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# Ordered Krasner Hyperrings

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Abstract. In this paper, we introduce the concept of Krasner hyperring $(R, +, \cdot)$ together with a suitable partial order relation $\leq$. Also, we consider some Krasner hyperrings and define a binary relation on them which become ordered Krasner hyperrings. By using the notion of pseudoorder on an ordered Krasner hyperring $(R, +, \cdot, \leq)$, we obtain an ordered ring. Moreover, we give some results on ordered Krasner hyperrings.

**Keywords:** Algebraic hyperstructure, Ordered ring, Ordered Krasner hyperring, Strongly regular relation, Pseudoorder.

**2000 Mathematics subject classification:** 16Y99.

## 1. Introduction

Hyperstructure theory was introduced by Marty [19] in 1934, at the 8th congress of Scandinavian Mathematicians. The notion of a hyperstructure and hypergroup has been studied in the following decades and nowadays by many mathematicians. A short review of the theory of hyperstructures appears in [4, 5, 8, 9, 10, 23]. In [11], Heidari and Davvaz studied a semihypergroup $(H, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an *ordered semihypergroup* $(H, \circ, \leq)$ is a semihypergroup $(H, \circ)$ together with a partial order $\leq$ such that satisfies the monotone condition as follows:

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$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z, \text{ for all } x, y, z \in H.$$

Here, $z \circ x \leq z \circ y$ means that for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly. Indeed, the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. The concept of ordering hypergroups introduced by Chvalina [6] as a special class of hypergroups and studied by many authors, for example, see [7, 12, 13]. In [3], polygroups which are partially ordered are introduced and some properties and related results are given. In [14], Iampan studied some properties of ordered bi-ideals in ordered $\Gamma$-semigroups.

We can consider several definitions for a hyperring, by replacing at least one of the two operations by a hyperoperation. In general case, $(R, +, \cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup, $(R, \cdot)$ is a semihypergroup and the hyperoperation $\cdot$ is distributive over the hyperoperation $+$, which means that for all $x, y, z$ of $R$ we have: $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. We call $(R, +, \cdot)$ a hyperfield if $(R, +, \cdot)$ is a hyperring and $(R, \cdot)$ is a hypergroup. There are different types of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a usual operation, then we say that $R$ is an additive hyperring. A special case of this type is the Krasner hyperring. For more details about hyperrings we refer to [10]. In the theory of hyperrings, fundamental relations make a connection between hyperrings and ordinary rings [22]. An equivalence relation $\rho$ is called strongly regular over a hyperring $R$, if the quotient $R/\rho$ is a ring. In 2013, Asokkumar [1] defined derivations in Krasner hyperrings and obtained some results in this respect.

## 2. Preliminaries

In this section, we recall some definitions and notations.

Let $H$ be a non-empty set. Then a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* on $H$, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of $H$. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if $A$ and $B$ are two non-empty subsets of $H$ and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

An element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$, for all $x \in H$ and it is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$, for all $x \in H$. If $e$ is a scalar identity of $(H, \circ)$, then $e$ is the unique identity of $(H, \circ)$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$, for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction



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axiom. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup.

**Definition 2.1.** [20] A *canonical hypergroup* is a non-empty set $H$ endowed with an additive hyperoperation $+ : H \times H \to \mathcal{P}^*(H)$, satisfying the follwing properties:

(1) for any $x, y, z \in H, x + (y + z) = (x + y) + z$,
(2) for any $x, y \in H, x + y = y + x$,
(3) there exists $0 \in H$ such that $0 + x = x + 0 = x$, for any $x \in H$,
(4) for every $x \in H$, there exists a unique element $x' \in H$, such that $0 \in x + x'$ (we shall write $-x$ for $x'$ and we call it the opposite of $x$),
(5) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, that is $(H, +)$ is reversible.

*Remark* 2.2. Every canonical hypergroup is a hypergroup.

**Definition 2.3.** [10, 18] A *Krasner hyperring* is an algebraic hypersructure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup,
(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$,
(3) The multiplication is distributive with respect to the hyperoperation $+$.

A Krasner hyperring $R$ is called *commutative* (with unit element) if $(R, \cdot)$ is a commutative semigroup (with unit element). A Krasner hyperring $R$ is called a *Krasner hyperfield*, if $(R \setminus \{0\}, \cdot)$ is a group. A Krasner hyperring $R$ is called a *hyperdomain*, if $R$ is a commutative hyperring with unit element and $a \cdot b = 0$ implies that $a = 0$ or $b = 0$ for all $a, b \in R$. A *subhyperring* of a Krasner hyperring $(R, +, \cdot)$ is a non-empty subset $A$ of $R$ which forms a Krasner hyperring containing 0 under the hyperoperation $+$ and the operation $\cdot$ on $R$, that is, $A$ is a canonical subhypergroup of $(R, +)$ and $A \cdot A \subseteq A$. Then a non-empty subset $A$ of $R$ is a subhyperring of $(R, +, \cdot)$ if and only if, for all $x, y \in A$, $x + y \subseteq A$, $-x \in A$ and $x \cdot y \in A$. A non-empty subset $I$ of $(R, +, \cdot)$ is called a *left* (resp. *right*) *hyperideal* of $(R, +, \cdot)$ if $(I, +)$ is a canonical subhypergroup of $(R, +)$ and for every $a \in I$ and $r \in R$, $r \cdot a \in I$ (resp. $a \cdot r \in I$). A *hyperideal* $I$ of $(R, +, \cdot)$ is one which is a left as well as a right hyperideal of $R$, that is, $x + y \subseteq I$ and $-x \in I$, for all $x, y \in I$ and $x \cdot y, y \cdot x \in I$, for all $x \in I$ and $y \in R$.

**Definition 2.4.** [2, 15] An *ordered ring* is a ring $(R, +, \cdot)$, together with a compatible partial order, i.e., a partial order $\leq$ on the underlying set $R$ that is compatible with the ring operations in the sense that it satisfies:

(1) For all $a, b, c \in R$, $a \leq b$ implies $a + c \leq b + c$.



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(2) For all $a, b \in R$, $0 \leq a$ and $0 \leq b$, we have $0 \leq a \cdot b$.

*Remark* 2.5. It is an easy consequence of (1) and (2) above that if $a, b, c \in R$ with $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

## 3. Ordered Krasner Hyperrings

In this section, we introduce the notion of ordered Krasner hyperring, giving several examples that illustrate the significance of this new hyperstructure and some related results are given.

**Definition 3.1.** An algebraic hypersructure $(R, +, \cdot, \leq)$ is called an *ordered Krasner hyperring* if $(R, +, \cdot)$ is a Krasner hyperring with a partial order relation $\leq$, such that for all $a, b$ and $c$ in $R$:

(1) If $a \leq b$, then $a + c \leq b + c$, meaning that for any $x \in a + c$, there exists $y \in b + c$ such that $x \leq y$.
(2) If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

Note that the concept of ordered Krasner hyperrings is a generalization of the concept of ordered rings. Indeed, every ordered ring is an ordered Krasner hyperring.

*Remark* 3.2. An ordered Krasner hyperring $R$ is *positive* if $0 \leq x$ for any $x \in R$.

If we remove the restriction $0 \leq c$ from (2), then Definition 3.1 is equivalent to positive ordered Krasner hyperring.

**Definition 3.3.** An *ordered Krasner hyperfield* is an ordered Krasner hyperring $R$, where $R$ is also a Krasner hyperfield.

**Definition 3.4.** An *ordered hyperdomain* is an ordered Krasner hyperring $R$, where $R$ is a commutative hyperring with unit element and $a \cdot b = 0$ implies that $a = 0$ or $b = 0$ for all $a, b \in R$.

Example 3.5. Let $(R, +, \cdot)$ be a Krasner hyperring. Define a partial order on $R$ by

$$x \leq_R y \text{ if and only if } x = y \text{ for all } x, y \in R.$$

Then, $(R, +, \cdot, \leq_R)$ forms an ordered Krasner hyperring.

Example 3.6. Let $R = \{a, b, c\}$ be a set with the hyperoperation $\oplus$ and the binary operation $\odot$ defined as follow:

| $\oplus$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $b$ | $\{a, b, c\}$ |
| $c$ | $c$ | $\{a, b, c\}$ | $c$ |

| $\odot$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $c$ |
| $c$ | $a$ | $b$ | $c$ |



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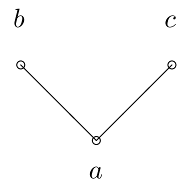
Then, $(R, \oplus, \odot)$ is a Krasner hyperring [10]. We have $(R, \oplus, \odot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:

$$\leq := \{(a, a), (b, b), (c, c), (a, b), (a, c)\}.$$

The covering relation and the figure of $R$ are given by:

$$\prec = \{(a, b), (a, c)\}.$$




Example 3.7. Define the hyperoperation $\oplus$ on the unit interval [0,1] by

$$x \oplus y = \begin{cases} \{max\{x, y\}\}, & x \neq y \\ [0, x] & x = y \end{cases}$$

Then, $([0, 1], \oplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication [17]. Consider $[0, 1]$ as a poset with the natural ordering. Thus, $([0, 1], \oplus, \cdot, \leq)$ is an ordered Krasner hyperring.

Example 3.8. Define the hyperoperation $\boxplus$ on the set $\{0, 1\}$ by

$$0 \boxplus 0 = \{0\},\ 1 \boxplus 0 = 0 \boxplus 1 = \{1\},\ 1 \boxplus 1 = \{0, 1\}.$$

Then, $(\{0, 1\}, \boxplus, \cdot)$ is a Krasner hyperring where $\cdot$ is the usual multiplication [21]. Consider $(\{0, 1\}, \boxplus, \cdot)$ as a poset with the natural ordering. Thus, $(\{0, 1\}, \boxplus, \cdot, \leq)$ is an ordered Krasner hyperring.

Example 3.9. Let $(\mathbb{Q}^+, +, \cdot)$ be the canonical hypergroup defined on $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x \geq 0\}$, $\forall x$, $x + x = \{y \mid y \leq x\}$, if $x \neq y$, $x + y = max\{x, y\}$ where max is intended with respect to the natural order. If one considers in $\mathbb{Q}^+$ as product the ordinary multiplication $\cdot$, then $(\mathbb{Q}^+, +, \cdot)$ has the structure of Krasner hyperring. We define $\leq$ the natural ordering on $\mathbb{Q}^+$, then $(\mathbb{Q}^+, +, \cdot, \leq)$ is an ordered Krasner hyperring.

**Definition 3.10.** Let $(R_1, +_1, \cdot_1, \leq_1)$ and $(R_2, +_2, \cdot_2, \leq_2)$ be two ordered Krasner hyperrings. A *homomorphism* from $R_1$ into $R_2$ is a function $\varphi : R_1 \to R_2$ such that we have:

(1) $\varphi(a +_1 b) \subseteq \varphi(a) +_2 \varphi(b)$,
(2) $\varphi(a \cdot_1 b) = \varphi(a) \cdot_2 \varphi(b)$,
(3) If $a \leq_1 b$, then $\varphi(a) \leq_2 \varphi(b)$.

Also $\varphi$ is called a good (strong) homomorphism if in the previous condition (1), the equality is valid. An *isomorphism* from $(R_1, +_1, \cdot_1, \leq_1)$ into $(R_2, +_2, \cdot_2, \leq_2)$ is a bijective good homomophism from $(R_1, +_1, \cdot_1, \leq_1)$ onto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$(R_2, +_2, \cdot_2, \leq_2)$. The kernel of $\varphi$, ker$\varphi$, is defined by ker$\varphi = \{x \in R_1 \mid \varphi(x) = 0_2\}$ where $0_2$ is the zero of $(R_2, +_2, \cdot_2)$. In fact ker$\varphi$ may be empty.

EXAMPLE 3.11. Let $([0,1], \oplus, \cdot, \leq)$ be the ordered Krasner hyperring defined as in Example 3.7 and $(\{0,1\}, \boxplus, \cdot, \leq)$ be the ordered Krasner hyperring defined as in Example 3.8. Define $\varphi : [0,1] \longrightarrow \{0,1\}$ by $\varphi(0) = 0$ and $\varphi(x) = 1$, for all $x \in (0,1]$. Then we have

$$\varphi(x \oplus y) = \begin{cases} \varphi(\{0\}) = \{0\} = 0 \boxplus 0 = \varphi(x) \boxplus \varphi(y), & x = y = 0 \\ \varphi(\{x\}) = \{1\} = 1 \boxplus 0 = \varphi(x) \boxplus \varphi(y), & x > y = 0 \\ \varphi(\{x\}) = \{1\} \subsetneqq \{0,1\} = 1 \boxplus 1 = \varphi(x) \boxplus \varphi(y), & x > y > 0 \\ \varphi([0,x]) = \{0,1\} = 1 \boxplus 1 = \varphi(x) \boxplus \varphi(y), & x = y > 0 \end{cases}$$

and

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = \begin{cases} 0, & x = 0 \text{ or } y = 0 \\ 1, & x \neq 0 \text{ and } y \neq 0 \end{cases}$$

and

$$\text{if } x \leq_{[0,1]} y, \text{ then } \varphi(x) \leq_{\{0,1\}} \varphi(y).$$

It follows that $\varphi$ is a homomorphism from $([0,1], \oplus, \cdot)$ into $(\{0,1\}, \boxplus, \cdot)$ which is not a good (strong) homomorphism.

EXAMPLE 3.12. Let $([0,1], \oplus, \cdot, \leq)$ be the ordered Krasner hyperring defined as in Example 3.7. Define $\varphi : [0,1] \longrightarrow [0,1]$ by $\varphi(x) = 1$, for all $x \in [0,1]$. Since $1 \oplus 1 = [0,1]$ and $1 \cdot 1 = 1$ and $x \leq_{[0,1]} y$ implies $\varphi(x) \leq_{[0,1]} \varphi(y)$, it follows that $\varphi$ is a homomorphism from the ordered Krasner hyperring $([0,1], \oplus, \cdot, \leq)$ into itself. Notice that $ker\varphi = \emptyset$. Thus, the kernel of an ordered Krasner hyperring homomorphism may be empty.

Let $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ be two Krasner hyperrings. Then $(R_1 \times R_2, +, \cdot)$ is a Krasner hyperring, where the hyperoperation $+$ and operation $\cdot$ defined as follows:

$$(x_1, x_2) + (y_1, y_2) = \{(a, b) \mid a \in x_1 +_1 y_1, b \in x_2 +_2 y_2\},$$
$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot_1 y_1, x_2 \cdot_2 y_2).$$

The lexicographical order defined on $R_1 \times R_2$ as follows: $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 <_1 y_1$ or $x_1 = y_1$ and $x_2 <_2 y_2$. In the following, we prove that $(R_1 \times R_2, +, \cdot, \leq)$ is an ordered Krasner hyperring and is called the direct product of ordered Krasner hyperrings $(R_1, +_1, \cdot_1, \leq_1)$ and $(R_2, +_2, \cdot_2, \leq_2)$.

**Theorem 3.13.** *Let $(R_1, +_1, \cdot_1, \leq_1)$ and $(R_2, +_2, \cdot_2, \leq_2)$ be two ordered Krasner hyperrings. Then $(R_1 \times R_2, +, \cdot, \leq)$ is an ordered Krasner hyperring.*

*Proof.* Suppose that $(x_1, x_2) \leq (y_1, y_2)$ for $(x_1, x_2), (y_1, y_2) \in R_1 \times R_2$ and $(t_1, t_2) \in (a_1, a_2) + (x_1, x_2)$ for $(a_1, a_2) \in R_1 \times R_2$. Then $t_1 \in a_1 +_1 x_1$ and $t_2 \in a_2 +_2 x_2$. Since $(x_1, x_2) \leq (y_1, y_2)$, So we have two cases:



[ DOI: 10.7508/ijmsi.2017.2.003 ]

**Case 1.** $x_1 <_1 y_1$. Then $t_1 \in a_1 +_1 x_1 \leq_1 a_1 +_1 y_1$, so there exists $s_1 \in a_1 +_1 y_1$ such that $t_1 \leq_1 s_1$. Now, if $s_2 \in a_2 +_2 y_2$, then $(t_1, t_2) \leq (s_1, s_2) \in (a_1, a_2) + (y_1, y_2)$.
**Case 2.** $x_1 = y_1$ and $x_2 <_2 y_2$. Then $t_2 \in a_2 +_2 x_2 \leq_2 a_2 +_2 y_2$, so there exists $s_2 \in a_2 +_2 y_2$ such that $t_2 \leq_2 s_2$. Thus $(t_1, t_2) \leq (t_1, s_2) \in (a_1, a_2) + (y_1, y_2)$.

Now, suppose that $(x_1, x_2) \leq (y_1, y_2)$ for $(x_1, x_2), (y_1, y_2) \in R_1 \times R_2$ and $(t_1, t_2) = (a_1, a_2) \cdot (x_1, x_2)$ for $(a_1, a_2) \in R_1 \times R_2$. Then $t_1 = a_1 \cdot_1 x_1$ and $t_2 = a_2 \cdot_2 x_2$. Since $(x_1, x_2) \leq (y_1, y_2)$, so we have two cases:
**Case 1.** $x_1 <_1 y_1$. Then $t_1 = a_1 \cdot_1 x_1 \leq_1 a_1 \cdot_1 y_1 = s_1$. Now, if $s_2 = a_2 \cdot_2 y_2$, then $(t_1, t_2) \leq (s_1, s_2) = (a_1, a_2) \cdot (y_1, y_2)$.
**Case 2.** $x_1 = y_1$ and $x_2 <_2 y_2$. Then $a_2 \cdot_2 x_2 \leq_2 a_2 \cdot_2 y_2$. Thus $(a_1, a_2) \cdot (x_1, x_2) \leq (a_1, a_2) \cdot (y_1, y_2)$ for $(a_1, a_2) \in R_1 \times R_2$. Therefore, $(R_1 \times R_2, +, \cdot, \leq)$ is an ordered Krasner hyperring. □

**Definition 3.14.** Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. A non-empty subset $I$ of $R$ is called a *hyperideal* of $R$ if it satisfies the following conditions:

(1) $(I, +)$ is a canonical subhypergroup of $(R, +)$;
(2) $x \cdot y \in I$ and $y \cdot x \in I$ for all $x \in I$ and $y \in R$;
(3) When $x \in I$ and $y \in R$ such that $y \leq x$, imply that $y \in I$.

**Theorem 3.15.** *Suppose that $\varphi$ is a homomorphism from an ordered Krasner hyperring $(R, +, \cdot, \leq)$ into a positive ordered Krasner hyperring $(T, \oplus, \odot, \preceq)$. Then, $ker\varphi$ is a hyperideal of $R$.*

*Proof.* Let $r \in ker\varphi$. Then we have $\varphi(r) = 0$. Since $\varphi$ is a homomorphism, it follows that $\{\varphi(0)\} = \varphi(0) \oplus 0 = \varphi(0) \oplus \varphi(r) \supseteq \varphi(0 + r) = \varphi(\{r\}) = \{\varphi(r)\}$. This implies that $\varphi(0) = \varphi(r) = 0$. Thus we have $0 \in ker\varphi$. Since $0 \in ker\varphi$, it follows that $ker\varphi \neq \emptyset$.

Let $x, y \in ker\varphi$. Then $\varphi(x) = 0 = \varphi(y)$. By Definition 3.10, we have $\varphi(x+y) \subseteq \varphi(x) \oplus \varphi(y) = 0 \oplus 0 = \{0\}$. Hence $x + y \subseteq ker\varphi$. Now, let $x \in ker\varphi$. Since $0 \in x - x$, it follows that $\varphi(0) \in \varphi(x - x) \subseteq \varphi(x) \oplus \varphi(-x)$. So, $0 \in \varphi(x) \oplus \varphi(-x)$. Thus, $\varphi(-x)$ is the inverse of $\varphi(x)$ in the canonical hypergroup $(T, \oplus)$. Therefore, $\varphi(-x) = -\varphi(x)$. So, $\varphi(-x) = -0 = 0$. Hence $-x \in ker\varphi$. Now, let $z \in R$ and $x \in ker\varphi$. Then $\varphi(z \cdot x) = \varphi(z) \odot \varphi(x) = \varphi(z) \odot 0 = 0$ and $\varphi(x \cdot z) = \varphi(x) \odot \varphi(z) = 0 \odot \varphi(z) = 0$. Thus we have $z \cdot x, x \cdot z \in ker\varphi$. Now, let $x \in ker\varphi$, $r \in R$ and $r \leq x$. Since $x \in ker\varphi$, it follows that $\varphi(x) = 0$. By Definition 3.10, we have $\varphi(r) \preceq \varphi(x) = 0$. Since $T$ is positive, we get $\varphi(r) = 0$. Therefore, $r \in ker\varphi$. This proves that $ker\varphi$ is a hyperideal of $R$. □

**Definition 3.16.** Suppose that $(R, +, \cdot, \leq)$ is an ordered Krasner hyperring and $A \subseteq R$ is a subhyperring. Then $A$ is *convex* if for all $a \in A$ and all $r \in R$, the inequality $-a \leq r \leq a$ implies $r \in A$.


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Example 3.17. Let $([0,1],\oplus,\cdot,\leq)$ be the ordered Krasner hyperring defined as in Example 3.7. We can see that 0 is the zero of $([0,1],\oplus,\cdot)$ and for every $x\in[0,1]$, $x$ is the inverse of $x$ in $([0,1],\oplus)$. The set $A=\{[0,a]|a\in[0,1]\}\cup\{[0,a)|a\in(0,1]\}$ is of all canonical subhypergroups of the canonical hypergroup $([0,1],\oplus)$. We can see that $A$ is the set of all subhyperrings of $R$. Now, let $a\in A$, $r\in R$, $-a\leq r\leq a$. Then, we have $a\leq r\leq a$, that implies $r=a\in A$. Therefore, $A$ is convex.

*Remark* 3.18. A relation $\rho^*$ is the *transitive closure* of a binary relation $\rho$ if

(1) $\rho^*$ is transitive,
(2) $\rho\subseteq\rho^*$,
(3) For any relation $\rho'$, if $\rho\subseteq\rho'$ and $\rho'$ is transitive, then $\rho^*\subseteq\rho'$, that is, $\rho^*$ is the smallest relation that satisfies (1) and (2).

**Definition 3.19.** [22] Let $(R,+,\cdot)$ be a hyperring. We define the relation $\gamma$ as follows: $x\gamma y\Leftrightarrow\exists n\in\mathbb{N},\exists k_i\in\mathbb{N},\exists(x_{i1},\cdots,x_{ik_i})\in R^{k_i},1\leq i\leq n$, such that

$$\{x,y\}\subseteq\sum_{i=1}^{n}\Big(\prod_{j=1}^{k_i}x_{ij}\Big).$$

**Theorem 3.20.** [22] *Let $R$ be a hyperring and $\gamma^*$ be the transitive closure of $\gamma$. Then, we have:*

(1) *$\gamma^*$ is a strongly regular relation both on $(R,+)$ and $(R,\cdot)$.*
(2) *The quotient $R/\gamma^*$ is a ring.*
(3) *The relation $\gamma^*$ is the smallest equivalence relation such that the quotient $R/\gamma^*$ is a ring.*

**Question.** Let $(R,+,\cdot,\leq)$ be an ordered Krasner hyperring. Is there a strongly regular relation $\rho$ on $R$ for which $R/\rho$ is an ordered ring?

Our main aim in the next section is reply to the above question.

## 4. Main Results

The notion of pseudoorder on an ordered semigroup $(S,\cdot,\leq)$ was introduced and studied by Kehayopulu and Tsingelis [16]. Now, we extend this concept for ordered Krasner hyperrings. We begin this section with the following definition.

**Definition 4.1.** Let $(R,+,\cdot,\leq)$ be an ordered Krasner hyperring. A relation $\rho$ on $R$ is called *pseudoorder* if the following conditions hold:

(1) $\leq\subseteq\rho$;
(2) $a\rho b$ and $b\rho c$ imply $a\rho c$;
(3) $a\rho b$ implies $a+c\overline{\overline{\rho}}b+c$ and $c+a\overline{\overline{\rho}}c+b$, for all $c\in R$;
(4) $a\rho b$ implies $a\cdot c\rho b\cdot c$ and $c\cdot a\rho c\cdot b$, for all $c\in R$.



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**Theorem 4.2.** *Let $(R,+,\cdot,\leq)$ be an ordered Krasner hyperring and $\rho$ be a pseudoorder on $R$. Then there exists a strongly regular relation $\rho^*$ on $R$ such that $R/\rho^*$ is an ordered ring.*

*Proof.* Suppose that $\rho^*$ is the relation on $R$ defined as follows:

$$\rho^* = \{(a,b) \in R \times R \mid a\rho b \text{ and } b\rho a\}.$$

First we show that $\rho^*$ is a strongly regular relation on $(R,+)$ and $(R,\cdot)$. Clearly, $(a,a) \in \leq \subseteq \rho$, so $a\rho^* a$. If $(a,b) \in \rho^*$, then $a\rho b$ and $b\rho a$. Hence, $(b,a) \in \rho^*$. If $(a,b) \in \rho^*$ and $(b,c) \in \rho^*$, then $a\rho b$, $b\rho a$, $b\rho c$ and $c\rho b$. Hence, $a\rho c$ and $c\rho a$, which imply that $(a,c) \in \rho^*$. Thus $\rho^*$ is an equivalence relation. Now, let $a\rho^* b$ and $c \in R$. Then $a\rho b$ and $b\rho a$. Since $\rho$ is pseudoorder on $R$, by conditions (3) and (4) of Definition 4.1, we conclude that

$$a + c\overline{\overline{\rho}}b + c,$$

$$b + c\overline{\overline{\rho}}a + c,$$

$$a \cdot c\rho b \cdot c,\ c \cdot a\rho c \cdot b,$$

$$b \cdot c\rho a \cdot c,\ c \cdot b\rho c \cdot a.$$

Hence, for every $x \in a+c$ and $y \in b+c$, we have $x\rho y$ and $y\rho x$ which imply that $x\rho^* y$. So, $a+c\overline{\overline{\rho^*}}b+c$. Thus $\rho^*$ is a strongly regular relation on $(R,+)$. Clearly, $\rho^*$ is a strongly regular relation on $(R,\cdot)$. Hence $R/\rho^*$ with the following operations is a ring:

$$\rho^*(x) \oplus \rho^*(y) = \rho^*(z), \text{ for all } z \in \rho^*(x) + \rho^*(y);$$

$$\rho^*(x) \odot \rho^*(y) = \rho^*(x \cdot y).$$

Now, we define a relation $\preceq$ on $R/\rho^*$ as follows:

$$\preceq := \{(\rho^*(x), \rho^*(y)) \in R/\rho^* \times R/\rho^* \mid \exists a \in \rho^*(x), \exists b \in \rho^*(y) \text{ such that } (a,b) \in \rho\}.$$

We show that

$$\rho^*(x) \preceq \rho^*(y) \Leftrightarrow x\rho y.$$

Let $\rho^*(x) \preceq \rho^*(y)$. We show that for every $a \in \rho^*(x)$ and $b \in \rho^*(y)$, $a\rho b$. Since $\rho^*(x) \preceq \rho^*(y)$, there exist $x' \in \rho^*(x)$ and $y' \in \rho^*(y)$ such that $x'\rho y'$. Since $a \in \rho^*(x)$ and $x' \in \rho^*(x)$, we obtain $a\rho^* x'$, and so $a\rho x'$ and $x'\rho a$. Since $b \in \rho^*(y)$ and $y' \in \rho^*(y)$, we obtain $b\rho^* y'$, and so $b\rho y'$ and $y'\rho b$. Now, we have $a\rho x'$, $x'\rho y'$ and $y'\rho b$, which imply that $a\rho b$. Since $x \in \rho^*(x)$ and $y \in \rho^*(y)$, we conclude that $x\rho y$. Conversely, let $x\rho y$. Since $x \in \rho^*(x)$ and $y \in \rho^*(y)$, we have $\rho^*(x) \preceq \rho^*(y)$.

Finally, we prove that $(R/\rho^*, \oplus, \odot, \preceq)$ is an ordered ring. Suppose that $\rho^*(x) \in R/\rho^*$, where $x \in R$. Then, $(x,x) \in \leq \subseteq \rho$. Hence, $\rho^*(x) \preceq \rho^*(x)$. Let



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$\rho^*(x) \preceq \rho^*(y)$ and $\rho^*(y) \preceq \rho^*(x)$. Then, $x\rho y$ and $y\rho x$. Thus, $x\rho^* y$, which means that $\rho^*(x) = \rho^*(y)$. Now, let $\rho^*(x) \preceq \rho^*(y)$ and $\rho^*(y) \preceq \rho^*(z)$. Then, $x\rho y$ and $y\rho z$. So, $x\rho z$. This implies that $\rho^*(x) \preceq \rho^*(z)$.

Now, let $\rho^*(x) \preceq \rho^*(y)$ and $\rho^*(z) \in R/\rho^*$. Then $x\rho y$ and $z \in R$. By conditions (3) and (4) of Definition 4.1, we have $x + z\overline{\overline{\rho}}y + z$, $x \cdot z\rho y \cdot z$ and $z \cdot x\rho z \cdot y$. So, for all $a \in x + z$ and $b \in y + z$, we have $a\rho b$. This implies that $\rho^*(a) \preceq \rho^*(b)$. Hence, $\rho^*(x) \oplus \rho^*(z) \preceq \rho^*(y) \oplus \rho^*(z)$. Similarly, we get, $\rho^*(z) \oplus \rho^*(x) \preceq \rho^*(z) \oplus \rho^*(y)$. By the similar argument, we can show that $\rho^*(x) \odot \rho^*(z) \preceq \rho^*(y) \odot \rho^*(z)$ and $\rho^*(z) \odot \rho^*(x) \preceq \rho^*(z) \odot \rho^*(y)$. □

**Theorem 4.3.** *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring and $\rho$ be a pseudoorder on $R$. Let $\mathcal{X} = \{\theta \mid \theta$ is pseudoorder on $R$ such that $\rho \subseteq \theta\}$. Let $\mathcal{Y}$ be the set of all pseudoorder on $R/\rho^*$. Then, $card(\mathcal{X}) = card(\mathcal{Y})$.*

*Proof.* For $\theta \in \mathcal{X}$, we define a relation $\theta'$ on $R/\rho^*$ as follows:

$$\theta' := \{(\rho^*(x), \rho^*(y)) \in R/\rho^* \times R/\rho^* \mid \exists a \in \rho^*(x), \exists b \in \rho^*(y) \text{ such that } (a, b) \in \theta\}.$$

First, we show that

$$(\rho^*(x), \rho^*(y)) \in \theta' \Leftrightarrow (x, y) \in \theta.$$

Let $(\rho^*(x), \rho^*(y)) \in \theta'$. We show that for every $a \in \rho^*(x)$ and $b \in \rho^*(y)$, $(a, b) \in \theta$. Since $(\rho^*(x), \rho^*(y)) \in \theta'$, there exist $x' \in \rho^*(x)$ and $y' \in \rho^*(y)$ such that $(x', y') \in \theta$. Since $a\rho^* x$ and $x\rho^* x'$, we have $a\rho^* x'$. So, $a\rho x'$. Since $\rho \subseteq \theta$, it follows that $a\theta x'$. Similarly, we obtain $y'\theta b$. Now, we have $a\theta x'$, $x'\theta y'$ and $y'\theta b$. Thus we have $a\theta b$. Since $x \in \rho^*(x)$ and $y \in \rho^*(y)$, we conclude that $(x, y) \in \theta$. Conversely, let $(x, y) \in \theta$. Since $x \in \rho^*(x)$ and $y \in \rho^*(y)$, we obtain $(\rho^*(x), \rho^*(y)) \in \theta'$.

Now, let $(\rho^*(x), \rho^*(y)) \in \preceq$. Then by Theorem 4.2, $(x, y) \in \rho \subseteq \theta$. This implies that $(\rho^*(x), \rho^*(y)) \in \theta'$. Hence, $\preceq \subseteq \theta'$. Now, suppose that $(\rho^*(x), \rho^*(y)) \in \theta'$ and $(\rho^*(y), \rho^*(z)) \in \theta'$. Then, $(x, y) \in \theta$ and $(y, z) \in \theta$ which imply that $(x, z) \in \theta$. Thus we have $(\rho^*(x), \rho^*(z)) \in \theta'$. Also, if $(\rho^*(x), \rho^*(y)) \in \theta'$ and $\rho^*(z) \in R/\rho^*$, then $(x, y) \in \theta$ and $z \in R$. Thus $x + z\overline{\overline{\theta}}y + z$, $x \cdot z\theta y \cdot z$ and $z \cdot x\theta z \cdot y$. So, for all $a \in x + z$ and $a' = x \cdot z$ and for all $b \in y + z$ and $b' = y \cdot z$, we have $a\theta b$ and $a'\theta b'$. This implies that $\theta'(\rho^*(a)) = \theta'(\rho^*(b))$ and $\theta'(\rho^*(a')) = \theta'(\rho^*(b'))$ and so $\theta'(\rho^*(x) \oplus \rho^*(z)) = \theta'(\rho^*(y) \oplus \rho^*(z))$ and $\theta'(\rho^*(x) \odot \rho^*(z)) = \theta'(\rho^*(y) \odot \rho^*(z))$. Thus, $(\rho^*(x) \oplus \rho^*(z))\theta'(\rho^*(y) \oplus \rho^*(z))$ and $(\rho^*(x) \odot \rho^*(z))\theta'(\rho^*(y) \odot \rho^*(z))$. Similarly, we obtain $(\rho^*(z) \oplus \rho^*(x))\theta'(\rho^*(z) \oplus \rho^*(y))$ and $(\rho^*(z) \odot \rho^*(x))\theta'(\rho^*(z) \odot \rho^*(y))$. Therefore, if $\theta \in \mathcal{X}$, then $\theta'$ is a pseudoorder on $R/\rho^*$.

Now, we define the map $\psi : \mathcal{X} \to \mathcal{Y}$ by $\psi(\theta) = \theta'$. Let $\theta_1, \theta_2 \in \mathcal{X}$ and $\theta_1 = \theta_2$. Suppose that $(\rho^*(x), \rho^*(y)) \in \theta'_1$ is an arbitrary element. Then, $(x, y) \in \theta_1$ and so $(x, y) \in \theta_2$. This implies that $(\rho^*(x), \rho^*(y)) \in \theta'_2$. Thus we have $\theta_1' \subseteq \theta_2'$. Similarly, we obtain $\theta_2' \subseteq \theta_1'$. Therefore, $\psi$ is well defined.

Let $\theta_1, \theta_2 \in \mathcal{X}$ and $\theta_1' = \theta_2'$. Suppose that $(x, y) \in \theta_1$ is an arbitrary element. Then, $(\rho^*(x), \rho^*(y)) \in \theta_1'$ and so $(\rho^*(x), \rho^*(y)) \in \theta_2'$. This implies that $(x, y) \in \theta_2$. Thus we have $\theta_1 \subseteq \theta_2$. Similarly, we obtain $\theta_2 \subseteq \theta_1$. Therefore, $\psi$ is one to one.

Finally, we prove that $\psi$ is onto. Consider $\Sigma \in \mathcal{Y}$. We define a relation $\theta$ on $R$ as follows:

$$\theta = \{(x, y) \mid (\rho^*(x), \rho^*(y)) \in \Sigma\}.$$

We show that $\theta$ is a pseudoorder on $R$ and $\rho \subseteq \theta$. Suppose that $(x, y) \in \rho$. By Theorem 4.2, $(\rho^*(x), \rho^*(y)) \in \preceq \subseteq \Sigma$, and so $(x, y) \in \theta$. If $(x, y) \in \leq$, then $(x, y) \in \rho \subseteq \theta$. Hence, $\leq \subseteq \theta$. Let $(x, y) \in \theta$ and $(y, z) \in \theta$. Then, $(\rho^*(x), \rho^*(y)) \in \Sigma$ and $(\rho^*(y), \rho^*(z)) \in \Sigma$. So, $(\rho^*(x), \rho^*(z)) \in \Sigma$. This implies that $(x, z) \in \theta$.

Now, let $(x, y) \in \theta$ and $z \in R$. Then, $(\rho^*(x), \rho^*(y)) \in \Sigma$ and $\rho^*(z) \in R/\rho^*$. Thus, $(\rho^*(x) \oplus \rho^*(z), \rho^*(y) \oplus \rho^*(z)) \in \Sigma$ and $(\rho^*(x) \odot \rho^*(z), \rho^*(y) \odot \rho^*(z)) \in \Sigma$. Therefore, for all $a \in x + z$ and $a' = x \cdot z$ and for all $b \in y + z$ and $b' = y \cdot z$, we have $(\rho^*(a), \rho^*(b)) \in \Sigma$ and $(\rho^*(a'), \rho^*(b')) \in \Sigma$. This means that $(a, b) \in \theta$ and $(a', b') \in \theta$. Therefore, $x + z\overline{\overline{\theta}}y + z$ and $x \cdot z\theta y \cdot z$. Similarly, we obtain $z \cdot x\theta z \cdot y$. Now, obviously we have $\theta' = \Sigma$. □

*Remark* 4.4. In Theorem 4.3, it is easy to see that $\theta_1 \subseteq \theta_2$ if and only if $\theta_1' \subseteq \theta_2'$.

*Remark* 4.5. Let $(R, +, \cdot, \leq_R)$ and $(T, \uplus, \diamond, \leq_T)$ be two ordered rings and $\varphi : R \to T$ a homomorphism. We denote by $k$ the pseudoorder on $R$ defined by $k = \{(a, b) | \varphi(a) \leq_T \varphi(b)\}$. Then, we have $ker\varphi = k^*$.

**Corollary 4.6.** *Let $(R, +, \cdot, \leq_R)$ and $(T, \uplus, \diamond, \leq_T)$ be two ordered rings and $\varphi : R \to T$ a homomorphism. Then, $R/ker\varphi \cong Im\varphi$.*

Let $(R, +, \cdot, \leq_R)$ be an ordered Krasner hyperring, $\rho, \theta$ be pseudoorders on $R$ such that $\rho \subseteq \theta$. We define a relation $\theta/\rho$ on $R/\rho^*$ as follows:

$$\theta/\rho := \{(\rho^*(a), \rho^*(b)) \in R/\rho^* \times R/\rho^* \mid \exists x \in \rho^*(a), \exists y \in \rho^*(b) \text{ such that } (x, y) \in \theta\}.$$

Then, we can see that

$$(\rho^*(a), \rho^*(b)) \in \theta/\rho \Leftrightarrow (a, b) \in \theta.$$

**Theorem 4.7.** *Let $(R, +, \cdot, \leq_R)$be an ordered Krasner hyperring, $\rho, \theta$ be pseudoorders on $R$ such that $\rho \subseteq \theta$. Then,*

(1) *$\theta/\rho$ is a pseudoorder on $R/\rho^*$.*
(2) *$(R/\rho^*)/(\theta/\rho)^* \cong R/\theta^*$.*

*Proof.* (1): If $(\rho^*(a), \rho^*(b)) \in \preceq_\rho$, then $(a, b) \in \rho$. So, $(a, b) \in \theta$ which implies that $(\rho^*(a), \rho^*(b)) \in \theta/\rho$. Thus, $\preceq_\rho \subseteq \theta/\rho$. Let $(\rho^*(a), \rho^*(b)) \in \theta/\rho$ and



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$(\rho^*(b),\rho^*(c)) \in \theta/\rho$. Then $(a,b) \in \theta$ and $(b,c) \in \theta$. Hence, $(a,c) \in \theta$ and so $(\rho^*(a),\rho^*(c)) \in \theta/\rho$. Now, let $(\rho^*(a),\rho^*(b)) \in \theta/\rho$ and $\rho^*(c) \in R/\rho^*$. Then, $(a,b) \in \theta$. Since $\theta$ is a pseudoorder on $R$, we obtain $a + c\overline{\overline{\theta}}b + c$, $a \cdot c\theta b \cdot c$ and $c \cdot a\theta c \cdot b$. Hence, for all $x \in a + c$ and $x' = a \cdot c$ and for all $y \in b + c$ and $y' = b \cdot c$, we have $(x,y) \in \theta$ and $(x',y') \in \theta$. This implies that $(\rho^*(x),\rho^*(y)) \in \theta/\rho$ and $(\rho^*(x'),\rho^*(y')) \in \theta/\rho$. Since $\rho^*$ is a strongly regular relation on $R$, $\rho^*(x) = \rho^*(a) \oplus \rho^*(c)$, $\rho^*(y) = \rho^*(b) \oplus \rho^*(c)$, $\rho^*(x') = \rho^*(a) \odot \rho^*(c)$ and $\rho^*(y') = \rho^*(b) \odot \rho^*(c)$. So, we obtain $(\rho^*(a) \oplus \rho^*(c), \rho^*(b) \oplus \rho^*(c)) \in \theta/\rho$ and $(\rho^*(a) \odot \rho^*(c), \rho^*(b) \odot \rho^*(c)) \in \theta/\rho$. Similarly, we obtain $(\rho^*(c) \oplus \rho^*(a), \rho^*(c) \oplus \rho^*(b)) \in \theta/\rho$ and $(\rho^*(c) \odot \rho^*(a), \rho^*(c) \odot \rho^*(b)) \in \theta/\rho$. Therefore, $\theta/\rho$ is a pseudoorder on $R/\rho^*$.

(2): We define the map $\psi : R/\rho^* \to R/\theta^*$ by $\psi(\rho^*(a)) = \theta^*(a)$. If $\rho^*(a) = \rho^*(b)$, then $(a,b) \in \rho^*$. Hence, by the definition of $\rho^*$, $(a,b) \in \rho \subseteq \theta$ and $(b,a) \in \rho \subseteq \theta$. This implies that $(a,b) \in \theta^*$ and so $\theta^*(a) = \theta^*(b)$. Thus, $\theta$ is well defined. For all $\rho^*(x), \rho^*(y) \in R/\rho^*$, we have

$$\rho^*(x) \oplus \rho^*(y) = \rho^*(z), \text{ for all } z \in x + y,$$

$$\theta^*(x) \uplus \theta^*(y) = \theta^*(z), \text{ for all } z \in x + y,$$

$$\rho^*(x) \odot \rho^*(y) = \rho^*(x \cdot y),$$

$$\theta^*(x) \otimes \theta^*(y) = \theta^*(x \cdot y).$$

Thus,

$$\begin{aligned} \psi(\rho^*(x) \oplus \rho^*(y)) &= \psi(\rho^*(z)), \text{ for all } z \in x + y \\ &= \theta^*(z), \text{ for all } z \in x + y \\ &= \theta^*(x) \boxplus \theta^*(y) \\ &= \psi(\rho^*(x)) \boxplus \psi(\rho^*(y)), \end{aligned}$$

and

$$\begin{aligned} \psi(\rho^*(x) \odot \rho^*(y)) &= \psi(\rho^*(x \cdot y)) \\ &= \theta^*(x \cdot y) \\ &= \theta^*(x) \otimes \theta^*(y) \\ &= \psi(\rho^*(x)) \otimes \psi(\rho^*(y)), \end{aligned}$$

and if $\rho^*(x) \preceq_\rho \rho^*(y)$, then $(x,y) \in \rho$. So, $(x,y) \in \theta$ and this implies that $\theta^*(x) \preceq_\theta \theta^*(y)$. Therefore, $\psi$ is a homomorphism. It is easy to see that $\psi$ is onto, since

$$Im\psi = \{\psi(\rho^*(x)) \mid x \in R\} = \{\theta^*(x) \mid x \in R\} = R/\theta^*.$$

So, by Corollary 4.6, we obtain

$$(R/\rho^*)/ker\psi \cong Im\psi = R/\theta^*.$$

Suppose that

$$k := \{(\rho^*(x), \rho^*(y)) \mid \psi(\rho^*(x)) \preceq_\theta \psi(\rho^*(y)).$$



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Then,

$$\begin{aligned}(\rho^*(x),\rho^*(y))\in k \quad &\Leftrightarrow \psi(\rho^*(x)) \preceq_\theta \psi(\rho^*(y))\\ &\Leftrightarrow \theta^*(x) \preceq_\theta \theta^*(y)\\ &\Leftrightarrow (x,y)\in\theta\\ &\Leftrightarrow (\rho^*(x),\rho^*(y))\in \theta/\rho.\end{aligned}$$

Hence, $k=\theta/\rho$ and by Remark 4.5, we have $k^*=(\theta/\rho)^*=ker\psi$. □

**Definition 4.8.** Let $(R,+,\cdot,\leq_R)$ and $(T,\uplus,\diamond,\leq_T)$ be two ordered Krasner hyperrings, $\rho_1$, $\rho_2$ be two pseudoorders on $R,T$, respectively, and the map $f:R\to T$ be a homomorphism. Then, $f$ is called a $(\rho_1,\rho_2)$-homomorphism if

$$(a,b)\in\rho_1 \Rightarrow (f(a),f(b))\in\rho_2.$$

**Lemma 4.9.** *Let $(R,+,\cdot,\leq_R)$ and $(T,\uplus,\diamond,\leq_T)$ be two ordered Krasner hyperrings, $\rho_1$, $\rho_2$ be two pseudoorders on $R,T$, respectively, and the map $f:R\to T$ be a $(\rho_1,\rho_2)$-homomorphism. Then, the map $\overline{f}:R/\rho_1^*\to T/\rho_2^*$ defined by*

$$\overline{f}(\rho_1^*(x))=\rho_2^*(f(x)), \text{ for all } x\in R$$

*is a homomorphism of rings.*

*Proof.* Suppose that $\rho_1^*(x)=\rho_1^*(y)$. Then we have $(x,y)\in\rho_1$. Since $f$ is a $(\rho_1,\rho_2)$-homomorphism, it follows that $(f(x),f(y))\in\rho_2$. This implies that $\rho_2^*(f(x))=\rho_2^*(f(y))$ or $\overline{f}(\rho_1^*(x))=\overline{f}(\rho_1^*(y))$. Therefore, $\overline{f}$ is well defined. Now, we show that $\overline{f}$ is a homomorphism. Suppose that $\rho_1^*(x),\rho_1^*(y)$ be two arbitrary elements of $R/\rho_1^*$. Then,

$$\overline{f}(\rho_1^*(x)\oplus\rho_1^*(y))=\overline{f}(\rho_1^*(z))=\rho_2^*(f(z)), \text{ for all } z\in x+y.$$

Since $z\in x+y$, it follows that $f(z)\in f(x)\uplus f(y)$. Since $\rho_2^*$ is a strongly regular relation, we obtain $\rho_2^*(f(z))=\rho_2^*(f(x))\boxplus\rho_2^*(f(y))$. Thus we have

$$\overline{f}(\rho_1^*(x)\oplus\rho_1^*(y))=\rho_2^*(f(x))\boxplus\rho_2^*(f(y))=\overline{f}(\rho_1^*(x))\boxplus\overline{f}(\rho_1^*(y)).$$

Now, suppose that $\rho_1^*(x),\rho_1^*(y)$ be two arbitrary elements of $R/\rho_1^*$. Then,

$$\overline{f}(\rho_1^*(x)\odot\rho_1^*(y))=\overline{f}(\rho_1^*(z))=\rho_2^*(f(z)), \text{ for } z=x\cdot y.$$

Since $z=x\cdot y$, it follows that $f(z)=f(x)\diamond f(y)$. Since $\rho_2^*$ is a strongly regular relation, we obtain $\rho_2^*(f(z))=\rho_2^*(f(x))\otimes\rho_2^*(f(y))$. Thus, we have

$$\overline{f}(\rho_1^*(x)\odot\rho_1^*(y))=\rho_2^*(f(x))\otimes\rho_2^*(f(y))=\overline{f}(\rho_1^*(x))\otimes\overline{f}(\rho_1^*(y)).$$

□

**Theorem 4.10.** *Let $(R,+,\cdot,\leq_R)$ and $(T,\uplus,\diamond,\leq_T)$ be two ordered Krasner hyperrings, $\rho_1$, $\rho_2$ be two pseudoorders on $R,T$, respectively, and the map $f:R\to T$ be a $(\rho_1,\rho_2)$-homomorphism. Then, the relation $\rho_f$ defined by*

$$\rho_f:=\{(\rho_1^*(x),\rho_1^*(y))\mid \rho_2^*(f(x))\preceq_T\rho_2^*(f(y))\}$$

*is a pseudoorder on $R/\rho_1^*$.*



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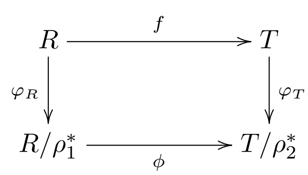
*Proof.* Suppose that $(\rho_1^*(x), \rho_1^*(y)) \in \preceq_R$. By Lemma 4.9, $\overline{f}$ is a homomorphism. Since $\rho_1^*(x) \preceq_R \rho_1^*(y)$, it follows that $\overline{f}(\rho_1^*(x)) \preceq_T \overline{f}(\rho_1^*(y))$. Thus we have $\rho_2^*(f(x)) \preceq_T \rho_2^*(f(y))$. This means that $(\rho_1^*(x), \rho_1^*(y)) \in \rho_f$.

Let $(\rho_1^*(x), \rho_1^*(y)) \in \rho_f$ and $(\rho_1^*(y), \rho_1^*(z)) \in \rho_f$. Then, we have $\rho_2^*(f(x)) \preceq_T \rho_2^*(f(y))$ and $\rho_2^*(f(y)) \preceq_T \rho_2^*(f(z))$. So, $\rho_2^*(f(x)) \preceq_T \rho_2^*(f(z))$. This implies that $(\rho_1^*(x), \rho_1^*(z)) \in \rho_f$. Now, let $(\rho_1^*(x), \rho_1^*(y)) \in \rho_f$ and $\rho_1^*(z) \in R/\rho_1^*$. We show that $(\rho_1^*(x) \oplus \rho_1^*(z), \rho_1^*(y) \oplus \rho_1^*(z)) \in \rho_f$. Since $(\rho_1^*(x), \rho_1^*(y)) \in \rho_f$, it follows that $\overline{f}(\rho_1^*(x)) \preceq_T \overline{f}(\rho_1^*(y))$. Thus, by the definition of ordered Krasner hyperring, we get $\overline{f}(\rho_1^*(x)) \boxplus \overline{f}(\rho_1^*(z)) \preceq_T \overline{f}(\rho_1^*(y)) \boxplus \overline{f}(\rho_1^*(z))$. So, $\overline{f}(\rho_1^*(x) \oplus \rho_1^*(z)) \preceq_T \overline{f}(\rho_1^*(y) \oplus \rho_1^*(z))$. Hence, for all $u \in x+z$ and for all $v \in y+z$, we have $\overline{f}(\rho_1^*(u)) \preceq_T \overline{f}(\rho_1^*(v))$. This implies that $(\rho_1^*(u), \rho_1^*(v)) \in \rho_f$. Therefore, we have $(\rho_1^*(x) \oplus \rho_1^*(z), \rho_1^*(y) \oplus \rho_1^*(z)) \in \rho_f$. Similarly, we obtain $(\rho_1^*(z) \oplus \rho_1^*(x), \rho_1^*(z) \oplus \rho_1^*(y)) \in \rho_f$. Now, we show that $(\rho_1^*(x) \odot \rho_1^*(z), \rho_1^*(y) \odot \rho_1^*(z)) \in \rho_f$. Since $(\rho_1^*(x), \rho_1^*(y)) \in \rho_f$, we have $\overline{f}(\rho_1^*(x)) \preceq_T \overline{f}(\rho_1^*(y))$. Thus, by the definition of ordered Krasner hyperring, we get $\overline{f}(\rho_1^*(x)) \otimes \overline{f}(\rho_1^*(z)) \preceq_T \overline{f}(\rho_1^*(y)) \otimes \overline{f}(\rho_1^*(z))$. So, $\overline{f}(\rho_1^*(x) \odot \rho_1^*(z)) \preceq_T \overline{f}(\rho_1^*(y) \odot \rho_1^*(z))$. Hence, $\overline{f}(\rho_1^*(x \cdot z)) \preceq_T \overline{f}(\rho_1^*(y \cdot z))$. This implies that $(\rho_1^*(x \cdot z), \rho_1^*(y \cdot z)) \in \rho_f$. Therefore, we have $(\rho_1^*(x) \odot \rho_1^*(z), \rho_1^*(y) \odot \rho_1^*(z)) \in \rho_f$. Similarly, we obtain $(\rho_1^*(z) \odot \rho_1^*(x), \rho_1^*(z) \odot \rho_1^*(y)) \in \rho_f$. □

**Corollary 4.11.** *Let us follow the notations and definitions used in Lemma 4.9 and Theorem 4.10. We have* $ker\overline{f} = \rho_f^*$.

*Proof.* It is straightforward. □

**Corollary 4.12.** *Let* $(R, +, \cdot, \leq_R)$ *and* $(T, \uplus, \diamond, \leq_T)$ *be two ordered Krasner hyperrings,* $\rho_1$, $\rho_2$ *be two pseudoorders on* $R, T$*, respectively, and the map* $f : R \to T$ *be a* $(\rho_1, \rho_2)$*-homomorphism. Then, the following diagram is commutative.*




*Proof.* It is straightforward. □

**Open Problem.** Is there a regular relation $\rho$ on an ordered Krasner hyperring $(R, +, \cdot, \leq_R)$ for which $R/\rho$ is an ordered Krasner hyperring?

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