p(t) \left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2 \right] \right| \\ & \leq \frac{1}{4}(L-l) \left[\int_{\Omega} p(t) \left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2 \right] \end{aligned}$$

for any $x \in [r, R]$, which is equivalent to (5.12). $\square$

Utilising Corollary 4.10 we can state the following result as well:

Proposition 5.2. *With the assumptions in Proposition 5.1, we have*

$$\begin{aligned} & \left| I_f(p, q) - f(x) - f'(1)(1-x) - \frac{1}{4}(L+l) \left[D_{\chi^2}(p, q) - (1-x)^2 \right] \right| \\ & \leq \frac{1}{4}(L-l) \left[|x-1| \int_{\Omega} |q-xp| d\mu + \int_{\Omega} |q-xp| \left| \frac{q}{p} - 1 \right| d\mu \right] \\ & \leq \frac{1}{4}(L-l) \left[|x-1| + \left\| \frac{q}{p} - 1 \right\|_{\Omega, \infty} \right] \int_{\Omega} |q-xp| d\mu \end{aligned} \quad (5.15)$$

for any $x \in [r, R]$.

If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t \ln t$ then

$$\begin{aligned} I_f(p, q) &:= \int_{\Omega} p(t) \frac{q(t)}{p(t)} \ln \left[\frac{q(t)}{p(t)} \right] d\mu(t) = \int_{\Omega} q(t) \ln \left[\frac{q(t)}{p(t)} \right] d\mu(t) \\ &= D_{KL}(q, p). \end{aligned}$$

We have $f'(t) = \ln t + 1$ and $f''(t) = \frac{1}{t}$ and then we can choose $l = \frac{1}{R}$ and $L = \frac{1}{r}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(q, p) - \left(\frac{R+r}{4rR} \right) D_{\chi^2}(p, q) \right| \leq \frac{R-r}{4rR} D_{\chi^2}(p, q). \quad (5.16)$$

If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = -\ln t$ then

$$\begin{aligned} I_f(p, q) &:= - \int_{\Omega} p(t) \ln \left[\frac{q(t)}{p(t)} \right] d\mu(t) = \int_{\Omega} p(t) \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t) \\ &= D_{KL}(p, q). \end{aligned}$$

We have $f'(t) = -\frac{1}{t}$ and $f''(t) = \frac{1}{t^2}$ and then we can choose $l = \frac{1}{R^2}$ and $L = \frac{1}{r^2}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(p, q) - \frac{R^2 + r^2}{4R^2r^2} D_{\chi^2}(p, q) \right| \leq \frac{R^2 - r^2}{4R^2r^2} D_{\chi^2}(p, q). \quad (5.17)$$

ACKNOWLEDGMENTS

The author would like to thank the referees for giving fruitful advices.

REFERENCES

1. S. M. Ali, S. D. Silvey, A general class of coefficients of divergence of one distribution from another, *J. Roy. Statist. Soc. Sec B*, **28**, (1966), 131-142.
2. G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.*, **135**(3), (2002), 175-189.
3. A. Bhattacharyya, On a measure of divergence between two statistical populations defined by their probability distributions, *Bull. Calcutta Math. Soc.*, **35**, (1943), 99-109.
4. M. Beth Bassat, f -entropies, probability of error and feature selection, *Inform. Control*, **39**, (1978), 227-242.
5. I. Burbea, C. R. Rao, On the convexity of some divergence measures based on entropy function, *IEEE Trans. Inf. Th.*, **28**(3), (1982), 489-495.
6. P. Cerone, S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, (2000), 135-200.
7. P. Cerone, S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, *Advances in Statistics Combinatorics and Related Areas*, (2002), 53-62.
8. P. Cerone, S. S. Dragomir, C. E. M. Pearce, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.*, **24**(2), (2000), 147-163.
9. P. Cerone, S. S. Dragomir, J. Roumeliotis, Some Ostrowski type inequalities for n -time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2), (1999), 697-712.
10. C. H. Chen, L. F. Pau, P. S. P. Wang *Statistical Pattern Recognition*, Rocelle Park, New York, Hoyderc Book Co., 1973.
11. C. K. Chow, C. N. Lin, Approximating discrete probability distributions with dependence trees, *IEEE Trans. Inf. Th.*, **14**(3), (1968), 462-467.
12. I. Csiszár, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica*, **2**, (1967), 299-318.

13. I. I. Csiszár, On topological properties of f -divergences, *Studia Math. Hungarica*, **2**, (1967), 329-339.
14. I. I. Csiszár, J. Körner, *Information Theory: Coding Theorem for Discrete Memoryless Systems*, Academic Press, New York, 1981.
15. S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM*, **3**(1), (1999), 127-135.
16. S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.*, **38**, (1999), 33-37.
17. S. S. Dragomir, A converse result for Jensen's discrete inequality via Grüss' inequality and applications in information theory. *An. Univ. Oradea Fasc. Mat.*, **7**, (1999), 178-189.
18. S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.*, **60**(1), (1999), 495-508.
19. S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications, *Kragujevac J. Math.*, **22**, (2000), 13-18.
20. S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.*, **7**, (2000), 477-485.
21. S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.*, **4**(1), (2001), 33-40.
22. S. S. Dragomir, On a reverse of Jensen's inequality for isotonic linear functionals, *J. Ineq. Pure & Appl. Math.*, **2**(3), (2001), Article 36.
23. S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM*, **5**(1), (2001), 35-45.
24. S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.*, **3**(5), (2002), Art. 68.
25. S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.*, **3**(2), (2002), Article 31, 8 pp.
26. S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications, *Libertas Math.*, **22**, (2002), 49-63.
27. S. S. Dragomir, Some companions of Ostrowski's inequality for absolutely continuous functions and applications, *Bull. Korean Math. Soc.*, **42**(2), (2005), 213-230.
28. S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2), (2003), 373-382.
29. S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications, *Demonstratio Math.*, **36**(3), (2003), 551-562.
30. S. S. Dragomir, Bounds for the deviation of a function from the chord generated by its extremities, *Bull. Aust. Math. Soc.*, **78**(2), (2008), 225-248.
31. S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.*, Springer Briefs in Mathematics, Springer, New York, 2012.
32. S. S. Dragomir, Perturbed companions of Ostrowski's inequality for absolutely continuous functions (I), Preprint *RGMA Res. Rep. Coll.*, **17** Art 7, (2014), 15 pp.
33. S. S. Dragomir, Reverses of the Jensen inequality in terms of first derivative and applications, *Acta Math. Vietnam*, **38**(3), (2013), 429-446.
34. S. S. Dragomir, Jensen and Ostrowski type inequalities for general Lebesgue integral with applications (I), *RGMA Res. Rep. Coll.*, **17**, (2014).
35. S. S. Dragomir, P. Cerone, J. Roumeliotis, S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Roumanie*, **42(90)** (4), (1999), 301-314.

36. S. S. Dragomir, N. M. Ionescu, Some converse of Jensen's inequality and applications, *Rev. Anal. Numér. Théor. Approx.*, **23**(1), (1994), 71–78.
37. S. S. Dragomir, Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.
38. S. S. Dragomir, S. Wang, A new inequality of Ostrowski's type in L_1 -norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28**, (1997), 239–244.
39. S. S. Dragomir, S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11**, (1998), 105–109.
40. S. S. Dragomir, S. Wang, A new inequality of Ostrowski's type in L_p -norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.*, **40**(3), (1998), 245–304.
41. A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.*, **42**(117), (1992), 298–310.
42. D. V. Gokhale, S. Kullback, *Information in Contingency Tables*, New York, Marcel Dekker, 1978.
43. A. Guessab and G. Schmeisser, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Th.*, **115**, (2002), 260–288.
44. J. H. Havrda, F. Charvat, Quantification method classification process: concept of structural α -entropy, *Kybernetika*, **3**, (1967), 30–35.
45. E. Hellinger, Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen, *J. für reine und Angew. Math.*, **36**, (1909), 210–271.
46. A. Sheikh Hossein, A generalized singular value inequality for Heinz means, *Iranian Journal of Mathematical Sciences and Informatics*, **10**(2), (2015), 23–27.
47. H. Jeffreys, An invariant form for the prior probability in estimating problems, *Proc. Roy. Soc. London*, **186A**, (1946), 453–461.
48. T. T. Kadota, L. A. Shepp, On the best finite set of linear observables for discriminating two Gaussian signals, *IEEE Trans. Inf. Th.*, **13**, (1967), 288–294.
49. T. Kailath, The divergence and Bhattacharyya distance measures in signal selection, *IEEE Trans. Comm. Technology*, **COM-15**, (1967), 52–60.
50. J. N. Kapur, A comparative assessment of various measures of directed divergence, *Advances in Management Studies*, **3**, (1984), 1–16.
51. D. Kazakos, T. Cotsidas, A decision theory approach to the approximation of discrete probability densities, *IEEE Trans. Perform. Anal. Machine Intell.*, **1**, (1980), 61–67.
52. S. Kullback, R. A. Leibler, On information and sufficiency, *Annals Math. Statist.*, **22**, (1951), 79–86.
53. J. Lin, Divergence measures based on the Shannon entropy, *IEEE Trans. Inf. Th.*, **37**(1) (1991), 145–151.
54. M. Mei, The theory of genetic distance and evaluation of human races, *Japan J. Human Genetics*, **23** (1978), 341–369.
55. A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.
56. E. C. Pielou, *Ecological Diversity*, Wiley, New York, 1975.
57. C. R. Rao, Diversity and dissimilarity coefficients: a unified approach, *Theoretic Population Biology*, **21**, (1982), 24–43.
58. A. Rényi, On measures of entropy and information, *Proc. Fourth Berkeley Symp. Math. Stat. and Prob.*, University of California Press, **1**, (1961), 547–561.
59. A. W. Roberts, D. E. Varberg, *Convex Functions*, Academic Press, 1973.
60. A. Sen, *On Economic Inequality*, Oxford University Press, London, 1973.

61. M. Z. Sarikaya, A. Saglamb and H. Yildirim, On generalization of Čebyšev type inequalities, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(1), (2010), 41–48.
62. B. D. Sharma, D. P. Mittal, New non-additive measures of relative information, *Journ. Comb. Inf. Sys. Sci.*, **2**(4), (1977), 122–132.
63. H. Shioya, T. Da-Te, A generalisation of Lin divergence and the derivative of a new information divergence, *Elec. and Comm. in Japan*, **78** (7), (1995), 37–40.
64. I. J. Taneja, *Generalised Information Measures and their Applications*, (<http://www.mtm.ufsc.br/~taneja/bhtml/bhtml.html>).
65. F. Topsoe, Some inequalities for information divergence and related measures of discrimination, *Res. Rep. Coll., RGMIA*, **2** (1), (1999), 85–98.
66. H. Theil, *Economics and Information Theory*, North-Holland, Amsterdam, 1967.
67. H. Theil, *Statistical Decomposition Analysis*, North-Holland, Amsterdam, 1972.
68. I. Vajda, *Theory of Statistical Inference and Information*, Dordrecht-Boston, Kluwer Academic Publishers, 1989.