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Fixed Point Results on b-Metric Space via Picard Sequences and b-Simulation Functions

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Abstract. In a recent paper, Khojasteh *et al.* [F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theorems via simulation functions, Filomat, 29 (2015), 1189-1194] presented a new class of simulation functions, say $\mathcal{Z}$-contractions, with unifying power over known contractive conditions in the literature. Following this line of research, we extend and generalize their results on a b-metric context, by giving a new notion of b-simulation function. Then, we prove and discuss some fixed point results in relation with existing ones.

Keywords: b-Metric space, Partial order, Nonlinear contraction, Fixed point, b-Simulation function.

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1. Introduction

The source of metric fixed point theory is the contraction mapping principle, presented in Banach's Ph.D. dissertation, and later published in 1922 [5]. This

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fundamental principle was largely applied in dealing with various theoretical and practical problems, arising in a number of branches of mathematics. This potentiality attracted many researchers and hence the literature is reach in fixed point results, see for example [7, 13, 33, 35, 36, 39, 40].

In this exciting context, Bakhtin [6] and Czerwik [14, 15] developed the notion of b-metric space and proved some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces. Successively, this notion has been reintroduced by Khamsi [22] and Khamsi and Hussain [23], with the name of metric-type space. In the literature, there are a lot of consequences of this study, see for example [10, 13, 20, 21, 22, 23].

On the other hand, a pioneering paper for the success of fixed point theory in applied science is the paper of Ran and Reurings [32], where they established a fixed point result by dealing with partially ordered sets. Further, several results appeared in this direction, we refer to [1, 11, 12, 18, 19, 25, 26, 30, 31, 42] and the references therein.

Finally, Khojasteh *et al.* [24] introduced the notion of $\mathcal{Z}$-contraction which is a new type of nonlinear contractions defined by using a specific simulation function. Then, they proved existence and uniqueness of fixed points for $\mathcal{Z}$-contraction mappings. In fact, the advantage of this technique is the possibility to treat several fixed point problems from a unique common point of view. In this direction, we recall that Roldán *et al.* [38] used simulation functions to studying the existence and uniqueness of coincidence points of a pair of contractive nonlinear operators. Also, Argoubi *et al.* [4] studied the existence of coincidence and common fixed point results of a pair of nonlinear operators satisfying a certain contractive condition involving simulation functions, in the setting of ordered metric spaces.

In this paper, we introduce the notion of b-simulation function in the setting of b-metric spaces and consider nonlinear operators satisfying a nonlinear contractive condition involving a b-simulation function in a b-metric space or in a b-metric space endowed with a partial order. For this kind of contractions, we establish existence and uniqueness of fixed points. As consequences of this study, we deduce several related results in fixed point theory in a b-metric space.

2. Preliminaries

The aim of this section is to present and collect some notions used in the paper.

Definition 2.1. Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty[$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

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(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \leq b[d(x,y) + d(y,z)]$.

A triplet (X,d,b), is called a b-metric space.

We observe that a metric space is included in the class of b-metric spaces. In fact, the notions of convergent sequence, Cauchy sequence and complete space are defined as in metric spaces.

Next, we give some examples of b-metric spaces.

EXAMPLE 2.2. Let $X = [0,1]$ and $d : X \times X \to [0,+\infty[$ be defined by $d(x,y) = (x-y)^2$, for all $x, y \in X$. Clearly, $(X,d,2)$ is a b-metric space.

EXAMPLE 2.3. Let $C_b(X) = \{f : X \to \mathbb{R} \ : \ \|f\|_\infty = \sup_{x\in X} |f(x)| < +\infty\}$ and let $\|f\| = \sqrt[3]{\|f^3\|_\infty}$. The function $d : C_b(X) \times C_b(X) \to [0, +\infty[$ defined by

$$d(f,g) = \|f-g\|, \quad \text{for all} \quad f, g \in C_b(X)$$

is a b-metric with constant $b = \sqrt[3]{4}$ and so $(C_b(X), d, \sqrt[3]{4})$ is a b-metric space.

Let X be a non-empty set. If (X,d,b) is a b-metric space and $(X,\preceq)$ is a partially ordered set, then $(X,d,b,\preceq)$ is called an ordered b-metric space. Two elements x and y of X are called comparable if $x \preceq y$ or $y \preceq x$ holds. A self-mappings f on $(X,\preceq)$ is said to be dominated if $fx \preceq x$ for all $x \in X$ and non-decreasing if $fx \preceq fy$ whenever $x \preceq y$ for all $x, y \in X$. An ordered b-metric space $(X,d,b,\preceq)$ has a sequential limit comparison property if the following holds:

(S) for every decreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, we have $x \preceq x_n$.

Khojasteh *et al.* gave the following definition of simulation function, see [24].

Definition 2.4. A simulation function is a mapping $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ satisfying the following conditions:
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$, for all $t, s > 0$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $]0,+\infty[$ such that $\lim_{n\to\infty} t_n = \lim_{n\to+\infty} s_n = \ell \in]0,+\infty[$, then

$$\limsup_{n\to+\infty} \zeta(t_n, s_n) < 0.$$

Then, they proved a theorem of existence and uniqueness of fixed point.

Theorem 2.5 ([24])**.** *Let (X,d) be a complete metric space and $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function ζ, that is,*

$$\zeta(d(fx,fy), d(x,y)) \geq 0, \quad \text{for all } x, y \in X. \tag{2.1}$$

Then f has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges to this fixed point.

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In [4], Argoubi *et al.* note that the condition (ζ_1) was not used for the proof of Theorem 2.5. Also they observe that taking $x = y$ in (2.1), we obtain $\zeta(0,0) \geq 0$ and hence, if $\zeta(0,0) < 0$, then the set of operators $f : X \to X$ satisfying (2.1) is empty.

Taking in consideration the above remarks, Argoubi *et al.* slightly modified the previous definition, by removing the condition (ζ_1). Precisely, we have to consider the following definition.

Definition 2.6. A simulation function is a mapping $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ satisfying the conditions (ζ_2) and (ζ_3).

Clearly, any simulation function in the original Khojasteh *et al.* sense (Definition 2.4) is also a simulation function in Argoubi *et al.* sense (Definition 2.6), but the converse is not true, as we show in the following example.

Example 2.7 ([4], Example 2.4). Let $\zeta_\lambda : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ be the function defined by

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (s,t) = (0,0), \\ \lambda s - t & \text{otherwise,} \end{cases}$$

where $\lambda \in]0,1[$. Then ζ_λ satisfies (ζ_2) and (ζ_3) with $\zeta_\lambda(0,0) > 0$.

Now, we give the definition of b-simulation function in the setting of b-metric space.

Definition 2.8. Let (X,d,b) be a b-metric space. A b-simulation function is a function $\xi : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ satisfying the following conditions:

(ξ_1) $\xi(t,s) < s - t$, for all $t, s > 0$;

(ξ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $]0,+\infty[$ such that

$$0 < \lim_{n\to+\infty} t_n \leq \liminf_{n\to+\infty} s_n \leq \limsup_{n\to+\infty} s_n \leq b \lim_{n\to+\infty} t_n < +\infty,$$

then

$$\limsup_{n\to+\infty} \xi(b\, t_n, s_n) < 0.$$

Following are some examples of b-simulation functions.

Example 2.9. Let $\xi : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$, be defined by

(i) $\xi(t,s) = \lambda\, s - t$ for all $t, s \in [0,+\infty[$, where $\lambda \in [0,1[$.

(ii) $\xi(t,s) = \psi(s) - \varphi(t)$ for all $t, s \in [0,+\infty[$, where $\varphi, \psi : [0,+\infty[\to [0,+\infty[$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \varphi(t)$ for all $t > 0$.

(iii) $\xi(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0,+\infty[$, where $f, g : [0,+\infty[\times[0,+\infty[\to]0,+\infty[$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.

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(iv) $\xi(t,s) = s - \varphi(s) - t$ for all $t, s \in [0, +\infty[$, where $\varphi : [0, +\infty[\to [0, +\infty[$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

(v) $\xi(t,s) = s\varphi(s) - t$ for all $t, s \in [0, +\infty[$, where $\varphi : [0, +\infty[\to [0, 1[$ is such that $\lim_{t \to r^+} \varphi(t) < 1$ for all $r > 0$.

Each of the function considered in (i)-(v) is a b-simulation function.

3. Fixed Points via b-Simulation Functions

The following lemmas, on Picard sequence, are needed to establish the main result. Let $X \neq \emptyset$ and f a self-mapping on X. Let $x_0 \in X$ and $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is called a Picard sequence of initial point at x_0. Denote with $Fix(f) = \{x \in X : x = fx\}$, that is, the set of fixed points of f.

Lemma 3.1. *Let (X, d, b) be a b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that*

$$\xi\left(b\, d(fx, fy), d(x, y)\right) \geq 0 \quad \text{for all } x, y \in X. \tag{3.1}$$

Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0 \in X$. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then

$$\lim_{n \to +\infty} d(x_{n-1}, x_n) = 0.$$

Proof. It follows from (3.1) and (ξ_1) that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \xi(b\, d(x_n, x_{n+1}), d(x_{n-1}, x_n)) \\ &< d(x_{n-1}, x_n) - b\, d(x_n, x_{n+1}). \end{aligned}$$

The above inequality shows that

$$b\, d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N},$$

which implies that $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence of positive real numbers. So there is some $r \geq 0$ such that

$$\lim_{n \to +\infty} d(x_{n-1}, x_n) = r.$$

Suppose that $r > 0$. It follows from the condition (ξ_2), with $t_n = d(x_n, x_{n+1})$ and $s_n = d(x_{n-1}, x_n)$, that

$$0 \leq \limsup_{n \to +\infty} \xi\left(b\, d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right) < 0,$$

which is a contradiction. Then we conclude that $r = 0$, which ends the proof. □

Lemma 3.2. *Let (X, d, b) be a b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that (3.1) holds. Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0 \in X$. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a bounded sequence.*

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Proof. Let us assume that $\{x_n\}$ is not a bounded sequence. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$d(x_m, x_{n_k}) \leq 1, \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

By the triangle inequality, we obtain

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq b\, d(x_{n_{k+1}}, x_{n_{k+1}-1}) + b\, d(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq b\, d(x_{n_{k+1}}, x_{n_{k+1}-1}) + b. \end{aligned}$$

Letting $k \to +\infty$ in the above inequality and using Lemma 3.1, we get

$$1 \leq \liminf_{k\to+\infty} d(x_{n_{k+1}}, x_{n_k}) \leq \limsup_{k\to+\infty} d(x_{n_{k+1}}, x_{n_k}) \leq b. \tag{3.2}$$

Again, from (3.1), we deduce

$$\begin{aligned} b\, d(x_{n_{k+1}}, x_{n_k}) &\leq d(x_{n_{k+1}-1}, x_{n_k-1}) \\ &\leq b\, d(x_{n_{k+1}-1}, x_{n_k}) + b\, d(x_{n_k}, x_{n_k-1}) \\ &\leq b + b\, d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Letting $k \to +\infty$ in the above inequality and using (3.2), we deduce that there exist

$$\lim_{k\to+\infty} d(x_{n_{k+1}}, x_{n_k}) = 1 \quad \text{and} \quad \lim_{k\to+\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = b.$$

Then by condition (ξ_2), with $t_k = d(x_{n_{k+1}}, x_{n_k})$ and $s_k = d(x_{n_{k+1}-1}, x_{n_k-1})$, we obtain

$$0 \leq \limsup_{k\to+\infty} \xi\left(b\, d(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})\right) < 0,$$

which is a contradiction. This ends the proof.

□

Lemma 3.3. *Let (X, d, b) be a b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (3.1) *holds. Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0 \in X$. Suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. Let

$$C_n = \sup\{d(x_i, x_j) : i, j \geq n\}, \ n \in \mathbb{N}.$$

From Lemma 3.2, we know that $C_n < +\infty$ for every $n \in \mathbb{N}$. Since $\{C_n\}$ is a positive decreasing sequence, there is some $C \geq 0$ such that

$$\lim_{n\to+\infty} C_n = C. \tag{3.3}$$

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Let us suppose that $C > 0$. By the definition of C_n, for every $k \in \mathbb{N}$, there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq C_k.$$

Letting $k \to +\infty$ in the above inequality, we get

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = C. \tag{3.4}$$

Again, from (3.1) and the definition of C_n, we deduce

$$b\, d(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) \leq C_{k-1}.$$

Letting $k \to +\infty$ in the above inequality, using (3.3) and (3.4), we get

$$bC \leq \liminf_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) \leq \limsup_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) \leq C. \tag{3.5}$$

Now, if $b > 1$, the previous inequality implies $C = 0$. If $b = 1$, by the condition (ξ_2), with $t_k = d(x_{m_k}, x_{n_k})$ and $s_k = d(x_{m_k-1}, x_{n_k-1})$, we get

$$0 \leq \limsup_{k\to+\infty} \xi(b\, d(x_{m_k}, x_{n_k}), d(x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Thus we have $C = 0$, that is,

$$\lim_{n\to+\infty} C_n = 0 \quad \text{for all } b \geq 1.$$

This proves that $\{x_n\}$ is a Cauchy sequence.

□

Now, we present our first main result.

Theorem 3.4. *Let (X, d, b) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (3.1) *holds, that is,*

$$\xi\left(b\, d(fx, fy), d(x, y)\right) \geq 0, \quad \textit{for all } x, y \in X.$$

Then f has a unique fixed point.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence of Picard with initial point at x_0. At first, observe that if $x_m = x_{m+1}$ for some $m \in \mathbb{N}$, then $x_m = x_{m+1} = fx_m$, that is, x_m is a fixed point of f. In this case, the existence of a fixed point is proved. So, we can suppose that $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}$.

Now, by Lemma 3.3, the sequence $\{x_n\}$ is Cauchy and since (X, d, b) is complete, then there exists some $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z. \tag{3.6}$$

We claim that z is a fixed point of f. Using (3.1) with $x = x_n$ and $y = z$, we deduce that

$$0 \leq \xi(b\, d(fx_n, fz), d(x_n, z)) < d(x_n, z) - bd(fx_n, fz).$$

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This implies

$$b\,d(fx_n, fz) \leq d(x_n, z) \quad \text{for all } n \in \mathbb{N}$$

and consequentily

$$d(z, fz) \leq b\,d(z, x_{n+1}) + b\,d(fx_n, fz) \leq b\,d(z, x_{n+1}) + d(x_n, z).$$

Letting $n \to +\infty$ in the above inequality, we obtain that $d(z, fz) = 0$, that is, $z = fz$.

Now, we establish uniqueness of the fixed point. Suppose that there exists $w \in X$ such that $w = fw$ and $z \neq w$. Using (3.1) with $x = w$ and $y = z$, we get that

$$0 \leq \xi(b\,d(fw, fz), d(w, z)) < d(w, z) - b\,d(w, z) \leq 0,$$

which ia a contradiction and hence $w = z$. This ends the proof of Theorem 3.4. □

4. Consequences

We show the unifying power of b-simulation functions by applying Theorem 3.4 to deduce different kinds of contractive conditions in the existing literature.

The following corollary give a result of Banach type [5].

Corollary 4.1 ([21], Theorem 3.3)**.** Let (X, d, b) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists $\lambda \in]0, 1[$ such that

$$b\,d(fx, fy) \leq \lambda\,d(x, y) \quad \text{for all } x, y \in X.$$

Then f has a unique fixed point.

Proof. The result follows from Theorem 3.4, by taking as b-simulation function

$$\xi(t, s) = \lambda\,s - t,$$

for all $t, s \geq 0$. □

The following corollary give a result of Rhoades type [37].

Corollary 4.2. Let (X, d, b) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a lower semi-continuous function $\varphi : [0, +\infty[\to [0, +\infty[$ with $\varphi^{-1}(0) = \{0\}$ such that

$$b\,d(fx, fy) \quad \leq d(x, y) - \varphi(d(x, y)) \quad \text{for all } x, y \in X.$$

Then f has a unique fixed point.

Proof. The result follows from Theorem 3.4, by taking as b-simulation function

$$\xi(t, s) = s - \varphi(s) - t,$$

for all $t, s \geq 0$. □

We have also the following corollary, see [34].

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Corollary 4.3. Let (X, d, b) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a function $\varphi : [0, +\infty[\to [0, 1[$ with $\limsup_{t \to r^+} \varphi(t) < 1$ for all $r > 0$ such that

$$b\, d(fx, fy) \quad \leq \quad \varphi(d(x, y))\, d(x, y) \qquad \text{for all } x, y \in X.$$

Then f has a unique fixed point.

Proof. The result follows from Theorem 3.4, by taking as b-simulation function

$$\xi(t, s) = s\varphi(s) - t,$$

for all $t, s \geq 0$. □

The following corollary give a result of Boyd-Wong type [8].

Corollary 4.4. Let (X, d, b) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists an upper semi-continuous function $\eta : [0, +\infty[\to [0, +\infty[$ with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ such that

$$b\, d(fx, fy) \leq \eta(d(x, y)) \qquad \text{for all } x, y \in X.$$

Then f has a unique fixed point.

Proof. The result follows from Theorem 3.4, by taking as simulation function

$$\zeta(t, s) = \eta(s) - t,$$

for all $t, s \geq 0$. □

Following example shows that the above Theorem 3.4 is a proper generalization of Banach contraction principle in the setting of b-metric spaces.

Example 4.5. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = (x - y)^2$. Then $(X, d, 2)$ is a complete b-metric space. Define a mapping $f : X \to X$ by

$$fx = \frac{ax}{1 + x} \qquad \text{for all } x \in X \text{ and } a \in]0, \frac{1}{\sqrt{2}}].$$

For all $x, y \in X$ with $x \geq y$, we have

$$d(fx, fy) = a^2 \frac{(x - y)^2}{[(1 + x)(1 + y)]^2} \leq a^2 \frac{(x - y)^2}{[1 + (x - y)]^2} \leq a^2 \frac{(x - y)^2}{1 + (x - y)^2}.$$

Now, let $\eta : [0, +\infty[\to [0, +\infty[$ be defined by $\eta(t) = 1/(1 + t)$ for all $t \geq 0$. From the previous inequality, we get

$$2\, d(fx, fy) \leq \eta(d(x, y)) \qquad \text{for all } x, y \in X.$$

Since all the conditions of Corollary 4.4 are satisfied, then f has a unique fixed point.

Note that for $a = \frac{1}{\sqrt{2}}$, there does not exist $\lambda \in [0, 1[$ such that

$$2\, d(fx, fy) \leq \lambda\, d(x, y) \qquad \text{for all } x \in X.$$

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In fact, the previous inequality for $x > 0$ and $y = 0$ implies

$$\frac{x^2}{(1+x)^2} \leq \lambda\, x^2 \quad \text{for all } x \in]0,1],$$

that is, $1 \leq \lambda$.

5. Fixed Points in Ordered b-Metric Spaces

The existence of fixed points of self-mappings defined on certain type of ordered sets plays an important role in the order theoretic approach for applications in differential and matrix equations. This approach has been initiated by Ran and Reurings [32], and further studied by Nieto and Rodríguez-Lopez [26]. Other contributions can be found in [2, 3, 9, 16, 17, 27, 28, 29, 41].

First, we formulate Lemmas 3.1-3.3 in ordered b-metric spaces as follows.

Lemma 5.1. *Let $(X, d, b, \preceq)$ be an ordered b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that for every $x, y \in X$ with $x \preceq y$, we have*

$$\xi\left(b\, d(fx, fy), d(x, y)\right) \geq 0. \tag{5.1}$$

Let $\{x_n\}$ be a sequence of Picard with initial point at $x_0 \in X$ such that

$$x_{n+1} = fx_n \prec x_n, \text{ for all } n \in \mathbb{N}. \tag{5.2}$$

Then

(i) $\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0$;
(ii) $\{x_n\}$ *is a bounded sequence;*
(iii) $\{x_n\}$ *is a Cauchy sequence.*

Clearly, one can prove Lemma 5.1 by proceeding as in the proofs of Lemmas 3.1-3.3.

Theorem 5.2. *Let $(X, d, b, \preceq)$ be a complete ordered b-metric space and let $f : X \to X$ be a dominated mapping. Suppose that there exists a b-simulation function ξ such that*

$$\xi\left(b\, d(fx, fy), d(x, y)\right) \geq 0$$

for all $x, y \in X$ with $x \preceq y$. If the following condition is satisfied:

(i) *X has the property (S),*

then f has a fixed point. Moreover, the set $Fix(f)$ of fixed points of f is well ordered if and only if f has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be a Picard sequence of initial point at $x_0 \in X$. If $x_{m-1} = x_m$ for some $m \in \mathbb{N}$, then $x_{m-1} = x_m = fx_{m-1}$ and so x_{m-1} is a fixed point of f. Assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Using the property of mapping f, we deduce

$$x_n = fx_{n-1} \prec x_{n-1}, \qquad \text{for all } n \in \mathbb{N}.$$

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Then $x_n \prec x_{n-1}$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is a decreasing sequence and by Lemma 5.1 the sequence $\{x_n\}$ is Cauchy. Then there exists $z \in X$ sucht that $x_n \to z$. Note that condition (S) ensures that $z \prec x_{n-1}$ for all $n \in \mathbb{N}$.

Now, we show that z is a fixed point of f. Using (5.1) with $x = z$ and $y = x_{n-1}$, we deduce that

$$0 \leq \xi(b\, d(fz, fx_{n-1}), d(z, x_{n-1})) < d(z, x_{n-1}) - bd(fz, fx_{n-1}).$$

This implies

$$b\, d(fz, fx_{n-1}) \leq d(z, x_{n-1}) \quad \text{for all } n \in \mathbb{N}$$

and consequently

$$d(fz, z) \leq b\, d(fz, fx_{n-1}) + b\, d(x_n, z) \leq d(z, x_{n-1}) + b\, d(x_n, z).$$

Letting $n \to +\infty$ in the above inequality, we obtain that $d(fz, z) = 0$, that is, $z = fz$.

Now, assume that the set of fixed points of f is well ordered and establish uniqueness of the fixed point. Suppose that there exists $w \in Fix(f)$ such that $z \neq w$. Assume that $w \prec z$. Using (5.1) with $x = w$ and $y = z$, we get that

$$0 \leq \xi(b\, d(fw, fz), d(w, z)) < d(w, z) - b\, d(w, z) \leq 0,$$

which is a contradiction and hence $w = z$.

Conversely, if f has a unique fixed point, then the set $Fix(f)$ being singleton is well ordered. This ends the proof of Theorem 5.2. □

Theorem 5.3. *Adding to the hypotheses of Theorem 5.2 the following condition:*

(H) *for all $z, w \in Fix(f)$ that are not comparable there exists $v \in X$ such that $v \prec z$ and $v \prec w$,*

then f has a unique fixed point in X.

Proof. If z and w are two comparable fixed points of f, then $z = w$ by condition (ξ_1). Assume that z and w are not comparable, then by condition (H) there exists $v \in X$ such that $v \prec z$ and $v \prec w$. Since f is a dominated mapping, we deduce that $v_n = f^n v \preceq z$ for all $n \in \mathbb{N}$. Now, using (5.1) with $x = v_{n-1}$ and $y = z$, we obtain

$$\xi(b\, d(v_n, z), d(v_{n-1}, z)) < d(v_{n-1}, z) - b\, d(v_n, z).$$

This implies

$$d(v_n, z) < \frac{1}{b} d(v_{n-1}, z) \quad \text{for all } n \in \mathbb{N}$$

and hence

$$d(v_n, z) < \frac{1}{b^n} d(v, z) \quad \text{for all } n \in \mathbb{N}.$$

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Now, if $b > 1$, we get $\lim_{n\to+\infty} d(v_n, z) = 0$. If $b = 1$, from the previous inequality, we deduce that $d(v_n, z) \to r$ with $r \geq 0$. If $r > 0$, by the property (ξ_2), with $t_n = d(v_{n+1}, z)$ and $s_n = d(v_n, z)$, we have

$$0 \leq \limsup_{n\to+\infty} \xi(b\, d(v_{n+1}, z), d(v_n, z)) < 0$$

which is a contradiction and so $r = 0$,

Similar, we deduce that $\lim_{n\to+\infty} d(v_n, w) = 0$. From

$$d(z, w) \leq b\, d(z, v_n) + b\, d(v_n, w),$$

letting $n \to +\infty$, we get $d(z, w) = 0$, that is, $z = w$. This ends the proof of theorem. □

Now, we give a result of fixed point for non-decreasing self mappings in the setting of ordered b-metric spaces.

Theorem 5.4. *Let $(X, d, b, \preceq)$ be a complete ordered b-metric space and let $f : X \to X$ be a non-decreasing mapping. Suppose that there exists a b-simulation function ξ such that*

$$\xi\left(b\, d(fx, fy), d(x, y)\right) \geq 0$$

for all $x, y \in X$ with $x \preceq y$. If the following conditions are satisfied:

(i) *there exists $x_0 \in X$ such that $fx_0 \preceq x_0$;*
(S) *X has property (S),*

then f has a fixed point. Moreover, the set $Fix(f)$ is well ordered if and only if f has a unique fixed point.

Theorem 5.5. *Adding to the hypotheses of Theorem 5.4 the following condition:*

(H) *for all $z, w \in Fix(f)$ that are not comparable there exists $v \in X$ such that $v \prec z$ and $v \prec w$,*

then f has a unique fixed point in X.

Also in the setting of ordered b-metric space, we can deduce some results of fixed point analogous to Corollaries 4.1-4.4, via specific choices of b-simulation functions. In order to avoid repetition we omit the details.

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